$A = \begin{pmatrix} (J\varphi(s)\cdot e_1)^T \\ \vdots \\ (J\varphi(s)\, e_n)^T \\ N^T(p) \end{pmatrix}$, $|A|>0$, $B = \begin{pmatrix} (J\psi(t)\, e_1)^T \\ \vdots \\ (J\psi(t)\, e_n)^T \\ N^T(p) \end{pmatrix}$, $|B|>0$. But $J\psi(t) = J_{\varphi\circ h}(t) \cdot J_h(t) = J\varphi(s) J_h(t)$ as $\psi = \varphi \circ h$.

where $N(p)$ is the orientation of S.

$AB^T = \begin{pmatrix} J_\varphi^T(s) \\ N^T(p) \end{pmatrix} \left(J\varphi(s) J_h(t),\ N(p) \right) = \begin{pmatrix} J_\varphi^T(s)\cdot J\varphi(s) \cdot J_h(t) & 0 \\ 0 & 1 \end{pmatrix}$ So $|A|\cdot|B| = |J_\varphi^T(s) J_\varphi(s)| \cdot |J_h(t)|$

As $J_\varphi^T J_\varphi$ is positive semi-definite, $|J_\varphi^T(s) J_\varphi(s)| \geq 0$. But $|A|, |B| > 0$. So $|J_h(t)| > 0$.

Since p is any point on W and ψ is bijective, so $|J_h(t)|>0$ for any $t \in \psi^{-1}(W)$

Thus $\psi|_{\psi^{-1}(W)} = \varphi\circ h|_{\psi^{-1}(W)}$ is reparametrization of $\varphi|_{\varphi^{-1}(W)}$

17.19 Denote $x = X(p) = (x_1, x_2, x_3)$, $y = Y(p) = (y_1, y_2, y_3)$

$(W_X \wedge W_Y)(v,w) = W_X(v) W_Y(w) - W_X(w) W_Y(v) = (x\cdot v)\cdot(y\cdot w) - (x\cdot w)(y\cdot v)$

$= \left(\sum_{i=1}^3 x_i v_i\right)\left(\sum_{i=1}^3 y_i w_i\right) - \left(\sum_{i=1}^3 x_i w_i\right)\left(\sum_{i=1}^3 y_i v_i\right)$

$(X\times Y)(p)\cdot(v\times w) = (x_2y_3 - x_3y_2,\ x_3y_1 - x_1y_3,\ x_1y_2 - x_2y_1)\cdot(v_2w_3 - v_3w_2,\ v_3w_1 - v_1w_3,\ v_1w_2 - v_2w_1)$

$= x_2v_2y_3w_3 + x_3v_3y_2w_2 + x_3v_3y_1w_1 + x_1v_1y_3w_3 + x_1v_1y_2w_2 + x_2v_2y_1w_1$

$- x_2y_3v_3w_2 - x_3y_2v_2w_3 - x_3y_1v_1w_3 - x_1y_3v_3w_1 - x_1y_2v_2w_1 - x_2y_1v_1w_2$

$= \left(\sum_{i=1}^3 x_i v_i\right)\left(\sum_{i=1}^3 y_i w_i\right) - \left(\sum_{i=1}^3 x_i w_i\right)\left(\sum_{i=1}^3 y_i v_i\right)$

So $(W_X\wedge W_Y)(v,w) = (X\times Y)(p)\cdot(v\times w)$

18.1 $\varphi(t,\theta) = (t,\ y(t)\cos\theta,\ y(t)\sin\theta)$. $t\in I$, $\theta\in[0,2\pi)$

$E_1^\varphi = \frac{\partial\varphi}{\partial t} = (1,\ y'\cos\theta,\ y'\sin\theta)$, $E_2^\varphi = \frac{\partial\varphi}{\partial\theta} = (0,\ -y\sin\theta,\ y\cos\theta)$

$N = E_1^\varphi\times E_2^\varphi / \|E_1^\varphi\times E_2^\varphi\| = (1+y'^2)^{-1/2}(y', -\cos\theta, -\sin\theta)$

$L_p(E_1(p)) = -\frac{\partial N}{\partial t} = \left(\frac{-y''}{(1+y'^2)^{3/2}}, \frac{-y''y'\cos\theta}{(1+y'^2)^{3/2}}, \frac{-y''y'\sin\theta}{(1+y'^2)^{3/2}}\right) = -y''(1+y'^2)^{-3/2}(1, y'\cos\theta, y'\sin\theta)$

$L_p(E_2(p)) = -\frac{\partial N}{\partial\theta} = \left(0, \frac{-\sin\theta}{(1+y'^2)^{1/2}}, \frac{\cos\theta}{(1+y'^2)^{1/2}}\right) = -\frac{1}{y(1+y'^2)^{1/2}}(0, -y\sin\theta, y\cos\theta)$

So $k_1(t,\theta) = -y''(t)/(1+y'^2)^{3/2}$, $k_2(t,\theta) = -\left(y(1+y'^2)^{\frac12}\right)^{-1}$

18.2 $E_1 = \frac{\partial\varphi}{\partial t} = (\cos\theta, \sin\theta, 0)$, $E_2 = \frac{\partial\varphi}{\partial\theta} = (-t\sin\theta, t\cos\theta, 1)$

$N = \frac{1}{\sqrt{1+t^2}}(\sin\theta, -\cos\theta, t)$. $L_p(E_1) = -\frac{\partial N}{\partial t} = \frac{-1}{(1+t^2)^{3/2}}(-t\sin\theta, t\cos\theta, 1) = -(1+t^2)^{3/2}E_2$

$L_p(E_2) = -\frac{\partial N}{\partial\theta} = -(1+t^2)^{-1/2}(\cos\theta, \sin\theta, 0) = -(1+t^2)^{-1/2}E_1$

So the matrix of L_p wrt E_1, E_2 is $\begin{pmatrix} 0 & -(1+t^2)^{-3/2} \\ -(1+t^2)^{-1/2} & 0\end{pmatrix}$. $H=0$.

18.3 By Ex 10.1. Let $\alpha(t) = (x(t), y(t))$, $t\in I$, then $k\circ\alpha = (x'y'' - x''y')/(x'^2+y'^2)^{3/2}$

If $k\equiv 0$, then $x'y'' - y'x'' = 0$. Since α is regular, so either $x'\neq 0$ or $y'\neq 0$

Suppose $y'\neq 0$ in some subinterval of I, then $\left(\frac{x'}{y'}\right)' = 0$, $\frac{x'}{y'} = C_1$, $x - C_2 = C_1(y - C_3)$

So $X = C_1 Y + C_2$. Suppose $X' \neq 0$ in some subinterval of I, similarly $Y = C_1' X + C_2'$.
It is obvious that a line segment parallel to x_1-axis and a line segment parallel to x_2-axis do not fit together smoothly, so S is a segment of a straight line.

18.4. Suppose the two principal curvatures are k_1, k_2. Then minimal surface $\Rightarrow k_1 + k_2 = 0$

So $K = k_1 k_2 \leq 0$.

18.5 Suppose the Weingarten map L_p has two eigenvalues λ_1, λ_2 corresponding to two eigenvectors v_1, v_2 which are orthonormal. $\forall \hat{v} \in S_p$ $\exists \alpha_1, \alpha_2 \in \mathbb{R}$, s.t. $\hat{v} = \alpha_1 v_1 + \alpha_2 v_2$.

So $k(\hat{v}) = L_p(\hat{v}) \cdot \hat{v} = (\alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2)(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1^2 \lambda_1 + \alpha_2^2 \lambda_2$

As S is minimal surface, so $\lambda_1 + \lambda_2 = 0$, $\lambda_2 = -\lambda_1$, $k(\hat{v}) = \lambda_1(\alpha_1^2 - \alpha_2^2)$

Now let $v = \frac{\sqrt{2}}{2}(v_1 + v_2)$, $w = \frac{\sqrt{2}}{2}(v_1 - v_2)$, then $v \cdot w = 0$.

$k(v) = \lambda_1\left(\left(\frac{\sqrt{2}}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2\right) = 0$. $k(w) = \lambda_1\left(\left(\frac{\sqrt{2}}{2}\right)^2 - \left(-\frac{\sqrt{2}}{2}\right)^2\right) = 0$.

18.6 $\underbrace{dN_p(v)}_{\forall v \in S_p} = \nabla_v N\big|_p = -L_p(v)$. Suppose the principal curvatures are λ_1, λ_2 corresponding to principal curvature directions v_1, v_2. Since v_1, v_2 span S_p, so $\exists \alpha_1, \alpha_2 \in \mathbb{R}$, $v = \alpha_1 v_1 + \alpha_2 v_2$

$\|dN_p(v)\| = \|L_p(v)\| = \|L_p(\alpha_1 v_1 + \alpha_2 v_2)\| = \|\alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2\| = \sqrt{\alpha_1^2 \lambda_1^2 + \alpha_2^2 \lambda_2^2}$

As S is minimal surface, $\lambda_1 = -\lambda_2$, So $\|dN_p(v)\| = |\lambda_1| \sqrt{\alpha_1^2 + \alpha_2^2} = |\lambda_1| \cdot \|v\|$.

18.7 $\cancel{\frac{d}{ds}\Big|_0 \ell(\alpha_s) = \frac{d}{ds}\Big|_0 \int_a^b \|\dot{\alpha}_s(t)\| dt = \int_a^b \frac{d}{ds} \|\dot{\alpha}_s(t)\|}$

$\frac{d}{ds}\Big|_0 \ell(\alpha_s) = \frac{d}{ds}\Big|_0 \int_a^b \|\dot{\alpha}_s(t)\| dt = \int_a^b \frac{d}{ds}\Big|_0 \|\dot{\alpha}_s(t)\| dt$ (*)

$\frac{d}{ds}\Big|_0 \|\dot{\alpha}_s(t)\| = \frac{d}{ds}\Big|_0 \sqrt{\dot{\alpha}_s(t) \cdot \dot{\alpha}_s(t)} = 2 \dot{\alpha}_s(t) \frac{d}{ds} \dot{\alpha}_s(t) / 2\|\dot{\alpha}_s(t)\| \Big|_{s=0}$

As $\cancel{\dot{\alpha}_s(t)|_0 = \dot{\alpha}(t)}$ $\cdot$ $\|\dot{\alpha}_s(t)|_{s=0}\| = \|\dot{\alpha}(t)\| = 1$, $\dot{\alpha}_s(t)\big|_{s=0} = \dot{\alpha}(t)$

$\frac{d}{ds}\Big|_0 \dot{\alpha}_s(t) = \frac{\partial^2 \Psi(t,0)}{\partial s \partial t}$. So $\frac{d}{ds}\Big|_0 \|\dot{\alpha}(t)\| = \dot{\alpha}(t) \cdot \frac{\partial^2 \Psi(t,0)}{\partial s \partial t}$. Plugging into (*)

$\frac{d}{ds}\Big|_0 \ell(\alpha_s) = \int_a^b \dot{\alpha}(t) \cdot \frac{\partial^2 \Psi(t,0)}{\partial s \partial t} dt = \int_a^b \dot{\alpha}(t)\, d \frac{\partial \Psi(t,0)}{\partial s} = \int_a^b \dot{\alpha}(t)\, d X(t)$

$= \dot{\alpha}(t) X(t)\Big|_a^b - \int_a^b \ddot{\alpha}(t) X(t) dt$ (Note $X(t) = \frac{\partial \Psi(t,s)}{\partial s}\Big|_{s=0} := \frac{\partial \Psi(t,0)}{\partial s}$)

Using Ex 10.6, $\ddot{\alpha}(t) = k(t) N$, we have $\frac{d}{ds}\Big|_0 \ell(\alpha_s) = \dot{\alpha}(t) X(t)\Big|_a^b - \int_a^b (X \cdot N) k(t) dt$

If $\Psi(a,s) = \alpha(a)$, $\Psi(b,s) = \alpha(b)$ $\forall s$ (i.e., compactly supported), then

$X(a) = 0 = X(b)$ So $\frac{d}{ds}\Big|_0 \ell(\alpha_s) = -\int_a^b (X \cdot N) k(t) dt$.