16.5 (a) \( J_0 = \frac{\pi}{2} \frac{\partial}{\partial \theta} \int_0^\theta \left( N^2(x(t_0)), x(t_0) + s(N \cdot \alpha)(0) \right) \)

so \( X(0) = J_0 \left|_{\theta=0} \right. = \dot{x}(0) + 5(N \cdot \alpha)(0) \)

\( X(0) = \dot{x}(0) = v \) \( X(\theta) = (N \cdot \alpha)(0) \) so \( \dot{x}(0) = (N \cdot \alpha)(0) = \frac{d}{d\theta}(u) \)

(b) \( \dot{x}(0) = 0 \). So \( X(\theta) = \left( \int_0^\theta \dot{x}(0), \dot{x}(0) + s(N \cdot \alpha)(0) \right) = (\phi(\theta), v + sw) \)

(c) \( X(\theta) = 0 \iff \frac{d}{d\theta} V = -s \frac{\partial}{\partial \theta} (N \cdot \alpha)(0) = s \frac{d}{d\theta}(u) \)

So \( \frac{d}{d\theta} \alpha \) is a principal curvature and \( V \) is a principal curvature direction.

By Thm. 1 \( \frac{d}{d\theta}(x(0) + \frac{1}{2} \cdot N(s(\theta)) = s(t_0) + s(N \cdot \alpha)(\theta)) = \beta(\theta) \) is focal point of \( S \) along \( \beta \).

17.1 \( V(\theta) = \int_0^\theta \int_0^r \left( \frac{-r \sin^2 \theta \cos \theta}{\sin \theta \cos \theta} \right) \right) \right) \left( \frac{1}{r d\theta d\phi} \right) = 2\pi a \)

17.2 \( E_1 = \frac{\partial}{\partial \theta} = (-r \sin \theta, r \cos \theta, 0), E_2 = \frac{\partial}{\partial r} = (r \cos \theta, r \sin \theta, -h) \)

\( E_1 \cdot E_2 = t^2 \sqrt{r^2 + h^2} \) \( E_1 \cdot E_1 = 0 \) \( V(\theta) = \int_0^\theta \int_0^r \sqrt{r^2 + h^2} d\theta d\phi = \pi r \sqrt{r^2 + h^2} \)

17.3 \( E_1 = \frac{\partial}{\partial \theta} = (-a \cos \theta + b \sin \theta, a \cos \theta + b \sin \theta, 0), E_2 = \frac{\partial}{\partial r} = (-b \sin \theta \cos \theta, a \sin \theta \cos \theta, -h \cos \theta) \)

\( E_1 \cdot E_2 = (a \cos \theta)^2 + (b \cos \theta)^2, E_1 \cdot E_1 = a^2, E_2 \cdot E_2 = b^2, E_1 \cdot E_2 = 0 \) \( V(\theta) = \int_0^\theta \int_0^a \sqrt{(a+b \cos \theta)^2} d\theta d\phi = 4\pi^2 ab \)

17.4 \( E_1 = \frac{\partial}{\partial \theta} = (-a \sin \theta, a \cos \theta, 0), E_2 = \frac{\partial}{\partial r} = (a \sin \theta, a \cos \theta, -h \sin \theta) \)

\( E_1 \cdot E_1 = (a \sin \theta)^2 + (a \cos \theta)^2, E_1 \cdot E_2 = a, E_2 \cdot E_2 = b^2, E_1 \cdot E_2 = 0 \) \( V(\theta) = \int_0^\theta \int_0^a \sin \theta (a \sin \theta)^2 d\theta d\phi = 4\pi^2 ab \)

17.5 \( E_1 = \frac{\partial}{\partial \theta} = (a \cos \sin \phi, -a \sin \sin \phi, 0), E_2 = \frac{\partial}{\partial r} = (a \cos \sin \phi, a \sin \sin \phi, -h \sin \phi) \)

\( E_1 \cdot E_1 = (a \cos \sin \phi)^2 + (a \sin \sin \phi)^2, E_1 \cdot E_2 = a, E_2 \cdot E_2 = b^2, E_1 \cdot E_2 = 0 \) \( V(\theta) = \int_0^\theta \int_0^a \sin \theta (a \sin \phi)^2 d\theta d\phi = 2\pi^2 \)

17.6 \( E_1 = (x(t), y(t), (x(t))) \), \( E_2 = (0, -y(t), x(t), y(t), (x(t)) \)

\( E_1 \cdot E_1 = (x(t))^2 + (y(t))^2, E_1 \cdot E_2 = x(t)^2, E_1 \cdot E_2 = 0 \)

\( V(\theta) = \int_0^\theta \int_0^r \sqrt{x(t)^2 + (y(t))^2} d\theta d\phi = 2\pi \int_0^\theta x(t) (x(t))^2 + (y(t))^2) dt \)

17.7 \( \text{Let } A_i = \frac{\partial}{\partial x_i}, \text{ then } E_i = \frac{\partial}{\partial x_i} = (0, 0, 0, \ldots, 0, a_i) \)

\( N = \pm (a, a, \ldots, a, a, \ldots) \left( \int_0^\theta a_i^2 \right)^{1/2} \)

\( a_i, a_i^2, a_i^3, \ldots, a_i^n \)
17.8 (a) Prove \( J_n \) is not singular. We prove by induction. When \( n=2 \), \( J_2 = \begin{pmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{pmatrix} \). Then rank \( J_2 = \text{rank} \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \end{pmatrix} = 2 \), so \( J_2 \) is fully ranked. Suppose rank \( J_{n-1} = n-1 \).

Denote the \((i,j)\)th component of \( J_n \) as \( J_{ij} \), \( i=1,...,n \), \( j=1,...,n \). Then
\[
J_n = \begin{pmatrix}
J_{11} & \cdots & J_{1n} \\
\vdots & \ddots & \vdots \\
J_{n1} & \cdots & J_{nn}
\end{pmatrix} = \begin{pmatrix}
\sin \theta_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \sin \theta_n
\end{pmatrix}.
\]
So rank \( J_n = n+1 = n+1 \), i.e., \( J_n \) is parametrized \( n \)-surface.

(b) \( \varphi_n \) maps \( U \) one to one onto a subset of unit \( n \)-sphere \( S^n \). Let \( \varphi_n^i \) be the \( i \)th component of \( \varphi_n \), then \( \frac{\partial}{\partial x^i} \varphi_n = 1 \). So \( \varphi_n \) maps \( U \) to a subset of \( S^n \).

We only need to prove one to one. If \( \varphi_n(\theta_1,...,\theta_n) = \varphi_n(\hat{\theta}_1,...,\hat{\theta}_n) \), then \( \cos \theta_1 = \cos \hat{\theta}_1 \). As \( \theta_1, \hat{\theta}_1 \in (0, \pi) \), \( \theta_1 = \hat{\theta}_1 \). Thus \( \varphi_n \) is one to one.

(c) If \( \gamma \in S^n \), then \( \gamma = (\sin \theta_1, \cos \theta_1) \). This is because if \( \int_0^{\pi} \sin \theta_1 = 0 \), \( \varphi_n : U \to S^n \) is obvious that \( \varphi_n : U \to S^n \) with \( U = \{(x_1,...,x_n) | \cos \theta_1 > 0, 0 \leq \theta_1 < \pi \} \).

(d) \( \int_{x=0}^{x=n} \sin \theta_n \cdot J_{n1} \varphi_n(\theta_1,...,\theta_{n-1},0) \cos \theta_n d\theta_n = \int_{x=0}^{x=n} \sin \theta_n \cdot J_{n1} \varphi_n(\theta_1,...,\theta_{n-1},0) \cos \theta_n d\theta_n = 0 \).

(e) Note the fact: \( I_n = \int_0^{\pi} \sin \theta_n d\theta_n = \frac{\pi}{n} \). So \( I_n = \frac{n!}{n^{n-1}} \) if \( n \) is even and \( I_n = \frac{(n-1)!}{n^{n-2}} \) if \( n \) is odd.

17.9 Denote \( V_i = \frac{\partial}{\partial u^i} \) \( i=1,2 \). \( N = V_i \times V_2 / ||V_i \times V_2|| \).

\[
A(\varphi) = \int_{V_1 / V_2} V_1 / ||V_1 / V_2|| = \int_{V_1 / V_2} (V_1 V_2) / ||V_1 / V_2|| = \int_{V_1 / V_2} 1 / ||V_1 / V_2||
\]

17.10 (a) By Ex. 14.9, \( W \) is normal vector field along \( \varphi \).

\[
E_V(\varphi) = \frac{\partial}{\partial u^i} \left| W_{V_i} \right| = \sum_{i=1}^{n} W_{V_i} = 0. \text{ So } W / ||W|| \text{ is the orientation vector field along } \varphi.
\]

(b) \( V(\varphi) = \int_{V_1 / V_2} V_1 / ||V_1 / V_2|| = \int_{V_1 / V_2} W / ||W|| = \int_{V_1 / V_2} 1 / ||W||
\]

17.11 Let \( \varepsilon = (e_1,...,e_n) \) with \( e_i = (0,...,0,1,0) \in \mathbb{R}^n \), \( A = \left( E_i^1, \cdots, E_i^n \right) \). \( B = \left( E_i^{\varepsilon_1}, \cdots, E_i^{\varepsilon_n} \right) \).
As \( N^p \) is orientation vector field along \( \Phi \), So \( |B| > 0 \).

As \( \Phi(t)=0 \), \( A= (J^p \Phi, J_\Phi \Phi, \ldots, J^{p-1} \Phi, J_\Phi \Phi, N^p \Phi) = (J^p \Phi, J_\Phi \Phi, N^p \Phi) \) if we assume \( N^p = N^p \Phi \).

Then \( B = (J^p \Phi, \Phi N^p \Phi), A^TB = \begin{pmatrix} J^p \Phi, J_\Phi \Phi \end{pmatrix} (J^p \Phi) = 0 \) \( B^TB = (J^p \Phi, J_\Phi \Phi) (J^p \Phi, J_\Phi \Phi) \)

The zeros are because: \( (N^p \Phi)^T J^p \Phi = 0 \) by definition of \( N^p \) and \( (N^p \Phi)^T J^p \Phi = 0 \) by the fact that \( \{e_1 \ldots e_n\} \) forms a basis of \( \mathbb{R}^n \) so \( J_\Phi \Phi e_i \) can be written as a linear combination of \( \{e_1 \ldots e_n\} \).

So \( |A^TB| = |J_\Phi \Phi| \), \( |B^TB| = (J^p \Phi)^T (J^p \Phi) \).

So \( |A||B| = |J_\Phi \Phi| |B| > 0 \) as \( |J_\Phi \Phi > 0, |B| > 0 \) so \( |A|=|J_\Phi \Phi| |B| > 0 \)

So \( N^p = N^p \Phi \) satisfies all the conditions to be orientation vector field.

17.12 (b) First prove \( W(V_1 \ldots V_i \ldots V_k) = W(V_1 \ldots V_i \ldots V_k) \) where \( x \in \mathbb{R} \).

This is because the latter \( = W(V_1 \ldots V_i \ldots V_k) + x W(V_1 \ldots V_i \ldots V_k) \) and \( W(V_1 \ldots V_i \ldots V_k) = 0 \) by skew-symmetry.

If \( (V_1 \ldots V_k) \) is linearly dependent, then exist \( \lambda_1 \ldots \lambda_k \in \mathbb{R} \) s.t. \( \sum_{i=1}^{k} \lambda_i V_i = 0 \) and \( \sum_{i=1}^{k} \lambda_i = 0 \), so \( W(V_1 \ldots V_k) = W(V_1 \ldots V_k) \) assume \( \lambda_i = 0 \), then \( W(V_1 \ldots V_i \ldots V_k) = W(V_1 \ldots V_i \ldots V_k) \).

(6) If \( k > n \), then \( (V_1 \ldots V_k) \) must be linearly dependent, so \( W = 0 \).

17.13 (a) \( \xi(V_1 \ldots V_n)^2 = \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} (V_1 \ldots V_n) = 1 \) \( So \( \xi(V_1 \ldots V_k) = \pm 1 \) and \( \xi(V_1 \ldots V_n) = 1 \) iff \( (V_1 \ldots V_n) \) is consistent with \( N \).

(6) We only need to prove \( W(U_1 \ldots U_n) = W(V_1 \ldots V_n) \xi(U_1 \ldots U_n) \) for any \( \{U_1 \ldots U_n\} \in S_p \) on \( S \). As \( \{V_1 \ldots V_n\} \) forms a basis of \( S_p \), so there exist \( \alpha_j \in \mathbb{R} \) s.t. \( V_i = \sum_j \alpha_j V_j \) and \( \sum_j \alpha_j = 0 \), then \( W(V_1 \ldots V_n) = 0 \) as \( \sum_j \alpha_j = 0 \).

So \( W(U_1 \ldots U_n) = \frac{\sum_j \alpha_j \alpha_j \alpha_j}{0} (V_1 \ldots V_n) \xi(U_1 \ldots U_n) \).

Likewise \( \xi(U_1 \ldots U_n) = \frac{\sum_j \alpha_j \alpha_j \alpha_j}{0} (V_1 \ldots V_n) \xi(U_1 \ldots U_n) \).

Notice \( W(V_1 \ldots V_n) = (\xi(V_1 \ldots V_n)) W(V_1 \ldots V_n) \).

So \( W(U_1 \ldots U_n) = \frac{\sum_j \alpha_j \alpha_j \alpha_j}{0} (V_1 \ldots V_n) \xi(U_1 \ldots U_n) W(V_1 \ldots V_n) = W(V_1 \ldots V_n) \xi(U_1 \ldots U_n) \).

Continuing (1) we have

and plugging (3)(4) into (1),

42
We only need to prove skew symmetry. To this end, we only need to prove for $i,j \in \{1, \ldots, k+l\}$, $(W_{1} \wedge W_{2})(V_{i} \wedge V_{j} \wedge V_{k} \wedge V_{l}) = - (W_{1} \wedge W_{2})(V_{j} \wedge V_{i} \wedge V_{k} \wedge V_{l})$.

For $V_{0}$, if let $p,q \not\in \mathbb{Z}(0) = i, q(0) = j$, If $p,q \leq k$, then $V_{i}, V_{j}$ both appear in $W_{i}$ under such $\hat{\sigma}$, so swapping $V_{i}, V_{j}$ will just reverse the sign. The same happens if $p,q > k$

If $p \leq k, q > k$, then look at $\hat{\sigma}$ which is the same as $\hat{\sigma}$ except $\hat{\sigma}(p) = j$, $\hat{\sigma}(q) = i$.

So $\text{sign } \hat{\sigma} = - \text{sign } \sigma$. For $(W_{1} \wedge W_{2})(V_{i} \wedge V_{j} \wedge V_{k} \wedge V_{l}), \text{ we have summands}$

$\text{sign } \sigma (W_{1} \wedge W_{2})(V_{i} \wedge V_{j} \wedge V_{k} \wedge V_{l}) = - (\text{sign } \hat{\sigma}) (W_{1} \wedge W_{2})(V_{j} \wedge V_{i} \wedge V_{k} \wedge V_{l}).$

For $(W_{1} \wedge W_{2})(V_{i} \wedge V_{j} \wedge V_{k} \wedge V_{l}), \text{ we have summands}$

$(\text{sign } \sigma) (W_{1} \wedge W_{2})(V_{i} \wedge V_{j} \wedge V_{k} \wedge V_{l}) = - (\text{sign } \hat{\sigma}) (W_{1} \wedge W_{2})(V_{j} \wedge V_{i} \wedge V_{k} \wedge V_{l}).$

So except the summands for swapped $V_{i}, V_{j}$, all summands have opposite sign.

This also happens to $p > k$, $q \leq k$. So in all $(W_{1} \wedge W_{2})(V_{i} \wedge V_{j} \wedge V_{k} \wedge V_{l}) = -(W_{1} \wedge W_{2})(V_{j} \wedge V_{i} \wedge V_{k} \wedge V_{l})$.

We only need to prove that if

$$\sigma(1, \ldots, \sigma(k), \ldots, \sigma(k+l)) = (\hat{\sigma}(1), \ldots, \hat{\sigma}(k+l), \hat{\sigma}(1), \ldots, \hat{\sigma}(l)),$$

$$W_{1}(V_{\sigma(1)} \wedge V_{\sigma(k)} \wedge V_{\sigma(k+l)}) \wedge W_{2}(V_{\sigma(k+l)} \wedge V_{\sigma(k+l)}) = \prod_{i=1}^{k+l} \frac{1}{2} \text{sign } \hat{\sigma} \text{sign } \sigma W_{1}(V_{\sigma(1)}) \wedge W_{2}(V_{\sigma(k+l)}) W_{3}(V_{\sigma(k+l)}),$$

then

$$\text{sign } \sigma = (-1)^{k+l} \text{sign } \hat{\sigma}.$$ This boils down to how many number of swaps is needed in order to change $(\ldots, a_{k+1}, a_{k}, \ldots)$ to $(\ldots, a_{k}, a_{k+1}, \ldots)$, and we only care about the odd/even of the number. One schedule is pushing $a_{k+1}$ ahead for by swapping with the element to its left for $k$ times, i.e.,

$$(\ldots, a_{k}, a_{k+1}) \rightarrow (\ldots, a_{k+1}, a_{k}) \rightarrow (\ldots, a_{k+1}, a_{k+1}, \ldots).$$

Doing the same for $2k+1, \ldots, k+l$, then we change $(\ldots, a_{k+l}, a_{k}, \ldots)$ to $(\ldots, a_{k}, a_{k+1}, \ldots)$ in $k+1$ steps.

Since the odd/even of step number is independent of schedule,

we prove $\text{sign } \sigma = (-1)^{k+l} \text{sign } \hat{\sigma}$.

(c) $(W_{1} \wedge (W_{2} \wedge W_{3})) = \frac{1}{k+l+1} \sum_{\sigma} \text{sign } \sigma W_{1}(V_{\sigma(1)}) \wedge W_{1}(V_{\sigma(2)}) \wedge W_{1}(V_{\sigma(3)}) = \frac{1}{k+l+1} \sum_{\sigma} \text{sign } \sigma W_{1}(V_{\sigma(1)}) \wedge W_{1}(V_{\sigma(2)}) \wedge W_{1}(V_{\sigma(3)})$.

(d) $(W_{1} \wedge W_{2}) \wedge W_{3} = \frac{1}{k+l+1} \sum_{\sigma} \text{sign } \sigma W_{1}(V_{\sigma(1)}) \wedge W_{1}(V_{\sigma(2)}) \wedge W_{1}(V_{\sigma(3)}) W_{2}(V_{\sigma(4)}) \wedge W_{2}(V_{\sigma(5)}) \wedge W_{2}(V_{\sigma(6)}) W_{3}(V_{\sigma(7)})$.

where $\sigma$ is a permutation of $1, \ldots, (k+l+m)$ and $\hat{\sigma}$ is a permutation of $1, \ldots, k+l$. Notice $\text{sign } \sigma \text{sign } \hat{\sigma} = \text{sign } (\sigma \circ \hat{\sigma})$.

For each $W_{1}(V_{i} \wedge V_{j} \wedge V_{k} \wedge V_{l})$, there exist $(k+l)!$ different combinations of $\sigma$ and $\hat{\sigma}$ which finally results in this order of subscript by permuting from $(1, \ldots, (k+l+m))$. In fact, for any $\hat{\sigma}$, there exists a unique $\sigma$, such that $\sigma \circ \hat{\sigma}$ yields above subscripts. Besides, all such combinations $\sigma$ have the same sign of $\sigma \circ \hat{\sigma}$. So (d) is
equal to \( \frac{1}{k!} \cdot \sum_{i=0}^{k} \binom{k}{i} \cdot W_1(V_0(V_1(V_2(\ldots(V_k)\ldots)))) \).

For the same reason, \( W_1 \wedge (W_2 \wedge W_3) \) is also equal to (1). Thus, \( (W_1 \wedge W_2) \wedge W_3 = W_1 \wedge (W_2 \wedge W_3) \).

(2) First prove for \( k \in \{1, m\} \) \( (W_1 \wedge \cdots \wedge W_k) (X_1 \cdots X_k) = 1 \). by induction

If \( k = 1 \), then \( W_1(X_1) = X_1 \cdot p \cdot X_1(p) = 1 \). If it's true for \( k \), then

\[ (W_1 \wedge \cdots \wedge W_{k+1}) (X_1 \cdots X_{k+1}) = \frac{1}{k!} \sum_{(\sigma,k)} (W_1 \wedge \cdots \wedge W_k) (X_{\sigma(1)} \cdots X_{\sigma(k+1)}) \]

If \( \sigma(k+1) = k+1 \), then \( W_{k+1}(X_{\sigma(k+1)}) = X_{\sigma(k+1)} \). Therefore, \( \sigma(k+1) = k+1 \). Then

\[ (W_1 \wedge \cdots \wedge W_{k+1}) (X_1 \cdots X_{k+1}) = \frac{1}{k!} \sum_{(\sigma,k)} (W_1 \wedge \cdots \wedge W_k) (X_{\sigma(1)} \cdots X_{\sigma(k+1)}) \]

So, \( (W_1 \wedge \cdots \wedge W_k) (X_1 \cdots X_k) = 1 \) for all \( k = 1 \). This step is not necessary. We just follow the hint on textbook.

Next prove for \( k \in \{1, m\} \). \( \neq 1 \), \( X_k \wedge (W_1 \wedge \cdots \wedge W_k) = 0 \). Actually, \( V_1 \cdots V_k \wedge S_p \),

\[ X_k (V_1 \cdots V_k) (V_1 \cdots V_k) = (W_1 \wedge \cdots \wedge W_k) (X_k (p), V_1, \ldots, V_k). \]

Expanding as in the definition,

\[ X_k (V_1 \cdots V_k) \]

if \( X_k (p) \) appear in \( W_1 \wedge \cdots \wedge W_k \), then \( W_x (X_k (p)) = X_k (p) \cdot X_k (p) = 0 \)

if \( X_k (p) \) appear in \( W_1 \wedge \cdots \wedge W_k \), then by some induction like proof, it's easy to show \( (W_1 \wedge \cdots \wedge W_k) (\ldots, X_k (p), \ldots) = 0 \). So \( X_k (V_1 \cdots V_k) = 0 \). Finally, as \( \{X_1 \cdots X_m\} \) is an orthonormal basis for \( S_p \), by Ex. 17.13, \( f(p) = 1 \)

because \( (W_1 \wedge \cdots \wedge W_k) (X_1 (p), \ldots X_k (p)) = 1 \). So \( W_1 \wedge \cdots \wedge W_k \).

17.15 (a) Multilinearity is obvious. \( f^{*}w(V_0(\cdots V_n)) = w(d(f(V_0(\cdots d(f(V_n)))) \]

\[ = \sum (\text{sign} \sigma) \cdot w(df(V_1), \ldots, df(V_n)) = (\text{sign} \sigma) f^{*}w(V_1, \ldots, V_n) \]

As \( w, df \) are smooth, \( f^{*}w \) is also smooth.

(b) \[ \int_{p}^{f^{*}w} = \int_{w}^{w} (df(E^p), \ldots, df(E^p)) = \int w(E^{f(p)}, \ldots E^{f(p)}) = \int_{f_p} w \]

(c) Suppose \( f_i \) is a partition of unity on \( S \) subordinate to a collection \( \{U_i\} \) of one-to-one local parametrizations of \( S \). We prove first that \( f_i \circ f^{*}f \) is a partition of unity of \( S \), subordinate to a collection \( \{f_0 f_i\} \) of one-to-one local parametrizations of \( S \).

(1) \( \forall \gamma \in S \), \( f^{-1}(\gamma) \in \gamma \) (as \( f \) is differentiable), \( \sum f_i f^{-1}(\gamma) = 1 \).

(2) \( \forall \gamma \in S \), \( f^{-1}(\gamma) \in S \), thus \( \frac{f_i}{f_i^{-1}} f^{-1}(\gamma) = 1 \).

(3) as \( f_i \) and \( f_i^{-1} \) are both one-to-one, so \( f_0 f_i \) is one-to-one. As \( f_i \) is regular and \( f_i^{-1} \) is invertible. \( f_i \) is differentiable, so \( f_0 f_i \) is regular. \( f_0 f_i \) is also local parametrization of \( S \).

Besides, it is orientation-preserving and \( f_0 \) is open.

Suppose \( f_i \) is identically zero outside the image under \( f_i \) of a compact subset of \( S \).
$B_i$ of $U_i$, then $f$ is smooth, $f(B_i)$ is also compact, $\forall x \in f(B_i)$, $f^{-1}(x)$ and $x$ are in $f(\psi_i(\psi_i^{-1}(x)))$, then if $f_i(f^{-1}(x)) = 0$, then $f_i \psi_i(B_i)$, then $x = f_i(f^{-1}(x)) \in f(\psi_i(B_i))$, which contradicts with our assumption. So $f_i(f^{-1}(x)) = 0$.

So $f_i \psi_i$ is identically 0 outside the image under $f \psi_i$ of a compact subset $B_i$ of $U_i$.

Combining $D-9$, we conclude $f_i \psi_i$ is a partition of unity of $S$, subordinate to a collection $(f \psi_i)$ of one-to-one local parametrization of $S$.

Finally, $\frac{\partial}{\partial u} \int_S f^*w = \frac{\partial}{\partial u} \int_S f_i \psi_i^*w = \frac{\partial}{\partial u} \int_{f_i \psi_i(B_i)} w \left( f_i \psi_i^{-1}(x), \psi_i^{-1}(x) \right) = \frac{\partial}{\partial u} \int_{f_i \psi_i(B_i)} w \left( \psi_i^{-1}(x), \psi_i^{-1}(x) \right) = \int_{f \psi_i(B_i)} w = \int_S w$

17.6

For $\forall \mu \in \mathbb{R}^n$, if $v_1 \ldots v_n \in \mathbb{R}^n$ is a basis of $\mathbb{R}^n$, and $\|v_i\| > 0$, then $\frac{\partial}{\partial v_i} N(p) = -N(p)$, so $\frac{\partial}{\partial v_i} = (-1)^{n+1} \frac{\partial}{\partial v_i}$, which is positive if $n$ is odd.

17.17(a) $\lim_{t \to 0} h(t) = 0$, $h'(t) = \frac{t}{2} \varepsilon^2$, so $\lim_{t \to 0} h'(t) = 0$. Generally, $h^{(n)}(t)$ must be in the form of $h^{(n)}(t) = P(t) \varepsilon^2$, where $P(t)$ is a polynomial function of $t$ with finite degree.

(b) $h(t) = h(u(t))$, where $u(t) = t^2 - t^2$. Since both $u(t)$ and $h(u)$ are smooth, so $h(t)$ is smooth.

In the proof of Thm 4, $\Psi_0$ is smooth, $\Psi_0^{-1}$ and $\Psi_0^{-1}$ is smooth, $\Psi_0^{-1}(0) = 1$ and $\Psi_0^{-1}(1)$ is also smooth, $\Psi_0^{-1}(0) = 1$ and $\Psi_0^{-1}(1)$ is also smooth, $\Psi_0^{-1}(0)$ is a diffeomorphism.

The textbook only defines "orientation preserving" for a map between two oriented $n$-surfaces in $\mathbb{R}^n$ at a regular point, so we don't know what it means by $h$ being orientation preserving, because $h$ maps from $\Psi_0^{-1}(w)$ (open set) to $\Psi_0^{-1}(w)$ (open set). However, we can still prove that $Jh > 0$, thus $\Psi_0^{-1}(w)$ is a reparametrization of $\Psi_0^{-1}(w)$.

For any point $p \in W$, suppose $s = \Psi_0^{-1}(p)$, $t = \Psi_0^{-1}(p)$. Since both $\Psi$ and $\Psi_0$ are local parametrizations of $S$, we have $\Psi_0^{-1}(\Psi_0(s)) = h(t)$ and
\[ A = \begin{pmatrix} (\psi(5))\mathcal{E}^T \\ (\psi(5))\mathcal{E}N(p) \end{pmatrix}, \quad |A| > 0, \quad B = \begin{pmatrix} (\psi(5))\mathcal{E}^T \\ (\psi(5))\mathcal{E}N(p) \end{pmatrix}, \quad |B| > 0. \] But \( J\psi(t) = J\psi(h(t)) \cdot J_h(t) = J\psi(s) \cdot J_h(t) \) as \( \psi = \psi \circ h \).

Where \( N(p) \) is the orientation of \( S \).

\[ AB^T = \begin{pmatrix} (\psi(5)) \quad (\psi(5)) \quad J_h(t) \quad N(p) \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial s} (\psi(s)) \cdot J_h(t) \quad 0 \end{pmatrix} \quad So \quad |A| \cdot |B| = \left| \frac{\partial}{\partial s} (\psi(s)) \right| \cdot |J_h(t)|. \]

As \( \psi^T \psi \) is positive semi-definite, \( |J\psi(s) \cdot J\psi(s)| \geq 0 \). But \( |A| \cdot |B| > 0 \), so \( J\psi(t) \) > 0.

Since \( p \) is any point on \( W \) and \( \psi \) is bijective, so \( |J_h(t)| > 0 \) for any \( t \in \mathcal{Y}(W) \).

Thus \( \psi^* \gamma \gamma (w) = \psi \circ h \cdot \gamma^*(w) \) is reparameterization of \( \psi \gamma^*(w) \).

17.9 Denote \( x = X(p) = (x_1, x_2, x_3) \), \( y = Y(p) = (y_1, y_2, y_3) \).

\[ (W \times W_Y)(v, w) = W_X(v) \cdot W_Y(w) - W_X(v) \cdot W_Y(w) = (x \cdot v) \cdot (y \cdot w) - (x \cdot w) \cdot (v \cdot w) \]

\[ = \left( \frac{3}{4} x_1 v_1 \right) (\frac{3}{4} y_1 w_1) - \left( \frac{3}{4} x_1 w_1 \right) (\frac{3}{4} y_1 v_1). \]

\[ (X \times Y)(p) \cdot (v \times w) = (x_1 y_3 - x_3 y_1, x_3 y_2 - x_2 y_3, x_2 y_1 - x_1 y_2), (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1). \]

\[ = x_1 v_1 y_3 w_2 + x_3 v_3 y_1 w_1 + x_3 v_3 y_1 w_1 + x_1 v_1 y_3 w_2 + x_1 v_1 y_3 w_2 + x_2 v_2 y_1 w_1 \]

\[ - x_2 y_2 v_3 w_2 - x_2 y_2 v_2 w_3 - x_2 y_2 v_1 w_3 - x_1 y_2 v_3 w_1 - x_1 y_2 v_2 w_1 - x_2 y_1 w_1 w_2 \]

\[ = \left( \frac{3}{4} x_1 v_1 \right) (\frac{3}{4} y_1 w_1) - \left( \frac{3}{4} x_1 w_1 \right) (\frac{3}{4} y_1 v_1). \]

So \( (W \times W_Y)(v, w) = (X \times Y)(p) \cdot (v \times w) \).