that first the restriction of $X$ to $C$ is a tangent vector field on $C$, because $\nabla X_{\mathbf{f}_1} = (\partial_1 X, \partial_2 X) = (0, 0)$ for $i = 1, 2$.

So $(f_{i,0})'(t) = \nabla X_{\mathbf{f}_i}(u(t)), \dot{u}(t) = \nabla X_{\mathbf{h}_i}(u(t)), X(u(t)) = 0$. So $f_{i,0}$ is constant, thus $\mathbf{f}_i \in C$.

To make a map $\mathbf{J}$ onto $C$, first of all, $C$ must be connected. The proof is similar to Theorem 1 in Chapter 1, except the construction of rectangle $B$, we now construct $A = \{ \mathbf{v}_1, \mathbf{v}_2 \} \subset \mathbb{R}^3$.

Denote $\mathbf{f}(s, s) = \mathbf{f}(\mathbf{p}_0 + s \mathbf{v}_1 + s \mathbf{v}_2 + r, u)$. Then $J_{\mathbf{f}} = \{(\mathbf{f}(s, s) = \mathbf{f}(\mathbf{p}_0 + s \mathbf{v}_1 + s \mathbf{v}_2 + r, u)) : s \in \mathbb{R}, \mathbf{f}(s, s) \in C \}$, when $\mathbf{v}_i = \nabla X_{\mathbf{f}_i}(u(t)) \mathbf{u}(t)$.

Then $\mathbf{E}$, $\mathbf{E}_2$, $\mathbf{E}_3$ are chosen as follows. First, so that $\mathbf{J}$ is fully ranked for all $p \in A$. This is possible because $\mathbf{J}_{\mathbf{f}}$ is fully ranked, and Denote $J_{\mathbf{f}}(s, s) = \mathbf{f}(\mathbf{p}_0 + s \mathbf{v}_1 + s \mathbf{v}_2 + ru)$, then $J_{\mathbf{f}} = \mathbf{f}(s, s) = (s, s) = (0, 0, 0)$. As rank$(\mathbf{P}\mathbf{P}) = \text{rank}(\mathbf{P}) = \text{rank}(\mathbf{P}) = 2$ for $\mathbf{P} \in \mathbb{R}$. So $J_{\mathbf{f}}$ is fully ranked for all $p \in A$. Applying inverse function, then, if $r$ is chosen such that $\exists s, s, s + \mathbf{f}(\mathbf{p}_0 + s \mathbf{v}_1 + s \mathbf{v}_2 + r, u) = \mathbf{p}$, then such $s, s, s$ are unique in $(t, x, y)$.

Now let $y(t) = \mathbf{p}_0 + t \mathbf{v}_1 + t \mathbf{h}_0(t) + t \mathbf{h}_1 + t \mathbf{h}_2$, where $\mathbf{h}_0(t) = (v(t) - \mathbf{p}_0) + \mathbf{u}/11v_1t^2^2$.

$\mathbf{h}_0(t) = \mathbf{h}_0(t), v(t) = \mathbf{h}_2(t), \mathbf{h}_1(t), \mathbf{h}_2(t)$ are all smooth and $\mathbf{h}(t) = 1, 0, 0, 0$.

$\mathbf{h}_0(t) = \mathbf{h}_0(t), v(t) = \mathbf{h}_2(t), \mathbf{h}_1(t), \mathbf{h}_2(t)$.

By definition that $\mathbf{h}(t) = \mathbf{X}(\mathbf{p}_0) = \mathbf{u}$. So we can choose $t < 0 < t_2$ (small enough), such that $\mathbf{h}(t) > 0$, set $r_1 = \mathbf{h}_1(t)$, $r_2 = \mathbf{h}_2(t)$, then for $\mathbf{v} \in C(r_1, r_2)$, $\exists! t \in (t_1, t_2)$, $s, t \in (t, x, y)$.

Let $\mathbf{v} = \mathbf{p}_0 + s \mathbf{v}_1 + s \mathbf{v}_2$. Then its belonging to $B \Rightarrow \exists! t \in (t, x, y)$, $\mathbf{h}(t) = 1, 0$. Its belonging to $C \Rightarrow \exists! t \in (t, x, y)$. Now that we know $\mathbf{p} \in C$, $\mathbf{p} \in C$, $\mathbf{p} \in C$.

Now $\mathbf{v} = \mathbf{p}_0 + s \mathbf{v}_1 + s \mathbf{v}_2 \in C$. Now that we know $\mathbf{p} \in C$, $\mathbf{p} \in C$.

16.1 (a) By using the rest of $\mathbf{E}5$ (i.e., 4), we have the curvature for outward orientation is $\kappa = (\frac{\partial X}{\partial u} + 1) \times \frac{\partial X}{\partial v}$.

By applying Theorem 1.1, the focal point on the normal line starting from $p$ is $P + \mathbf{F}(p, N(p))$.

(b) For example, for $a > 2, b = 1$, see http://xiahorgeous.ams.edu.au/
xizhang/reading3/1ex0101.jpg

16.2 (a) Only need to prove that for $q$ sufficiently close to $p$, $N(q)$ and $N(q)$ are not parallel in $\mathbb{R}^2$.

Otherwise, for $\forall \mathbf{e} \in \mathbb{R}^2$, $\exists! C \in \mathbb{R}^2$ such that $\mathbf{N}(q)$ and $N(q)$ are parallel. But as $N$ is smooth, $\|N(p) - N(q)\| < 2$, in the neighborhood close enough to $p$, $\|N(p) - N(q)\|^2$ must be less than an arbitrary small positive number. So $N(q) = N(p)$, so $\left(\frac{N(p) - N(q)}{\|N(p) - N(q)\|}\right) (\mathbf{e} \cdot \mathbf{p}) = 0$. As $\|N(p) - N(q)\| < 2$, $r \in S$, which is compact,
there must be a subsequence of \((\frac{\ln \gamma - p}{\ln \gamma - p})\) which converges to \(v(0) = 1\). Without loss of
generality, we assume that subsequence is \((\ln \gamma - p)\) itself. Let \(k \to \infty\), We have \(v_N = 0\),
be because \(\lim_{\ln \gamma - p} = v(\ln \gamma - p)\). So \(q_\gamma \to q_\gamma\), \(v_N = 0\) contradicts with the
assumption that the curvature \(k(0) \neq 0\). So for \(\gamma \in C\) sufficiently close to \(p\), the
normal line to \(C\) at \(p\) and \(q\) intersects at some point \(h_\gamma \in \mathbb{R}^2\).

(b) First derive \(h(\gamma)\). \(p + S_1 N(\gamma) = q + S_2 N(\gamma)\). Suppose there is a local parametrization
of \(C\) about \(p = q\), \(\alpha(0) = q\), \(\alpha(t) = q\) and suppose \(I\) is small enough at.
\(t \in I\), \(\alpha(t)\) satisfies (a). So to derive \(h(\alpha(t))\), suppose \(\alpha(t) + S_1 N(\alpha(t)) = \alpha(t) + S_1 N(\alpha(t))\)
\((\alpha(t), \alpha(t))\). Multiply both sides by \(\alpha(t)\) and Notice \(N(\alpha(t)) \cdot \alpha(t) = 0\), so
\(\alpha(t) \cdot \alpha(t) = (\alpha(t) \cdot \alpha(t)) + S_1 N(\alpha(t)) \cdot \alpha(t) = \alpha(t) \cdot \alpha(t). By assumption,\( N(\alpha(t)) \)
not parallel with \(\alpha(t)\), so \(N(\alpha(t)) \cdot \alpha(t) \neq 0\) so \(S_1 = \frac{(\alpha(t) \cdot \alpha(t)) \cdot (\alpha(t) \cdot \alpha(t))}{N(\alpha(t)) \cdot \alpha(t)}\).
\(\alpha(t) \cdot \alpha(t) = \alpha(t) \cdot \alpha(t)\).

Both numerator and denominator \(\to 0\) as \(t \to 0\). So using \(L'Hospital's rule,\) the derivative of
denominator is \(N(\alpha(t)) \cdot \alpha(t) \), which equals \(-k_o(0) || \alpha(t)||^2\).
the derivative of numerator is \(\alpha(t) \cdot \alpha(t)\) which equals \(-k(0) \alpha(t) \cdot \alpha(t)\),
\(N(\alpha(t)) \cdot \alpha(t) + \alpha(t) \cdot \alpha(t)\) \(N(\alpha(t)) \cdot \alpha(t)\)
\(+ N(\alpha(t)) \cdot \alpha(t)\) \(\alpha(t) \cdot \alpha(t)\) \(-N(\alpha(t)) \cdot \alpha(t)\) \(\alpha(t) \cdot \alpha(t)\)
\(= -||\alpha(t)||^2 (N(\alpha(t)) \alpha(t) \alpha(t) + k(0) \alpha(t) \cdot \alpha(t))\)
\(= \alpha(t) + \frac{k(0)}{||\alpha(t)||^2} (N(\alpha(t)) \alpha(t) \alpha(t) + k(0) \alpha(t) \cdot \alpha(t))\).

By \(\lim_{t \to 0} h(\alpha(t)) = \alpha(t) + \frac{k(0)}{||\alpha(t)||^2} (N(\alpha(t)) \alpha(t) \alpha(t) + k(0) \alpha(t) \cdot \alpha(t))\).

16.3 (a) \(\alpha(t) = \psi(t) + \frac{d}{dt} \alpha(t) \frac{1}{K(t)}\) \((N(\alpha(t)) \alpha(t) \alpha(t) - k(0) \alpha(t) \cdot \alpha(t))\).

As \(N(\alpha(t)) \alpha(t) \alpha(t)\) \(-k(0) \alpha(t) \cdot \alpha(t)\)
\(\alpha(t) = \frac{1}{K(t)} k'(t) (N(\alpha(t)) \alpha(t) \alpha(t) - k(0) \alpha(t) \cdot \alpha(t))\).

So \(\alpha(t) = 0\) iff \(k'(t) = 0\).

(b) As \(\alpha(t)\) is parallel to \(N(\alpha(t)) \alpha(t) \alpha(t)\) and by definition \(\alpha(t)\) is on the normal
to image \(\psi(t) \cdot \psi(t)\), So the latter is tangent at \(\alpha(t)\) to the focal locus of \(\psi\).

and by \(\lim_{t \to \infty} \alpha(t) = \alpha(t)\) is the focal locus of \(\psi\).

(c) The sum is \(\alpha = \frac{1}{K(t)} ||\alpha(t)|| dt + ||\alpha(t)-\psi(t)||\) \((N(\alpha(t)) \alpha(t) \alpha(t) - k(0) \alpha(t) \cdot \alpha(t))\).

Suppose \(k'(t) = b ||\alpha(t)|| \) and \(k(t) = 0 \alpha(t) ||\alpha(t)|| \) \((a, b) \in (0, 1)\) as both \(k(t)\) and \(k'(t)\) keep their sign
for \(t \in (0, 1)\). So \(\alpha = \frac{1}{k'(t)} \alpha(t) = \frac{1}{k'(t)} \alpha(t) + \alpha \frac{\frac{1}{K(t)} k'(t)}{||\alpha(t)||} \alpha(t)\).

Notice that if \(a = 0\) then the conclusion in this exercise doesn't hold. Otherwise if \(k(t) \cdot k'(t) < 0\), \(\frac{d}{dt} \alpha(t) = 0\) so \(\alpha = \text{constant}\).
An example of $ab=1$ is the focal parabola. If we parameterize by $\Psi(t) = \left(t, \frac{1}{2}t^2 \right) + c \left(-\infty \atop \infty \right)$
then $\Psi(t) = (1,t)$, $N = \frac{1}{\sqrt{1+4t^2}}$, $k(t) = \frac{(1+t^2)^{3/2}}{2t}$, $k > 0$,
then $|P_{\Psi(t)}| + \text{length of } \alpha \ldots \text{can't be constant}$.
So we will need $kk' < 0$, like what the Figure 16.6 shows.

16.4 (a) Let $\Psi(s) = \Psi(t) + sN(\Psi(t))$. For each $s \leq \frac{1}{k(t)}$, $\Psi(s)$ is not a focal point by Thm 1, so $I_s \neq I$. If $t \in I_s$, then $\Psi_s(t) \neq 0$. As $\Psi_s$ is continuous, there must be $\epsilon > 0$
s.t. $\Psi(t + \epsilon) \in I_s$, i.e. $t \in I_s$. Thus $I_s$ is open.

(b) Suppose $\Psi(t)$ is unit speed, which doesn't lose generality as the conclusion only takes care of to. $\Psi_s(t) = \Psi(t) + sN(\Psi(t)), k(t) = \langle \dot{\Psi}(t), N(\Psi(t)) \rangle$
by definition $\Psi(0) - \Psi(0) = 0$. $\|N(\Psi(t))\| = 0 \Rightarrow (N \circ \Psi)(N(\Psi(t))) = 0$, so $\Psi_s(t) \neq 0$.
So $N_\Psi \Psi_s(t) = N(\Psi(t))$. To check the direction, we notice that
$\Psi_s(t) = (\dot{\Psi}(t) + \frac{1}{k(t)} N(\Psi(t))) \cdot N = \|\dot{\Psi}(t)\|^2 + \frac{k(t)}{k(t)} = \|\dot{\Psi}(t)\|^2 + s(-k\|\dot{\Psi}(t)\|^2)$
This holds for $S < \frac{1}{k(t)}$. So if $k(t) > 0$, then $\Psi_s(t)$ is in the same direction as $\dot{\Psi}(t)$
and $N_s(\Psi_s(t)) = N(\Psi(t)).$ If $k(t) < 0$, then $N_s(\Psi_s(t)) = -N(\Psi(t))$

1. $k(t) > 0$, $k_s(t) = \frac{\dot{\Psi}(t)N_s(\Psi_s(t))}{\|\Psi_s(t)\|^2}$
$= (\dot{\Psi}(t) + s\langle \dot{\Psi}(t), N(\Psi(t)) \rangle) \cdot \frac{N(\Psi(t))}{\|\Psi(t)\|^2}$
$= \frac{k(t)}{k(t)} \cdot \frac{k(t)}{k(t)} \cdot \frac{k(t)}{k(t)} \cdot \frac{k(t)}{k(t)}$,

So $k_s(t) = \langle (\dot{\Psi}(t) - k \dot{\Psi}(t)) \rangle / \|\Psi_s(t)\|^2$

2. $k(t) < 0$, $k_s(t) = -\frac{\dot{\Psi}(t) \cdot N(\Psi(t))}{\|\Psi(t)\|^2}$
$= \langle (\dot{\Psi}(t) - k \dot{\Psi}(t)) \rangle / \|\Psi(t)\|^2$
similar to above, we have
$\dot{\Psi}(t) = k_s(t) = -\left(\frac{1}{k(t)} - S \right)^{-1}$
We suspect that it should be $|S| \leq \frac{1}{k(t)}$

or simply assume $k(t) > 0$. To double check we are correct:

see parabola again and $\Psi(t) = (t, \frac{1}{2}t^2)$, $\dot{\Psi}(t) = (1, t)$, $k(t) = \frac{1}{t} \cdot \frac{1}{t}$,
$\dot{\Psi}(t) = (1, t)$.

Let $S = -2 < 1/k(t)$ at $P$, the curvature should still be negative, while the conclusion in the textbook exercise
insists $k(t) = \frac{1}{1-t^2} = 1 > 0$.
16.5 (a) \[ J_P |_{t=0} = (N^2(x(0)), \dot{x}(0) + 5(N \cdot \dot{x})(0)) \]

so \[ X(5) = J_P |_{t=0} (5) = \dot{x}(0) + 5(N \cdot \dot{x})(0) \]

\[ X(0) = \dot{x}(0) = V, \quad X(5) = (N \cdot \dot{x})(0) \quad \text{so} \quad \dot{X}(0) = (N \cdot \dot{x})(0) = 4V \]

(b) \[ \dot{X}(5) = 0 \quad \text{so} \quad X(5) = (\gamma(5, 0), \dot{x}(0) + 5(N \cdot \dot{x})(0)) = (\beta(5), v + 8w) \]

(c) \[ X(5) = 0 \iff \dot{v} = -5 (N \cdot \dot{x})(0) = 54V \]

So \( \frac{1}{s} \) is a principal curvature and \( \gamma \) is a principal curvature direction.

By Thm 1, \( \dot{x}(0) + \frac{1}{s} \cdot N^2(x(t)) = \dot{x}(0) + 5N \cdot \dot{x}(0) = \beta(5) \) is focal point of \( S \) along \( \beta \).