By Theorem 1 (Chapter 9), \( N(p) = -\nabla v N(p) \mathbf{v}(0) \) (\( N, N_i \) are orientation of \( \mathcal{C} \) and \( p = \mathbf{v}(0) \)).

\[
N(p) = N_i(p) = \lambda (p - \mathbf{v}(0))/r \quad \text{(Which determines orientation)} \quad \lambda = 1 \text{ outward, } \lambda = -1 \text{ inward.}
\]

\[
\nabla v N(p) = \nabla v N_i(p) = \lambda \frac{\mathbf{v}}{r}
\]

So \( N(\mathbf{v}(0)) \cdot \mathbf{v}(p) = \lambda \mathbf{v}(0) \cdot \mathbf{v}(p) = -\nabla v N_i(p) \mathbf{v}(0) = -\lambda \mathbf{v}(0) \cdot \mathbf{v}(0) = -\lambda
\]

So \( \lambda = -1 \). So \( f''(0) = -2 + 2(p - \mathbf{v}(0)) \mathbf{v}(0) = 0 \)

"If part" \( f''(0) = 0 \Rightarrow (p - \mathbf{v})(p - \mathbf{v})(0) = 0 \) as we are in 2D, and \( p - \mathbf{v} \in \mathcal{C}_p \). So \( \mathbf{v}(0) \) \( \mathcal{C}_p \).

But \( \mathbf{v}(0) \in \mathcal{C}_p \) as well and \( \mathcal{C}_p \) and \( \mathcal{C}_p \) are both one-dimensional, So \( \mathcal{C}_p = \mathcal{C}_p \); then we can easily choose an orientation of \( \mathcal{C} \) such that its orientation at \( p \) is the same as \( C_1 \).

Since \( \nabla v N \cdot \mathbf{N} = 0 \) \( \forall \mathbf{v} \in \mathcal{C}_p \), \( \mathbf{v} \in \mathcal{C}_p \) so \( \mathbf{v} \in \mathcal{C}_p \) as we are in 2D.

\[
\mathbf{a} = \nabla v N \cdot \mathbf{v}(0) = -\nabla v N(p) \mathbf{v}(0) = \lambda (p - \mathbf{v})(0) \mathbf{v}(0) = \lambda
\]

So \( \nabla v N \cdot \mathbf{v}(0) = \lambda \mathbf{v}(0) \cdot \mathbf{v}(0) \). But \( \nabla v N \mathbf{v}(0), N_i \mathbf{v}(0) \mathbf{v}(0) \) by Example in Chapter 9 on page 57.

So \( \nabla v N = \nabla v N_i \mathbf{v}(0) \mathbf{v}(0) \). Furthermore, \( \mathbf{v} \in \mathcal{C}_p \), \( \mathbf{v} \) must be \( \mathbf{v} = \mathbf{v}(0) \) \( \forall \mathbf{v} \in \mathcal{C}_p \) \( \mathcal{C}_p \).

But \( \nabla v N = \nabla v N \mathbf{v}(0) \mathbf{v}(0) = \mathbf{v}(0) \mathbf{v}(0) \mathbf{v}(0) = \mathbf{v}(0) \mathbf{v}(0) \mathbf{v}(0) \mathbf{v}(0) \mathbf{v}(0) \) \( \forall \mathbf{v} \in \mathcal{C}_p \).

Combining 6-6, \( \mathbf{v} \) is circle of curvature of \( \mathcal{C} \) at \( p \).

10.10 \( \mathbf{v}(t) = (\cos t, \sin t) \) (As \( t \) is local parameterization of \( \mathcal{C} \).)

\[
\mathbf{N}(\mathbf{v}(t)) = (-\sin t, \cos t) \cdot \mathbf{v}(t) = (\cos t, \sin t) \cdot \mathbf{v}(t).
\]

As \( u \) is unit speed, \( k = \mathbf{v} \cdot N(\mathbf{v}(t)) = \mathbf{v}(t) \cdot \frac{d\mathbf{v}}{dt} dt \).

11.1 \( \mathbf{v}(t) = \int_0^2 \sqrt{8 + 9 t^2} dt = \int_0^2 \sqrt{8 + 9 u} du = \frac{1}{2} \frac{u^{3/2}}{3} (u) \left[ \frac{1}{3} \frac{u^{3/2}}{3} (u) \right]_0^2 = \frac{1}{2} \frac{2}{3} \left( \frac{1}{3} (10 - 1) \right)
\]

11.2 \( \mathbf{v}(t) = \int_{-3}^{3} \sqrt{3} \sin 3t, 3 \cos 3t, 4 \right) dt = 10
\]

11.3 \( \mathbf{v}(t) = \int_0^\pi \sqrt{\sin 2t, 2 \cos t, 2 \cos t} \right) dt = \int_0^\pi -1 \sqrt{2} dt = 4 \sqrt{2}
\]

11.4 \( \mathbf{v}(t) = \int_0^\pi \sqrt{\sin t, \cos t, -\sin t, \sin t} \right) dt = 2 \sqrt{2} \pi
\]

11.5 \( \mathbf{v}(t) = (12t, 5t) \) \( t \in (-1, 1) \) \( \mathbb{R} \) \( \mathbb{R} \)

Actually, don't bother with orientation and \( \mathbf{v} \) component, because \( \mathcal{C}(t) \geq 0 \), and
1.6 \( \alpha(t) = (2\sin t, 1+2\cos t) \)
\[ \frac{d\alpha(t)}{dt} = \sqrt{\frac{d}{dt}(2\sin t)^2 + (1+2\cos t)^2} = 4\pi \]

1.7 \( \alpha(t) = (\sqrt{1+t^2}, t) \in (-\sqrt{2}, \sqrt{2}) \)
\[ \ell(c) = \int_0^\infty \| (t(t+1)^{1/2}, 1) \| dt = \sqrt{\frac{5}{3}} \int t^{1/2} \frac{dt}{(1+t)} \]

1.8 \( \alpha(t) = \left( \frac{2}{3} t^{3/2}, t \right) t \in (0.3) \)
\[ \ell(c) = \int_0^3 \| (t^{1/2}, 1) \| dt = \int_0^3 (1+t) dt = 14/3 \]

1.9 If \( \alpha(t) \) is consistent with \( N \), then \( \beta(t) = (x_1(t), x_2(t)) \) is consistent with \( N \)
\[ \beta(t) = R_{-1} (N(\alpha(t)), N_2(\alpha(t))) \]
So for \( \forall \in \pi \),
\[ \beta(t) = (-x_1(t), -x_2(t)) \]
\[ \int_{-\infty}^\infty \beta(t) dt = \int_{-\infty}^\infty \alpha(t) dt \]
So \( \ell(c) = \ell(c) \)

11.10 (a) \[ \int_{\beta} k_0 \omega(t) dt = \int_{\beta} \omega(\alpha(\tau)) dt = \int_{\beta} \omega(\alpha(t)) dt \]
Let \( \beta \) be a reparametrization of \( \alpha \).
\[ \beta = \alpha h \] (since \( \alpha \) and \( \beta \) are both consistent, there must exist \( h \) such that \( \beta = \alpha h \))
\[ \beta(t) = (x(t), h(t)) \] and \( h \) must be differentiable.
\[ h'(t) = \frac{d}{dt} (h(t)) \]
So \( \int_{-\infty}^\infty \omega(h(t)) dt = \int_{-\infty}^\infty \omega(t) dt \]
(b) \[ \int_{N(\alpha)} = \int_{N(\alpha)} \int_{\beta} \omega(\alpha(t)) dt \]

11.11 (a) \( d(f+g)(v) = (f+g)(v) = f(v) + g(v) \)
\[ \int_{\beta} (f+g)(v) = \int_{\beta} f(v) + g(v) \]
\[ \int_{\beta} f(v) + g(v) = \int_{\beta} f(v) + \int_{\beta} g(v) \]
So \( d(f+g) = d(f) + d(g) \)
(b) \( d(fg)(v) = f(g(v)) V + g(v) V \)
\[ \int_{\beta} f(g(v)) V + g(v) V = \int_{\beta} f(g(v)) V + \int_{\beta} g(v) V \]
So \( d(fg) = f d(g) + g d(f) \)
(c) \( d(h f)(v) = h'(f(v)) f'(v) \)
\[ \int_{\beta} h'(f(v)) f'(v) \]

11.12 (a) \[ \int_{\alpha} x_1 dx_1 + x_2 dx_2 = \int_{\alpha} 2 \sin t - 2 \cos t \]
\[ \int_{\alpha} 2 \sin t - 2 \cos t = -8 \pi \]
(b) \[ \int_{\alpha} x_1 dx_1 + x_2 dx_2 = \int_{\alpha} \frac{b \sin t}{(a \sin t + b \cos t) dt} = \int_{\alpha} \frac{b \sin t}{(a \sin t + b \cos t) dt} = 2 \pi a b \]
(c) \[ \int_{\alpha} x_1 dx_1 = f(x(0)) - f(x(1)) = \frac{1}{2} (x(1) - x(0)) \]
\[ df = \frac{1}{2} \sum_i x_i dx_i \]
\[ d(f) = \frac{1}{2} \sum_i x_i dx_i \]

11.13 \[ w(x(t)) = \frac{1}{\pi} \int_{x(t)} w(x(t)) \]
\[ \frac{\partial}{\partial t} \omega(x(t)) = \frac{\partial}{\partial t} (\omega(x(t))) \]
\[ \int w(x(t)) dt = \int_{\omega(x(t))} \]
11.15 Treat $x$ as $\theta$, then $x(t) = (\cos \theta(t), \sin \theta(t))$ by proof in Thm 3 ($\theta(t) = \theta_0 + \int_0^t \psi(s(t)) \, ds$) for $t$ in $I$. For uniqueness: If $\theta_1(t)$ and $\theta_2(t)$ satisfy $\cos \theta_1(t) = \cos \theta_2(t)$, $\sin \theta_1(t) = \sin \theta_2(t)$, $\theta_1(t) = \theta_2(t)$ for all $t \in I$. Proof: By first two equations, $\sin(\theta_1(t) - \theta_2(t)) = 0$ so $\cos(\theta_1(t) - \theta_2(t)) = 0$ for all $t \in I$. Hence $\theta_1(t) = \theta_2(t)$ as it holds for $t = 0$.

11.16 Let $\psi(t) = f(t) \cdot x(t)$. Define $\psi_1(t) = \psi_1(a) + \int_a^t \psi_1, \quad \psi_2(t) = \psi_2(a) + \int_a^t \psi_2$.

$\psi_1(a)$ is chosen so that $x(a) / ||x(a)|| = (\cos \psi_1(a), \sin \psi_1(a))$ and $\psi_1(a) \in [0, 2\pi)$.

$\psi_2(a)$ is chosen so that $x(a) / ||x(a)|| = (\cos \psi_2(a), \sin \psi_2(a))$ and $\psi_2(a) \in [0, 2\pi)$.

As $x(a) / ||x(a)|| = (\cos \psi_1(a), \sin \psi_1(a))$ and $x(a) / ||x(a)|| = (\cos \psi_2(a), \sin \psi_2(a))$, the choice of $\psi_1(a)$ is unique, we have $\psi_1(a) = \psi_2(a)$. Furthermore, by proof in Thm 3, $x(t) / ||x(t)|| = (\cos \psi_1(t), \sin \psi_1(t))$, $\psi_2(t) / ||\psi_2(t)|| = (\cos \psi_2(t), \sin \psi_2(t))$

As $x(a) / ||x(a)|| = (\cos \psi_1(a), \sin \psi_1(a))$, $\psi_2(a) / ||\psi_2(a)|| = (\cos \psi_2(a), \sin \psi_2(a))$, and $\psi_1(a) = \psi_2(a)$, we have $\psi_1(t) = \psi_2(t)$, $x(t) / ||x(t)|| = (\cos \psi(t), \sin \psi(t))$.

We now proceed to prove uniqueness in Ex 11.15. We have $\psi(t) = \psi_1(t)$, $x(t) / ||x(t)|| = (\cos \psi(t), \sin \psi(t))$.

11.17 Since by Ex 11.16, $x$ and $\psi(t)$ have the same winding number, it is now equivalent to proving that with $\psi(t,u)$ redefined as $\psi(t,u) = \psi(t) / ||\psi(t)||$, the result holds. Now $||\psi(t)|| = 1$ for all $u$ and $t$, and $\psi(t,u) / ||\psi(t,u)||$ is continuous as $||\psi(t,u)||$ is continuous, and $\hat{\psi}(t)$ is smooth on each $I_i, i \in I$.

As $[a_i, b_i] \times [0, 1]$ is compact, and $\hat{\psi}$ is continuous, $\hat{\psi}$ must be uniformly continuous, i.e., $\forall \epsilon > 0 \exists \delta > 0 \forall t, u \in I, \, ||t - u|| < \delta \Rightarrow ||\hat{\psi}(t) - \hat{\psi}(u)|| < \epsilon$. Specifically, let $\epsilon = \epsilon_2$ and $\epsilon_2 < \epsilon$, $\hat{\psi}(t) - \hat{\psi}(u) < \epsilon_2$.

Define $\hat{\psi}(t,v) = \psi_1(a) + \int_0^v \psi_1, \quad \hat{\psi}_x(t) = \psi_1(a) + \int_0^v \psi_1$. For $\forall t$, by proof in Thm 3, $\hat{\psi}_x(t) = (\cos \psi_x(t), \sin \psi_x(t))$.
So \( \hat{\varphi}(t) = \cos(\theta(t)) \). By (**) \( \cos(\theta(t) - \theta(t+\epsilon)) > 1 - \epsilon \)

Let \( \theta_0 = \arccos(1 - \epsilon) \). As \( \theta(0), \theta(t) \in [0, \pi] \), \( |\theta(t) - \theta(t+\epsilon)| < \theta_0 \)

So for all \( t \in [a, b] \), s.t. \( |\theta(t) - \theta(t+\epsilon)| < \theta_0 \)

If \( t_1, t_2 \) s.t. \( \theta(t_1) - \theta(t_2) = (2k_1\pi - \theta_0, 2k_1\pi + \theta_0) \)

Then \( \theta(t_1) - \theta(t_2) < \theta_0 \)

Note \( \theta_0 \in (a, \pi/2] \)

As \( \theta(t) - \theta(t+\epsilon) \) is continuous with respect to \( t \), there must exist \( t_0 \in (t_1, t_2) \)

s.t. \( |\theta(t_0) - \theta(t_0+\epsilon)| = 2\pi \cdot \min(|k_1, k_2|) + \epsilon \), if \( \epsilon \) is small enough and then \( \theta_0 \) small enough. But \( \cos(\theta(t_0) - \theta(t_0+\epsilon)) = 0 \) violating (**) . Thus there is a \( k_3 \in \mathbb{Z} \)

s.t. \( \theta(t_0+\epsilon) \in \theta_0 + \epsilon \), \( (2k_3\pi - \theta_0, 2k_3\pi + \theta_0) \)

But \( |\theta(t_0) - \theta(t_0+\epsilon)| = \frac{1}{2\pi} \mid (\theta(t_0) - \theta(t_0+\epsilon)) \mid < \epsilon \)

Thus \( \theta(t_0) = \theta(t_0+\epsilon) \)

Therefore \( \theta(t) = \theta(t+\epsilon) \) for all \( t \in [a, b] \)

Note \( \theta(t) \) can only assume integer value, \( \theta(t) = k(t) \)

Finally, as \( \theta(t) \) can only assume integer value, \( k(t) \) is a continuous function of \( t \)

11.18 (c) We define \( x(t) = (\phi(t) \cos t, \phi(t) \sin t) \) i.e. \( x(t) = (\phi(t) \cos t, \phi(t) \sin t) \)

Then following Example 2 on p. 75, \( \int_{-\infty}^{\infty} \sqrt{1 + \frac{\dot{x}^2(t)}{2 \pi}} dt = 2\pi \cdot \phi(t) \)

Let \( U \subset \mathbb{R}^2, U \neq 0 \) define \( \phi(t) = x(t) \cdot U \). Since \( x(t) \) is compact, \( U \) must attain its minimum \( \phi(t) \), say, at \( t_0 \). By Lagrange multiplier, \( \nabla x(t) = k(t) \)

So \( k(t_0) = x(t_0) \cdot U = 0 \), i.e. \( x(t) \perp U \). By definition, \( k(t_0) \) is continuous.

(As for \( k(t_0) \), when \( t \in (t_0, t_0+\epsilon/2) \)

Now the path \( \gamma \) is exact because \( \dot{\gamma}(t) \cdot u > 0 \) and \( \nabla U \cdot U = 0 \), because \( U \cdot U = 0 \).

So \( \int_{\gamma_{(t_0,t_0+\epsilon/2)}} \nabla U(x) \cdot \nabla U \gamma = 0 \)

For \( t \in (t_0, t_0+\epsilon/2) \)

\( x(t) = (2t-t_0-t_2, t_0+\epsilon/2) = (\phi(t_0) - \phi(t)) \cdot U(t) \)
Now we set \( V_1 = u, V_2 = R^2 - (r_1 v_1) r_2 \). \( \eta \) is exact on \( V \)

\[
\int_{\partial V} \eta = \int_{V} \Delta \tilde{\eta} = \int_{V} \Delta \eta = \int_{\partial V} \eta
\]

But for all \( u, \eta_u (2 \chi (t_0)) - \eta_u (\hat{\chi} (t_0)) + \eta_{-u} (\hat{\chi} (t_0)) - \eta_u (\hat{\chi} (t_0)) = \pm 2\pi \)

(1) If \( \chi \) is odd, then \( 2\pi \), if \( \chi \) is even, then \( -2\pi \)

Thus \( (\chi) \equiv \chi = \pm 1 \) i.e. rotation index is \( \pm 1 \).

11.19 (a) \( h'(t_0) = 0 \iff \chi (t_0) \cdot u = 0 \iff h(t_0) = 0 \).

\[\text{N}(\chi (t_0)) = \pm u, \quad \text{let} \quad \text{N}(\chi (t_0)) = d u \]

\[\delta = \pm 1. \]

\[h''(t_0) = 2 \chi (t_0) \cdot u = 2 \delta \]

(1) Since \( \chi (t) \) is periodic, \( h'(t_0) = 0 \) must have both minimum and maximum, say, \( t_0, t_0 \) resp.

\[h'(t_0) = h'(t_0); h''(t_0) = 0, h''(t_0) \\
\text{But since } N = \pm u, N(t_0) = \pm 1 \text{ so } h''(t_0) \]

So \( h''(t_0) < 0 \) \( \iff \delta = -1 \)

or \( \delta = 1 \) \( \iff \delta = 1 \)

By definition (b) \( \int_{t_0}^{t_0} \) is the rotation index of \( \chi \times 2\pi \).

As \( k > 0 \), \( t \) equals \( 2\pi \), which in turn equals \( \int_{t_0}^{t_0} \).

As \( k > 0 \), \( c = t_0 + T \). (a) has shown the Gaussian map in order.

Now we've proven that \( N(t) = N(t_0) \iff \chi (t) = \chi (t_0) \).

But \( T \) is period of \( \chi \).

So, Gaussian map is injection. In sum, it is one-to-one.

11.20 (a) \( \chi \) is just one point \( \omega_0 \), so obviously \( k(f) = 0 \) \( \text{const.} \neq \omega_0, 0 \)

(b) \( \chi (t) = (a \cos t, a \sin t) \) Similar to example 2 on p. 75, \( k(f) = 0 \).

(c) Construct \( \phi \): \( [0, \pi] \times [0, 1] \rightarrow R^2 - \{ 0 \} \)

\[\chi (t, u) = f(u (\cos t + i \sin t)) \]

\[\text{by } f(u) = 0 \text{ for } 0 \leq \theta \leq 1 \text{ obviously continuous.} \]

\[21\]
\[ \varphi(t) = f(t) = 0, \text{so } k(\varphi(t)) = 0. \]

Let \( \varphi_t'(t) = f(\cos t + i \sin t) \). Then

\[ k(f) = k(\varphi(t)) = k(\varphi(t)) = 0 \quad \text{(by Ex. 11.17)} \]

1d) Construct \( \varphi(t, u) = \left\{ \begin{array}{ll}
(\cos t + i \sin t) & \text{if } u \neq 0 \text{ on } [0, 2\pi] \times [0,1] \\
\mathbf{0} & \text{if } u = 0.
\end{array} \right. \]

By def of \( \varphi \) and \( \varphi(t) = 0 \), i.e. \( \mathbf{0} \), we have \( \varphi(t, u) = 0. \)

\( \triangle \) By \( \lim_{u \to 0} \mathbf{0} = \mathbf{0} \) \( \mathbf{0} \), we have \( \varphi(t, u) = 0. \) \( \triangle \) Says \( k(f) = 0 \). \( \triangle \) Says \( k(f) = 0. \) So either \( n = 0 \)

If \( n = 2 \), then \( \varphi(t) \) lies on positive \( x \)-axis, then choose \( v = (1,0) \), by \( \partial v \), we have \( k(v) = 0 \), correct.

Let \( \mathbf{a} = \mathbf{c} = \mathbf{b} \) be the set of all \( t \in (a, b) \). Define \( \varphi(t) = \mathbf{a} \) such that \( \varphi(t) \) lies on the positive \( x \)-axis. Note, \( \varphi(t) = \mathbf{a} \) is even if \( \mathbf{a} \) is on positive \( x \)-axis, we can still reparametrize \( \varphi(t) \) into \( \beta(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a}) \), then \( \beta(t) = x(t) \) which is on \( x \)-axis. Denote \( t_0 = 0 \). For all \( t = t_i \), \( i = 1, 2, \ldots, n \), if \( \varphi(t) \) crosses positive \( x \)-axis upward, define \( \delta_i = +1 \), if crosses downward, define \( \delta_i = -1 \).

If \( \varphi(t) = \mathbf{a} \) is on positive \( x \)-axis, define \( \delta_0 = \delta_{n+1} \) likewise in \( \delta_{13} \). \( \triangle \) \( \delta_i \) is \( \text{on positive } x \)-axis,

\[ k(v) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(t) \left[ \varphi(t) + \delta(t_0) \right] \, dt = \frac{1}{2\pi} \int_0^{2\pi} \varphi(t) \left[ \varphi(t) + \delta(t_0) \right] \, dt = \frac{1}{2\pi} \int_0^{2\pi} \varphi(t) \left[ \varphi(t) + \delta(t_0) \right] \, dt \]

where \( v = (1,0) \) and \( \partial v \) is defined as in proof of Thm 3. We check two consecutive crossings of \( x \)-axis: \( i = 1, \ldots, n \): \( \delta_i \) \( \text{inc} \) angle formula \( \text{at } t_i \)

\[ \text{angle means } \left[ \varphi(t_{i+1} - 2\pi), \varphi(t_{i+1}) \right] \equiv \sum_{i=1}^{m} \delta_i \]

So \( k(v) = \delta_0 + \delta_1 + \delta_2 + \ldots + \delta_{n+1} \)

If \( \varphi(t) \) is on positive \( x \)-axis, then \( k(v) = \frac{1}{2\pi} \left[ \varphi(t_0) + \delta(t_0) \right] = \frac{m}{2\pi} \delta_0 \) \( \text{(as } \delta_{n+1} = \delta_0 \text{)} \)

so the conclusion is correct in both cases.

Let \( \beta(t) = \varphi(t, -p) \)

1.22 \( \frac{\partial (c)}{\partial \beta} \): \( \beta(t) = \frac{1}{\partial \beta} \cdot \frac{\partial \beta(t)}{\partial (c)} \cdot \beta(t) = \frac{1}{\partial \beta} \cdot \frac{\partial \beta(t)}{\partial (c)} \cdot \beta(t) \)

So \( k(\beta) = \int \frac{\partial \beta(t)}{\partial (c)} \cdot k(\beta(t)) \, dt \). We know \( k(\beta) = 2k \pi \), so \( \frac{1}{\partial \beta} \cdot \frac{\partial \beta(t)}{\partial (c)} \) is \( \text{integer } \)

b) Suppose \( \alpha(t) = \alpha(t) \) and \( \alpha(t) \) are joined by \( \alpha(t) = \alpha(t) \)

Define \( \varphi(t, u) = \alpha(t) - \beta(t) \) on \( [a, b] \times [c, d] \rightarrow \mathbf{R}^2 \), \( \{0\} \). (Since \( \beta \rightarrow \mathbf{R}^2 \), \( \text{image} \) \( \beta \))

\[ \text{so } \varphi \neq 0 \]
Obviously, \( \varphi \) is continuous. \( \varphi(t, e) = \varphi(t) - P \). \( \varphi(t, d) = \varphi(t) - ? \)

\[ k(\varphi(t, e)) = k(\varphi(t, d)) \implies K_p(x) = K_p(x) \quad (a.s.) \]

12.1 The matrix corresponding to \( L_p \) is \( A = 1111^T(G, G^T - 1111^T I)_H \). \( (H \) is the Hessian) \( L_p(v) = -1111^T H \cdot v, K(v) = -1111^T H^T H \cdot v = y_p(v) \) for \( v \in S_p \).

12.2 \( \nabla f = (1, 1, \ldots, 1) \). \( V_i = \frac{\partial}{\partial x_i} (1, 0, \ldots, 1, 0, \ldots) \) where \( i \) is the \( i \)th spot after 1. \( i = 1, \ldots, n \)

\( \Delta f = \sum_{i=1}^{n+1} \frac{\partial f}{\partial x_i} x_i = 0 \). So \( \nabla f(v) = 0 \) \( \forall v \in S_p \) by \( \text{Ex. 12.1} \). So any \( v \in S_p \) is a principal curvature direction, with principal curvature 0.

\[ K(p) = 0 \quad H(p) = 0 \]

12.3 \( \nabla f = (2x_1, \ldots, 2x_n) \). \( \| \nabla f(p) \| = 2r \quad H = 2 \cdot I \). \( K(v) = 2a \frac{\xi \xi^T}{\xi^T \xi} \). \( \forall v \in S_p \), \( v = (v_1, v_2, v_3, \ldots, v_n) \), \( v_1^2 + \ldots + v_n^2 = 1 \). So \( K(v) \) attains its extremum at \((0, \pm 1, 0)\) and \((0, 0, \pm 1)\), i.e., principal curvature directions corresponding to curvatures \( \frac{\alpha}{b^2} \) and \( \frac{-\alpha}{b^2} \) respectively.

\[ K(p) = \frac{a}{b^2} \cdot c^2 \quad H(p) = \frac{\alpha}{b^2} \left( \frac{a}{b^2} + \frac{c^2}{b^2} \right) \]

12.4 \( \nabla f = \left( \frac{2x_1}{a^2}, \frac{2x_2}{b^2}, \frac{2x_3}{c^2} \right) \). \( H = \left( \begin{array}{ccc} \frac{2a^2}{b^2} & \frac{a}{b^2} & 0 \\ \frac{a}{b^2} & \frac{2b^2}{c^2} & 0 \\ 0 & 0 & \frac{2c^2}{a^2} \end{array} \right) \)

\[ \| \nabla f(p) \| = \frac{2a}{b} \quad \nabla f(p) = \left( \frac{2a}{b}, 0, 0 \right) \]

\[ K(v) = -\frac{a}{b} \left( \frac{2}{b^2} V_1^2 + \frac{2}{c^2} V_2^2 - \frac{2}{b^2} V_1^2 \right) \quad \forall v \in S_p \quad v = (0, V_2, V_3, \ldots, V_n) \quad V_2^2 + \ldots + V_n^2 = 1 \]

So \( K(v) \) attains its extremum at \((0, \pm 1, 0)\) and \((0, 0, \pm 1)\), i.e., principal curvature directions corresponding to curvatures \( \frac{\alpha}{b^2} \) and \( \frac{-\alpha}{b^2} \) respectively.

\[ K(p) = \frac{a}{b^2} \cdot c^2 \quad H(p) = \frac{\alpha}{b^2} \left( \frac{a}{b^2} + \frac{c^2}{b^2} \right) \]

12.5 \( \nabla f = (\frac{2x_1}{a^2}, \frac{2x_2}{b^2}, -\frac{2x_3}{c^2}) \). \( H = \frac{2}{a^2} \cdot \left( \begin{array}{ccc} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{array} \right) \)

\[ \| \nabla f(p) \| = \frac{2a}{b} \quad \nabla f(p) = \left( \frac{2a}{b}, 0, 0 \right) \]

\[ K(v) = -\frac{a}{b} \left( \frac{2}{a^2} V_1^2 + \frac{2}{b^2} V_2^2 - \frac{2}{c^2} V_3^2 \right) \quad \forall v \in S_p \quad v = (0, V_2, V_3, \ldots, V_n) \quad V_2^2 + \ldots + V_n^2 = 1 \]

So \( K(v) \) attains max when \( V_2^2 = 1 \), max = \( \frac{a}{b^2} \), attains min when \( V_2^2 = 0 \), min = \( \frac{a}{b^2} \).

12.6 \( \nabla f = (2x_1, 2x_2 (1 - 2(x_1^2 + x_3^2)^{3/2}), 2x_3 (1 - 2(x_2^2 + x_3^2)^{3/2})) \)

\[ H = \left( \begin{array}{ccc} \frac{3}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{3}{2} \end{array} \right) \]

For \( a \quad \nabla f(p) = (0, 2, 0) \) \( \| \nabla f(p) \| = 2 \), \( (p = (0, 0, 0)) \)

\[ H = \left( \begin{array}{ccc} \frac{3}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{3}{2} \end{array} \right) \]

12.6 \( b \quad \nabla f(p) = (0, 0, 2) \) \( \| \nabla f(p) \| = 2 \), \( (p = (0, 0, 0)) \)

\[ H = \left( \begin{array}{ccc} \frac{3}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{3}{2} \end{array} \right) \]