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Assignment 4

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1. **Cubic spline in $\mathbb{R}^2$.**

Solve the following variational problem

$$\min_{\gamma \in C^2([0,2],\mathbb{R}^2)} L(\gamma) \quad \text{with} \quad L(\gamma) = \int_0^2 \ddot{\gamma}^T \ddot{\gamma}\, dt$$

subject to

$$\gamma(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ \gamma(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \gamma(2) = \begin{pmatrix} 2 \\ 0 \end{pmatrix},\ \dot{\gamma}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \dot{\gamma}(2) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

Here $C^2([0,2],\mathbb{R}^2)$ denotes the set of twice continuously differentiable functions $[0,2] \to \mathbb{R}$.

Solution:

(a) $L(\gamma) = \int_0^2 \ddot{\gamma}^T \ddot{\gamma}\, dt = \int_0^1 \ddot{\gamma}^T \ddot{\gamma}\, dt + \int_1^2 \ddot{\gamma}^T \ddot{\gamma}\, dt$, so we break the problem into two subproblems:

$$\min_{\gamma_1 \in C^4([0,1],\mathbb{R}^2)} L(\gamma_1) \quad \text{with} \quad L(\gamma_1) = \int_0^1 \ddot{\gamma}_1{}^T \ddot{\gamma}_1\, dt,$$

subject to

$$\gamma_1(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ \gamma(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \dot{\gamma}_1(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \dot{\gamma}_1(1) = \begin{pmatrix} u \\ v \end{pmatrix}, \text{ where } u, v \in \mathbb{R},$$

and

$$\min_{\gamma_2 \in C^4([1,2],\mathbb{R}^2)} L(\gamma_2) \quad \text{with} \quad L(\gamma_2) = \int_1^2 \ddot{\gamma}_2{}^T \ddot{\gamma}_2\, dt,$$

subject to

$$\gamma_2(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \gamma_2(2) = \begin{pmatrix} 2 \\ 0 \end{pmatrix},\ \dot{\gamma}_2(1) = \begin{pmatrix} u \\ v \end{pmatrix},\ \dot{\gamma}_2(2) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

Using the result in the lecture, we know that

$$\gamma_1(t) = \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} t^3 + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} t^2 + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} t + \begin{pmatrix} a_0 \\ b_0 \end{pmatrix},\ \gamma_2(t) = \begin{pmatrix} c_3 \\ d_3 \end{pmatrix} t^3 + \begin{pmatrix} c_2 \\ d_2 \end{pmatrix} t^2 + \begin{pmatrix} c_1 \\ d_1 \end{pmatrix} t + \begin{pmatrix} c_0 \\ d_0 \end{pmatrix}.$$

Using the boundary conditions, we have

$$\gamma(0) = \gamma_1(0) = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \gamma(2) = \gamma_2(2) = \begin{pmatrix} 8c_3 + 4c_2 + 2c_1 + c_0 \\ 8d_3 + 4d_2 + 2d_1 + d_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\gamma(1) = \gamma_1(1) = \gamma_2(1) = \begin{pmatrix} \sum_{i=0}^3 a_i \\ \sum_{i=0}^3 b_i \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^3 c_i \\ \sum_{i=0}^3 d_i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\dot{\gamma}(0) = \dot{\gamma}_1(0) = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \dot{\gamma}(2) = \dot{\gamma}_2(2) = \begin{pmatrix} 12c_3 + 4c_2 + c_1 \\ 12d_3 + 4d_2 + d_1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix},$$

$$\dot{\gamma}(1) = \dot{\gamma}_1(1) = \dot{\gamma}_2(1) \Rightarrow \begin{pmatrix} 3a_3 + 2a_2 + a_1 \\ 3b_3 + 2b_2 + b_1 \end{pmatrix} = \begin{pmatrix} 3c_3 + 2c_2 + c_1 \\ 3d_3 + 2d_2 + d_1 \end{pmatrix},$$

$$\dot{\gamma}(1) = \ddot{\gamma}_1(1) = \ddot{\gamma}_2(1) \Rightarrow \begin{pmatrix} 6a_3 + 2a_2 \\ 6b_3 + 2b_2 \end{pmatrix} = \begin{pmatrix} 6c_3 + 2c_2 \\ 6d_3 + 2d_2 \end{pmatrix}$$

There are 16 variables and 16 linear equations. Solving it, we have

$a_3 = \frac{1}{2}, a_2 = -\frac{1}{2}, a_1 = 1, a_0 = 0$, $\qquad b_3 = 1, b_2 = -2, b_1 = 1, b_0 = 0$,

$c_3 = -\frac{3}{2}, c_2 = \frac{11}{2}, c_1 = -5, c_0 = 2$, $\qquad d_3 = -1, d_2 = 4, b_1 = -5, b_0 = 2$.

Thus,

$$\gamma(t)=\begin{cases}\gamma_1(t)=\begin{pmatrix}\frac{1}{2}\\1\end{pmatrix}t^3+\begin{pmatrix}-\frac{1}{2}\\-2\end{pmatrix}t^2+\begin{pmatrix}1\\1\end{pmatrix}t & for \quad t\in[0,1]\\ \gamma_2(t)=\begin{pmatrix}-\frac{3}{2}\\-1\end{pmatrix}t^3+\begin{pmatrix}\frac{11}{2}\\4\end{pmatrix}t^2+\begin{pmatrix}-5\\-5\end{pmatrix}t+\begin{pmatrix}2\\2\end{pmatrix} & for \quad t\in[1,2]\end{cases}.$$

2. **Geodesics on the set of real orthogonal (3×3)-matrices**

Consider the following constrained variational problem

$$\min_{X\in C^2([0,1],\mathbb{R}^{3\times3})} L(X) \quad \text{with} \quad L(\gamma)=\int_0^1 tr\dot{X}^T\dot{X}dt+\frac{1}{2}\int_0^1 trS\left(X^TX-I_3\right)dt$$

subject to

$$X(0)=I_3,\ X(1)=\exp\left(\begin{pmatrix}0 & \pi/2 & 0\\ -\pi/2 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right)=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&1\end{pmatrix}.$$

Here the smooth function $S:[0,1]\to Sym_3$ serves as a symmetric matrix valued Lagrange multiplier.

(a) Derive the Euler Lagrange equation.

(b) Solve the Euler Lagrange equation respecting the boundary conditions.

Solution:

(a) Let $X_\varepsilon=X+\varepsilon Y$, $S_\varepsilon=S+\varepsilon T$, where $Y(0)=Y(1)=0$ (0 matrix), $T(0)=T(1)=0$, $Y\in C^2\left([0,1],\mathbb{R}^{3\times3}\right)$, $T:[0,1]\to Sym_3$.

$$\left.\frac{d}{d\varepsilon}L(X_\varepsilon,S_\varepsilon)\right|_{\varepsilon=0}$$

$$=\left.\frac{d}{d\varepsilon}\left(\frac{1}{2}\int_0^1 tr\left(\dot{X}+\varepsilon\dot{Y}\right)^T\left(\dot{X}+\varepsilon\dot{Y}\right)dt+\frac{1}{2}\int_0^1 tr(S+\varepsilon T)\left((X+\varepsilon Y)^T(X+\varepsilon Y)-I_3\right)dt\right)\right|_{\varepsilon=0}$$

$$=\frac{1}{2}\int_0^1 tr\left(\dot{X}^T\dot{Y}+\dot{Y}^T\dot{X}\right)dt+\frac{1}{2}\int_0^1 trT\left(X^TX-I_3\right)dt+\frac{1}{2}\int_0^1 tr\left(SY^TX\right)dt+\frac{1}{2}\int_0^1 tr\left(SX^TY\right)dt$$

Notice that $tr\left(\dot{Y}^T\dot{X}\right)=tr\left(\dot{X}^T\dot{Y}\right)$, $tr\left(SY^TX\right)=tr\left(X^TYS^T\right)=tr\left(S^TX^TY\right)$, so go on:

$$=\int_0^1 tr\left(\dot{X}^T\dot{Y}\right)dt+\frac{1}{2}\int_0^1 trT\left(X^TX-I_3\right)dt+\frac{1}{2}\int_0^1 tr\left(S+S^T\right)X^TYdt$$

Notice that $\frac{d}{dt}tr\left(\dot{X}^TY\right)=tr\left(\ddot{X}^TY\right)+tr\left(\dot{X}^T\dot{Y}\right)$, so applying $Y(0)=Y(1)=0$, we have

$$\int_0^1 tr\left(\dot{X}^T\dot{Y}\right)dt=\int_0^1\frac{d}{dt}tr\left(\dot{X}^TY\right)dt-\int_0^1 tr\left(\ddot{X}^TY\right)dt=-\int_0^1 tr\left(\ddot{X}^TY\right)dt.$$

Thus, $\left.\frac{d}{d\varepsilon}L(X_\varepsilon,S_\varepsilon)\right|_{\varepsilon=0}=-\int_0^1 tr\left(\ddot{X}^TY\right)dt+\frac{1}{2}\int_0^1 trT\left(X^TX-I_3\right)dt+\frac{1}{2}\int_0^1 tr\left(S+S^T\right)X^TYdt$

$$=\frac{1}{2}\int_0^1 trT\left(X^TX-I_3\right)dt+\frac{1}{2}\int_0^1 tr\left(X\left(S+S^T\right)-2\ddot{X}\right)^TYdt$$

$= 0.$

Choose $T=\left(X^TX-I_3\right)t(1-t)\in Sym_3$, $Y=\left(X\left(S+S^T\right)-2\ddot{X}\right)t(1-t)$, we derive

$$\begin{cases}X^TX=I_3 & (1)\\ 2\ddot{X}=X\left(S+S^T\right) & (2)\end{cases}.$$

This is the Euler Lagrange equation.

(b) By (2), $S+S^T=2X^T\ddot{X}$. But $S+S^T$ is symmetric, thus $X^T\ddot{X}$ is also symmetric, i.e.,

$$\tfrac{1}{2}\left(S+S^T\right)=X^T\ddot{X}=\ddot{X}^TX. \quad (3)$$

We take the first derivative of (1): $\dot{X}^TX+X^T\dot{X}=0$. (4)

Take the second derivative of (1): $\ddot{X}^TX+2\dot{X}^T\dot{X}+X^T\ddot{X}=0$. (5)

By (3) and (5), we have $\tfrac{1}{2}\left(S+S^T\right)=X^T\ddot{X}=\ddot{X}^TX=-\dot{X}^T\dot{X}$. (6)

Now we prove that $\dot{X}^T\dot{X}$ is constant by showing that $\left(\dot{X}^T\dot{X}\right)^{\bullet}=0$.

$$\left(\dot{X}^T\dot{X}\right)^{\bullet}=\dddot{X}^T\dot{X}+\dot{X}^T\dddot{X}\overset{by\,(1)}{=}\ddot{X}^TXX^T\dot{X}+\dot{X}^TXX^T\ddot{X}\overset{by\,(6)}{=}-\dot{X}^T\dot{X}X^T\dot{X}-\dot{X}^TX\dot{X}^T\dot{X}$$

$$=-\dot{X}^T\left(\dot{X}X^T+X\dot{X}^T\right)\dot{X}\overset{by\,(4)}{=}0. \quad (7)$$

By (6) and (7), we get $\tfrac{1}{2}\left(S+S^T\right)$ is constant. As *S* is symmetric, we have *S* is constant. Now (1) and (2) gives:

$$\begin{cases} X^TX=I_3 & (8) \\ \ddot{X}-XS=0 & (9) \end{cases}.$$

where *S* is constant matrix in Sym_3.

By (6), we know that

$$S=-\dot{X}^T\dot{X}, \quad (10)$$

so $-S$ is real, symmetric and positive semi-definite. So there exists a ***real symmetric*** matrix *A*, such that

$$-S=A^2. \quad (11)$$

Plugging (11) into (9), we have

$$\ddot{X}+XA^2=0. \quad (12)$$

So the solution can be written as

$$X(t)=P\cos(At)+Q\sin(At). \quad (13)$$

where *P* and *Q* are two 3×3 constant matrices.

$X(0)=P\cos(0_3)+Q\sin(0_3)=P=I_3$, so (13) becomes

$$X(t)=\cos(At)+Q\sin(At). \quad (14)$$

Besides, $X(1)=\cos(A)+Q\sin(A)$. (15)

The solution (14) makes (9) fulfilled. To fulfil (8), we plug (14) into (8)

$$\left(\cos(At)+\sin(At)Q^T\right)\left(\cos(At)+Q\sin(At)\right)$$

$$=\cos(At)^2+\sin(At)Q^TQ\sin(At)+\sin(At)Q^T\cos(At)+\cos(At)Q\sin(At)$$

$$=I_3. \quad (16)$$

Setting $A=\begin{pmatrix}\pi/2 & 0 & 0\\ 0 & -\pi/2 & 0\\ 0 & 0 & 0\end{pmatrix}$, $Q=\begin{pmatrix}0 & -1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$, we can check that (15), (16) are fulfilled.

Note $\cos At=\begin{pmatrix}\cos(\pi t/2) & 0 & 0\\ 0 & \cos(\pi t/2) & 0\\ 0 & 0 & 1\end{pmatrix}$, $\sin At=\begin{pmatrix}\sin(\pi t/2) & 0 & 0\\ 0 & -\sin(\pi t/2) & 0\\ 0 & 0 & 0\end{pmatrix}$.

This gives that

$$X(t)=\begin{pmatrix}\cos(\pi t/2) & \sin(\pi t/2) & 0\\ -\sin(\pi t/2) & \cos(\pi t/2) & 0\\ 0 & 0 & 1\end{pmatrix}. \quad (17)$$

Finally, the solution to (8) (9) (a geodesic on the manifold O_3) with the boundary conditions (15) is known to be unique. So (17) is the solution.