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Assignment 2

14 August, 2006

1. Consider

$$f : \mathbb{R}^{2\times 2} \times \mathbb{R}^{2\times 2} \to \mathbb{R},\ f(X,Y) = tr\left(X^2 Y\right)$$

and

$$g : \mathbb{R} \to \mathbb{R}^{2\times 2},\ g(x) = \begin{pmatrix} x & 1 \\ 0 & x^2 \end{pmatrix}$$

and

$$h : \mathbb{R} \to \mathbb{R}^{2\times 2},\ h(x) = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}.$$

Consider $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $F(x,y) = f\left(g(x), h(y)\right)$.

Compute $D\,F(x,y)\cdot(\mu,\eta) = D_1F(x,y)\cdot\mu + D_2F(x,y)\cdot\eta$.

Verify that $D\,F(x,y)\cdot(0,\eta) = 0$.

Solution:

$$\begin{aligned}
&D\,F(x,y)\cdot(\mu,\eta) \\
&\quad = D_1F(x,y)\cdot\mu + D_2F(x,y)\cdot\eta \\
&\quad = D_1tr\left(g^2(x)h(y)\right)\cdot\mu + D_2tr\left(g^2(x)h(y)\right)\cdot\eta \\
&\quad = tr\left(D_1\left(g^2(x)h(y)\right)\right)\cdot\mu + tr\left(D_2\left(g^2(x)h(y)\right)\right)\cdot\eta \\
&\quad = tr\left(\left(g'(x)g(x) + g(x)g'(x)\right)h(y)\right)\cdot\mu + tr\left(g^2(x)h'(y)\right)\cdot\eta \\
&\quad = tr\left(\left(\begin{pmatrix} 1 & 0 \\ 0 & 2x \end{pmatrix}\begin{pmatrix} x & 1 \\ 0 & x^2 \end{pmatrix} + \begin{pmatrix} x & 1 \\ 0 & x^2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2x \end{pmatrix}\right)\begin{pmatrix} 1 & y \\ 0 & 0 \end{pmatrix}\right)\cdot\mu + tr\left(\begin{pmatrix} x & 1 \\ 0 & x^2 \end{pmatrix}^2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\cdot\eta \\
&\quad = 2x\mu\,.
\end{aligned}$$

So $D\,F(x,y)\cdot(0,\eta) = 0$.

To check, $F(x,y) = f\left(g(x),h(y)\right) = tr\left(\begin{pmatrix} x & 1 \\ 0 & x^2 \end{pmatrix}^2\begin{pmatrix} 1 & y \\ 0 & 0 \end{pmatrix}\right) = x^2$, so $D\,F(x,y)\cdot(\mu,\eta) = 2x\mu$.

2. Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $(x,y) \mapsto \left(x^3 + x, y^3 + 2y\right)$.

Obviously, $f(1,1) = (2,3)$ holds.

Compute $D\,f^{-1}(2,3)$ using Inverse Function Theorem.

Why can you apply the theorem in this case?

Solution:

$$Df = \begin{pmatrix} 3x^2+1 & 0 \\ 0 & 3y^2+2 \end{pmatrix},\ Df(1,1) = \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}.\ \text{So } Df^{-1}(2,3) = \left(Df(1,1)\right)^{-1} = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/5 \end{pmatrix}.$$

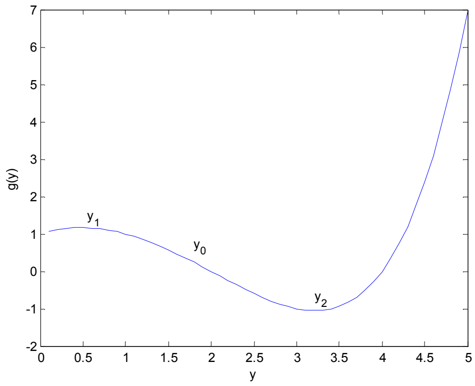
We can apply the Inverse Function Theorem in this case because:

(1) *f* is continuous differentiable in $\mathbb{R}^2$, so of course continuous differentiable in an open set containing (1, 1).

(2) $\det Df(1,1)=\det\begin{pmatrix}4 & 0\\ 0 & 5\end{pmatrix}=20\neq 0$.

3. Show that the equation $x^y-y^x=0$ is locally solvable for y as a smooth function of x in an open interval containing 2. Compute $y'(2)$.

Solution:
Define $f:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ as $f(x,y)=x^y-y^x$. Obviously, $f(2,2)=0$ and $f(2,4)=0$. Let $g(y)=f(2,y)=2^y-y^2$ ($y>0$). We can prove that 2 and 4 are the only two zeroes of $g(y)$. $g'(y)=2^y\ln 2-2y$, $g''(y)=2^y(\ln 2)^2-2$. $g''(y_0)=0$ iff $y_0=1-2\ln(\ln 2)/\ln 2$. When $y<y_0$, $g''(y)<0$ so $g'(y)$ decreases. When $y>y_0$, $g''(y)>0$ so $g'(y)$ increases. $g'(y_0)=2^{y_0}\ln 2-2y_0<0$. $\lim\limits_{y\to+\infty}g'(y)=+\infty$, $\lim\limits_{y\to 0}g'(y)=\ln 2>0$. So there exists unique y_1 and y_2: $0<y_1<y_0<y_2$, s.t. $g'(y_1)=g'(y_2)=0$. So $g(y)$ increases in $(0,y_1]$, decreases in $[y_1,y_2]$, and increases in $[y_2,+\infty)$. As $\lim\limits_{y\to 0}g(y)=1>0$, there are *at most* two zeroes of $g(y)$. Now that we have found two zeroes (2 and 4), there are two and only two zeroes. $g(y)$ is plotted below:



About (2, 2), $D_2f(2,2)=\left.\left(x^y\ln x-xy^{x-1}\right)\right|_{(2,2)}=4\ln 2-4\neq 0$. So by the implicit function theorem, there exists an open set *A* containing 2 and an open set *B* containing 2, such that for each $x\in A$, there exists a unique $y_1(x)\in B$, s.t. $f(x,y_1(x))=0$. Moreover,

$$y_1'(2)=-\left.\frac{D_1f(x,y)}{D_2f(x,y)}\right|_{(2,2)}=-\left.\frac{yx^{y-1}-y^x\ln y}{x^y\ln x-xy^{x-1}}\right|_{(2,2)}=1.$$

Similarly, about (2, 4), $D_2f(2,2)=\left.\left(x^y\ln x-xy^{x-1}\right)\right|_{(2,4)}=16\ln 2-8\neq 0$, thus there exists an open set A' containing 2 and an open set B' containing 4, such that for each $x\in A'$, there exists a unique $y_2(x)\in B'$, s.t. $f(x,y_2(x))=0$.

$$y_2'(2)=-\left.\frac{D_1f(x,y)}{D_2f(x,y)}\right|_{(2,4)}=-\left.\frac{yx^{y-1}-y^x\ln y}{x^y\ln x-xy^{x-1}}\right|_{(2,4)}=-\frac{32-32\ln 2}{16\ln 2-8}=\frac{4\ln 2-4}{2\ln 2-1}.$$

Since there are two solutions, if "solvable" requires uniqueness, then the conclusion that "$f(x,y)=0$ is locally solvable for *y* as a smooth function of *x* in an open interval containing 2" holds only in the sense of some specific open interval of *y*. If uniqueness is not required, then the conclusion is true, though there are still two answers to $y'(2)$.