**MATH3029: Probability Modelling and Applications** **5 October, 2006**

# Assignment 9

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1. Recall the definition of exponentially distributed random variables in Assignment 7. For $\lambda > 0$, the distribution function $F_\lambda : \mathbb{R}_0^+ \to [0,1]$ is given by

$$F_\lambda(x) = 1 - \exp(-\lambda x), \text{ for } x \geqslant 0$$

a) Give a "minimal" probability space $\left(\Omega^0, \mathcal{A}^0, \mathbb{P}^0\right)$ carrying a random variable $\tau$ that is exponentially distributed with parameter $\lambda > 0$. Here, choose $\mathbb{P}^0$ such that $\mathbb{P}^0(A) = \mathbb{P}^0(\tau \in A)$, for all $A \in \mathcal{A}^0$.
b) Using part a), explain briefly how to construct a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ carrying a family of random variables $(\tau_n)_{n\in\mathbb{N}}$ that are independent and identically distributed, where the distribution they follow is the exponential distribution with parameter $\lambda > 0$.
c) On the probability space in part b), define the random variables $(T_n)_{n\in\mathbb{N}}$ by

$$T_n = \sum_{k=1}^{n} \tau_k, \text{ for } n \in \mathbb{N}.$$

The random variable $\tau_n$ takes values in $\mathbb{R}_0^+$. Let $F_n(x) : \mathbb{R}_0^+ \to [0,1]$ be the distribution function of $T_n$ and $f_n$ the respective density, for each $n \in \mathbb{N}$. Prove by induction that

$$f_n(x) = \lambda^n e^{-\lambda x} \tfrac{x^{n-1}}{(n-1)!}, \text{ for } n \in \mathbb{N}.$$

For doing so, the density $f_1$ of $\tau_1 = T_1$ is needed and further the concept of the convolution. Let $X$and $Y$be two independent real valued random variables with densities $f_X$ and $f_Y$, respectively, then the random variable $Z = X + Y$ has a density $f_Z$ and

$$f_Z(z) = \textstyle\int f_X(z-y) f_Y(y)\, dy, \text{ for } z \in \mathbb{R}.$$

**Solution:**

(a) In Assignment 7, we have seen that $\left((0,1), \mathcal{B}((0,1)), \lambda_{(0,1)}\right)$ carries a uniform r.v. $U : (0,1) \to \mathbb{R}$ as $U(\omega) = \omega$, and r.v. $Y = -\log U/\lambda$ is an exponential r.v.. Define $\left(\Omega^0, \mathcal{A}^0, \mathbb{P}^0\right)$, where $\Omega^0 = \mathbb{R}_0^+$, $\mathcal{A}^0 = \mathcal{B}\left(\mathbb{R}_0^+\right)$, and $\mathbb{P}^0 = Q^Y$. Let r.v. $\tau : \left(\mathbb{R}_0^+, \mathcal{B}\left(\mathbb{R}_0^+\right)\right) \to \left(\mathbb{R}_0^+, \mathcal{B}\left(\mathbb{R}_0^+\right)\right)$ be $\tau(x) = x$ (identity). It is easy to see that $\left(\Omega^0, \mathcal{A}^0, \mathbb{P}^0\right)$ is a probability space, $\mathbb{P}^0(A) = \mathbb{P}^0(\tau \in A)$ $\forall A \in \mathcal{A}$, and $\tau$ is exponentially distributed with parameter $\lambda$.

(b) Choose $\Omega = \prod\limits_{n=1}^{\infty} \Omega_n^0$ ($\Omega_n^0 = \Omega^0$, countable Cartesian product), $\mathcal{A} = \bigotimes\limits_{n=1}^{\infty} \mathcal{A}_n^0$ ($\mathcal{A}_n^0 = \mathcal{A}^0$, the smallest $\sigma$-algebra on $\Omega$ generated by all sets of the form $A_1 \times A_2 \times \cdots \times A_k \times \Omega^0 \times \Omega^0 \times \cdots$, where $A_i \in \mathcal{A}^0$; $k = 1, 2, 3, \ldots$) and $\mathbb{P} = \bigotimes\limits_{n=1}^{\infty} \mathbb{P}_n^0$ ($\mathbb{P}_n^0 = \mathbb{P}^0$, existence by the Tonelli-Fubini Theorem).

(c) For $n = 1$, $T_1 = \tau_1$, $f_1(x) = \partial F_\lambda(x)/\partial x = \lambda e^{-\lambda x}$. If the conclusion holds for $n = k \geqslant 1$, then for $n = k+1$, $f_{k+1}(x) = \int_0^x f_1(x-y) f_k(y)\, dy = \int_0^x \lambda e^{-\lambda(x-y)} \lambda^n e^{-\lambda y} \frac{y^{k-1}}{(k-1)!} dy = \lambda^{k+1} e^{-\lambda x} \frac{x^k}{k!}$, where we utilized the fact that $\int_0^x y^{k-1} dy = x^k/k$. So the conclusion holds for $n = k+1$, and thus by induction it holds for all $n \in \mathbb{N}$.

2. The random variables $T_n$ can be interpreted as times associated to events that occur. For example, these events can be the incoming phone calls at a call center, or car accidents (for the purpose of car insurance). $T_n = t$ means then that at time $t$the $n$th event (car accident) occurs. The random variables $\tau_n$ give the time span between events and are called inter-arrival times.
For $t \geqslant 0$, the number of events that have occurred up to time $t$ can be counted by the random variable

$$N_t(\omega) = \sup\{n \in \mathbb{N} : T_n(\omega) \leqslant t\}, \text{ for } \omega \in \Omega.$$

Thus $N_t = n$if event $n$has occurred at time $t$, but the event $n+1$ has not occurred yet at time $t$, or more formally

$$\{N_t(\omega) = n\} = \{T_n(\omega) \leqslant t, T_{n+1}(\omega) > t\}$$

a) Use Theorem 10.3 b) (Tonelli-Fubini) to show the following statement is true:
Let $X$and $Y$be two *independent* real-valued random variables on the probability space $\left(\tilde{\Omega}, \tilde{\mathcal{A}}, \mathbb{Q}\right)$ and denote $\mathbb{Q}^X$ and $\mathbb{Q}^Y$ their respective distributions. Then for $u, v \in \mathbb{R}$ the following identity holds

$$\mathbb{Q}(X \leqslant u, X+Y > v) = \int_{-\infty}^{u} \mathbb{Q}^Y((v-x, \infty))\, \mathbb{Q}^X(dx).$$

b) Using Q2 a) and Q1 c), show that $N_t$ is Poisson distributed with parameter $\lambda t$, i.e.:

$$\mathbb{P}(N_t = n) = e^{-\lambda t} \tfrac{(\lambda t)^n}{n!}, \text{ for } n \in \mathbb{N}_0.$$

**Solution:**
(a) As $X$and $Y$are *independent* and real-valued, thus without loss of generality, we assume that $\tilde{\Omega} = \mathbb{R} \times \mathbb{R}$, $\tilde{\mathcal{A}} = \mathcal{B}(\mathbb{R} \times \mathbb{R})$, and $\mathbb{Q} = \mathbb{Q}^X \otimes \mathbb{Q}^Y$. Denote $f(X, Y) = 1_A$, where $A = \{X \leqslant u, X+Y > v\}$. Then by Tonelli-Fubini Theorem, we have

$$\int f d\mathbb{Q}^X \otimes \mathbb{Q}^Y = \int \left\{ \int f(x,y)\, \mathbb{Q}^Y(dy) \right\} \mathbb{Q}^X(dx).$$

But $\int f d\mathbb{Q}^X \otimes \mathbb{Q}^Y = \int 1_A d\mathbb{Q}^X \otimes \mathbb{Q}^Y = \mathbb{Q}(A) = \mathbb{Q}(X \leqslant u, X+Y > v)$, and the integrand of the term on the right-hand side is nonzero only if $x \leqslant u$ and $x + y > v$. So the right-hand integral is equal to $\int_{-\infty}^{u} \left\{ \int_{v-x}^{+\infty} f(x,y)\, \mathbb{Q}^Y(dy) \right\} \mathbb{Q}^X(dx) = \int_{-\infty}^{u} \mathbb{Q}^Y((v-x, \infty))\, \mathbb{Q}^X(dx)$. Thus

$$\mathbb{Q}(X \leqslant u, X+Y > v) = \int_{-\infty}^{u} \mathbb{Q}^Y((v-x, \infty))\, \mathbb{Q}^X(dx).$$

(b) Notice that $\{N_t(\omega) = n\} = \{T_n(\omega) \leqslant t, T_n(\omega) + \tau_{n+1}(\omega) > t\}$. Let $X = T_n$, $Y = \tau_{n+1}$. Then $X$ and $Y$ are independent. By Q1 c), the distribution of $X$ is $f_X(x) = \lambda^n e^{-\lambda x} \frac{x^{n-1}}{(n-1)!}$ and the distribution of $Y$ is $f_Y(y) = \lambda e^{-\lambda y}$. Thus by Q2 a), we have

$$\mathbb{P}(N_t = n) = \int_0^t \left\{ \int_{t-x}^{+\infty} \lambda e^{-\lambda y} dy \right\} \lambda^n e^{-\lambda x} \frac{x^{n-1}}{(n-1)!} dx = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$