MATH3029: Probability Modelling and Applications	26 October, 2006

Assignment 12

Student: Xinhua Zhang *StuID: U4263758* *Email: xinhua.zhang@anu.edu.au*

1. Let $B = (B_t)_{t \geqslant 0}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$.

(i) Let Z be a standard normal random variable. Define a stochastic process $X = (X_t)_{t \geqslant 0}$ by $X_t = t^{1/2} Z$, $t \geqslant 0$. Clearly, X has almost surely continuous sample paths. Show that $X_t \stackrel{d}{=} B_t$ for all $t \geqslant 0$, but X is not a Brownian motion.

(ii) Let $c > 0$ and show that $B^{(i)} = \left(B_t^{(i)}\right)_{t \geqslant 0}$, $i = 1, 2$, are Brownian motions where we set

$$B_t^{(1)} := c^{-1/2} B_{ct}, \quad t \geqslant 0; \qquad B_t^{(2)} := t B_{1/t} \quad (t > 0), \quad B_0^{(2)} := 0.$$

Solution:

(i) Since $X_t \sim N(0, t)$, and $B_t \sim N(0, t)$ by the definition of Brownian motion, $X_t \stackrel{d}{=} B_t$ for all $t \geqslant 0$. But $X_1 = Z_1$, $X_4 = 2Z_4$, and $Var(X_4 - X_1) = 5$ as Z_1 and Z_4 are standard normal and independent. (If the question means that Z is shared for all X_t, $t \geqslant 0$, then $Var(X_4 - X_1) = 1$). But the definition of Brownian motion requires that $X_4 - X_1 \sim N(0, 3)$. So X is not a Brownian motion.

(ii) Since B is Brownian motion so B is with almost surely continuous sample paths. As $c^{-1/2}$, c, t, $1/t$ are all continuous, it is obvious that $B^{(1)}$ and $B^{(2)}$ are processes with almost surely continuous sample paths. Now we make use of the following fact:

If $\tilde{B} = \left(\tilde{B}_t\right)_{t \geqslant 0}$ is a process with almost surely continuous sample paths defined on a probability space $(\Omega, \mathcal{F}, P)$, then $\tilde{B}$ is a Brownian motion if and only if $\tilde{B}$ is a Gaussian process with expectation function $E\tilde{B}_t = 0$ and covariance function $E\tilde{B}_t\tilde{B}_s = \min\{s, t\}$.

Since B is Brownian motion, so B is a Gaussian process with expectation function $EB_t = 0$ and covariance function $EB_tB_s = min\{t, s\}$. As $c^{-1/2}$ is constant, obviously $B^{(1)}$ is a Gaussian process with expectation $EB_t^{(1)} = c^{-1/2} EB_{ct} = 0$. Moreover, the covariance function

$$EB_t^{(1)} B_s^{(1)} = c^{-1} EB_{ct} B_{cs} = c^{-1} \min\{ct, cs\} = \min\{t, s\} \text{ as } c > 0.$$

Hence, $B^{(1)}$ is a Brownian motion.

As $B_s \sim N(0, s)$, so $B_s/s \sim N(0, 1/s)$, and $\lim_{s \to \infty} B_s/s = 0$ *a.s.*. Thus,

$$\lim_{t \to 0^+} B_t^{(2)} = \lim_{t \to 0^+} t B_{1/t} = \lim_{s \to +\infty} B_s/s = 0 = B_0^{(2)} \quad a.s.$$

To prove that $B^{(2)}$ is a Gaussian process, it suffices to show that for all $d \in \mathbb{N}$ and $0 \leqslant t_0 < t_1 < \cdots < t_d$, the random vector $\left(B_{t_0}^{(2)}, B_{t_1}^{(2)}, \cdots, B_{t_d}^{(2)}\right)$ is Gaussian. This is obvious if $t_0 > 0$ as

$\left(B^{(2)}_{t_0}, B^{(2)}_{t_1}, \cdots, B^{(2)}_{t_d}\right) = \left(t_0 B_{1/t_0}, t_1 B_{1/t_1}, \cdots, t_d B_{1/t_1}\right)$ and we know that $\left(B_{1/t_0}, B_{1/t_1}, \cdots, B_{1/t_d}\right)$ is Gaussian. But if $t_0 = 0$, then as $B^{(2)}_0 = 0$ and $\left(B^{(2)}_{t_0}, B^{(2)}_{t_1}, \cdots, B^{(2)}_{t_d}\right) = \left(0, t_1 B_{1/t_1}, \cdots, t_d B_{1/t_d}\right)$, we still have that $\left(B^{(2)}_{t_0}, B^{(2)}_{t_1}, \cdots, B^{(2)}_{t_d}\right)$ is Gaussian.

As for covariance, if $s, t > 0$, then

$$EB^{(2)}_t B^{(2)}_s = E\left(tB_{1/t} sB_{1/s}\right) = stE\left(B_{1/t}B_{1/s}\right) = st\min\{1/t, 1/s\} = \min\{t, s\}.$$

If s or t is 0, then $EB^{(2)}_t B^{(2)}_s = 0 = \min\{s, t\}$ $(s, t \geqslant 0)$.

Obviously $EB^{(2)}_0 = 0$, and $EB^{(2)}_t = E\left(tB_{1/t}\right) = t \cdot 0 = 0$ for all $t > 0$.

In sum, $B^{(2)}$ is a Gaussian process with expectation function $EB^{(2)}_t = 0$ and covariance function $EB^{(2)}_t B^{(2)}_s = \min\{s, t\}$. So $B^{(2)}$ is a Brownian motion.

2. The Brownian motion is used for modelling stock prices in the Black & Scholes model. The stock price process S_t is defined $S_t = s_0 \exp\left(\left(\mu - 1/2\sigma^2\right)t + \sigma B_t\right)$, for $t \geqslant 0$.

(i) Let s_0, $\sigma^2 > 0$, $r \in \mathbb{R}$. Find $\mu \in \mathbb{R}$ such that, for all $t \geqslant 0$

$$E\exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right) = e^{rt}.$$

(ii) Let Z be a standard normal random variable and set

$$\bar{\Phi}(x) = P(Z \geqslant x) = (2\pi)^{-1/2}\int_x^\infty \exp\left(-\frac{1}{2}z^2\right)dz, \quad x \in \mathbb{R}.$$

For $\mu = r$, the value of a contingent claim with pay off X at $T > 0$ is given by

$$Value(X) = e^{-rT}E(X),$$

where X is a random variable that is measurable w.r.t to the σ-algebra generated by$(B_t)_{0\leqslant t\leqslant T}$. Let $K > 0$ and consider the pay-off $X = 1_{S_T \geqslant K}$. Give a formula of the value of X in terms of $\bar{\Phi}$.

Solution:

(i) Since $B_t \sim N(0, t)$, so $\sigma B_t \sim N\left(0, \sigma^2 t\right)$, thus

$$\begin{aligned} E\exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right) &= \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t\right) \cdot E\exp(\sigma B_t) \\ &= \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t\right) \cdot E\exp(\sigma B_t) \\ &= \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t\right)\exp\frac{1}{2}\sigma^2 t \\ &= e^{rt} \end{aligned}$$

So $\left(\mu - \frac{1}{2}\sigma^2\right)t + \frac{1}{2}\sigma^2 t = rt$, thus $\mu = r$.

(ii)

$$
\begin{aligned}
Value\,(X) &= e^{-rT}E\,(X) = e^{-rT}E1_{S_T \geqslant K} = e^{-rT}P\,(S_T \geqslant K) \\
&= e^{-rT}P\left(s_0 \exp\left(\left(\mu - 1/2\sigma^2\right)T + \sigma B_T\right) \geqslant K\right) \\
&= e^{-rT}P\left(\frac{B_T}{\sqrt{T}} \geqslant \frac{1}{\sigma\sqrt{T}}\left(\ln\frac{K}{s_0} - \left(\mu - 1/2\sigma^2\right)T\right)\right) \\
&= e^{-rT}\bar{\Phi}\left(\frac{1}{\sigma\sqrt{T}}\left(\ln\frac{K}{s_0} - \left(\mu - 1/2\sigma^2\right)T\right)\right) \\
&= e^{-rT}\bar{\Phi}\left(\frac{1}{\sigma\sqrt{T}}\left(\ln\frac{K}{s_0} - \left(r - 1/2\sigma^2\right)T\right)\right).
\end{aligned}
$$