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Assignment 1

1. Let $(X_n)_{n\geq 1}$ be a sequence of *iid* Bernoulli random variables with

$$P(X_1 = 1) = p = 1 - P(X_1 = 0)\ ^1$$

and let $S_n = \sum_{k=1}^{n} X_k$ be the number of successes in *n* trials. Show that S_n has a binomial distribution by the following method:
(i) Prove for $n \geq 1$, $1 \leq k+1 \leq n+1$,

$$P(S_{n+1} = k) = pP(S_n = k-1) + (1-p)P(S_n = k).$$

(ii) Solve the recursion using generating functions.

Solution:
(i) For $n \geq 1$, $0 < k \leq n$, in order to make $S_{n+1} = k$, S_n can only be either *k* or $k-1$. So

$$\begin{aligned}
P(S_{n+1} = k) &= P(S_n = k-1, X_{n+1} = 1) + P(S_n = k, X_{n+1} = 0) \\
&\overset{(1)}{=} P(S_n = k-1)\cdot P(X_{n+1} = 1) + P(S_n = k)\cdot P(X_{n+1} = 0) \\
&= pP(S_n = k-1) + (1-p)P(S_n = k).
\end{aligned}$$

Step (1) is true because S_n and X_{n+1} are independent. The final equation also holds for $k = 0$ and $k \geq n+1$, because $P(S_n = i) = 0$ for all $n \geq 1$ and $i > n$ or $i < 0$.

(ii) Let the generating function of S_n be $P_n(s)$. Denote $q := 1 - p$. Then for $n \geq 1$:

$$\begin{aligned}
P_{n+1}(s) &= \sum_{k=0}^{\infty} P(S_{n+1} = k)s^k \\
&= \sum_{k=0}^{\infty}\big(pP(S_n = k-1) + (1-p)P(S_n = k)\big)s^k \\
&= p\sum_{k=1}^{\infty} P(S_n = k-1)s^k + q\sum_{k=0}^{\infty} P(S_n = k)s^k \\
&= ps\sum_{k=1}^{\infty} P(S_n = k-1)s^{k-1} + q\sum_{k=0}^{\infty} P(S_n = k)s^k \\
&= ps\sum_{k=0}^{\infty} P(S_n = k)s^k + q\sum_{k=0}^{\infty} P(S_n = k)s^k \\
&= (ps+q)P_n(s).
\end{aligned}$$

By definition, $P_1(s) = ps + q$, so $P_n(s) = (ps+q)^n$ for $n \geq 1$. This means that S_n has a binomial distribution.

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* As a PhD student from RSISE, I may not be on the *enrolled* student name list at present. Administrative steps are under way by RSISE and MSI for this cross faculty course enrolment.
[1] The original $P(X_1 = 0) = p = 1 - P(X_1 = 0)$ doesn't make sense. So we changed it.

2. Let $p \in (0,1)$. Consider a branching process which has as its off-spring distribution the geometric distribution $\left(g(k;p)\right)_{k\in\mathbb{N}_0}$ with parameter $p \in (0,1)$ [$g(k;p) = p(1-p)^k$, $k \in \mathbb{N}_0$]. Give a formula for its extinction probability.

Solution:

The generating function of geometric distribution $g(k;p)$ is $P(s) = p/(1-qs)$, where q = 1 – p. $m = E(Z_1)$ = Eg = q/p. So if $m \le 1$, i.e., $p \ge 0.5$, then extinction probability $\pi = 1$. Otherwise, π is the unique solution in $[0,1)$ to

$$s = P(s) = p/(1-qs).$$

Solving it, we get π = p/q. So in sum, the extinction probability

$$\pi = \begin{cases} 1 & if \ p \ge 0.5 \\ p/q & if \ p < 0.5 \end{cases}.$$