Assignment 6

1. ~~Show that if~~ Find an open convex set C, s.t. $\exists x_1, x_2 \in C$ and $C_\infty(x_1) \neq C_\infty(x_2)$

Solution: a not closed (but not open as well) example:

$S = (R \times (0,1)) \cup \{(0,1)\}$ $(1,0) \in C_\infty(0,\tfrac{1}{2})$, But $(1,0) \notin C_\infty(0,1)$. S is convex

Now we prove that if C is open convex, then $C_\infty(x_1) = C_\infty(x_2)$ $\forall x_1 \neq x_2 \in C$.

Suppose $d \in C_\infty(x_1)$, we want to show that for $\forall t > 0$, $x_2 + td \in C$.

Since C is open, so there exists a $\delta > 0$, s.t. $B(x_2, 2\delta \ell) \subseteq C$, where $\ell \triangleq \|x_2 - x_1\|$

Now pick $x_3 = x_2 + \delta(x_2 - x_1)$, so $x_3 \in B(x_2, 2\delta\ell) \subseteq C$.

Since $d \in C_\infty(x_1)$, so $x_4 \triangleq x_1 + \frac{1+\delta}{\delta} td \in C$.

Notice $x_2 + td = \frac{\delta}{1+\delta}\left(x_1 + \frac{1+\delta}{\delta} td\right) + \frac{1}{1+\delta}\left(x_2 + \delta \cdot (x_2 - x_1)\right) = \alpha x_4 + (1-\alpha) x_3$

where $\alpha = \frac{\delta}{1+\delta}$. By convexity $x_2 + td \in C$. So $d \in C_\infty(x_2)$. So $C_\infty(x_2) = C_\infty(x_1)$

for $\forall x_1 \neq x_2 \in C$. So $C_\infty(x)$ is independent of $x \in C$ if C is open.