Basic Reading Group: Convex Analysis **2006**

Lecture 6 — 26 October 2006

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In this lecture, we will try to approximate a set S near a point x by a simpler set. In classic differential geometry, one approximates a smooth surface S by an affine manifold "tangent" to S. Most commonly, this concept is used in the differentiation of a smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$, whose graph is "tangent" to an affine hyperplane in $\mathbb{R}^n \times \mathbb{R}$:

$$gr\ f \cong \{(y, r) : r - f(x) = \langle \nabla f(x), y - x \rangle\}.$$

Obviously, there is no reason to restrict/assume that convex sets are "smooth". One must find some substitute to the affine manifolds. Recall that affine manifolds are translations of subspaces; say we approximate S near x by:

$$S \cong H_S(x) = \{x\} + V_S(x),$$

where $V_S(x)$ is a subspace: the subspace tangent to S at x. In convex analysis, the natural substitutes for subspaces are the *closed convex cones*. In the next lecture, we may learn another important concept, the set of normals to S at x, i.e., the subspace orthogonal to $V_S(x)$, where the orthogonality will be replaced by polarity.

6.1 Convenient Definitions of Tangent Cones

In this subsection, we consider an arbitrary closed subset $S \subseteq \mathbb{R}^n$. A direction d is classically called tangent to S at $x \in S$ when it is the derivative at x of some curve drawn on S. So $-d$ is also tangent. Since we are rather interested in cones, we simply require a half-derivative (approaching from a single side) from the curve in question. Furthermore, sets of discrete type cannot have any tangent direction in the above sense, so we will replace curves by sequences. These are very typical ideas in math about how to extend existing concepts to new objects, without (much) loss of existing nice properties. In a word, our new definition of tangency is as follows:

Definition 6.1. (Definition 5.1.1) *Let $S \subseteq \mathbb{R}^n$ be nonempty. We say that $d \in \mathbb{R}^n$ is a direction tangent to S at $x \in S$ when there exists a sequence $\{x_k\} \subseteq S$ and a sequence $\{t_k\} \subseteq \mathbb{R}$ such that when $k \to +\infty$,*

$$x_k \to x, \quad t_k \downarrow 0, \quad \frac{x_k - x}{t_k} \to d.$$

The set of all such directions is called the tangent cone (also called the contingent cone, or Bouligand's cone) to S at $x \in S$, denoted by $T_S(x)$.

Observe: $0 \in T_S(x)$ (take $x_k \equiv x$); also, if d is tangent, so is αd for any $\alpha > 0$ (change t_k to t_k/α). Therefore the terminology "tangent cone" is legal ($T_S(x)$ is a cone). If $x \in int(S)$, then $T_S(x)$ is clearly the whole space $\mathbb{R}^n$, so the only interesting points are those on the boundary bdS.

Now we derive an equivalent formulation of tangent direction. Set in Definition 6.1 $d_k \triangleq \frac{x_k - x}{t_k}$, i.e., $x_k = x + t_k d_k$, so $\lim\limits_{k \to +\infty} d_k = d$, $x_k \in S$. Now we have the following proposition regarding the equivalence of the definition.

Theorem 6.2. **(Proposition 5.1.2)** *A direction d is tangent to S at $x \in S$ if and only if*

$$\exists \{d_k\} \to d, \quad \exists \{t_k\} \downarrow 0, \text{ such that } x + t_k d_k \in S \text{ for all } k.$$

Proof: Obvious. □

A tangent direction thus appears as a set of limits. A limit of tangent directions is therefore a "limit of limits", and is a limit itself:

Theorem 6.3. **(Proposition 5.1.3)** *The tangent cone is closed.*

Proof: Let $\{d_l\} \subseteq T_S(x)$ be converging to d. For each l, take sequences $\{x_{l,k}\}_k$ and $\{t_{l,k}\}_k$ associated with d_l in the sense of Definition 6.1. Fix $l > 0$ and we can find k_l, such that

$$\left\| \frac{x_{l,k_l} - x}{t_{l,k_l}} - d_l \right\| \leqslant \frac{1}{l}.$$

Letting $l \to +\infty$, we then obtain the sequences $\{x_{l,k_l}\}_l$ and $\{t_{l,k_l}\}_l$ which define d as an element of $T_S(x)$. □

The Figure 6.1 shows that our definition is more general than the classical one.

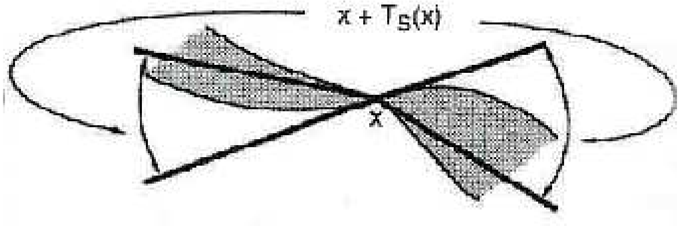

Fig. 5.1.1. Tangency to a "bad" set

Figure 6.1. New definition of tangency extends classical definition.

Clearly, the concept of tangency is local, as it depends only on the behaviour of S near x. From Definition 6.1, $T_S(x)$ appears as the set of all possible cluster points of the difference quotients $\{(y - x)/t\}$, with $y \in S$ and $t \downarrow 0$. We can also use set-valued notations

$$T_S(x) = \lim_{t \downarrow 0} \operatorname{ext} \frac{S - x}{t}$$

Or, Definition 6.1 can be interpreted in a set-formulation: for any $\varepsilon > 0$ and for any $\delta > 0$, there exists $t \in (0, \delta]$ such that

$$x + td \in S + B(0, t\varepsilon), \text{ i.e., } d \in \tfrac{S - x}{t} + B(0, \varepsilon).$$

6.2 Asymptotic Cone

Now we need some conical topology for convex sets.

Let x be a point in a closed convex cone K. For all $d \in K$, the half-line $x + \mathbb{R}^+ d$ is contained in K: $x + td \in K$ for all $t > 0$. ($x \in K \Rightarrow 2x \in K$ by cone, $d \in K \Rightarrow 2td \in K$ by cone, so $x + td = \frac{1}{2}(2x) + \frac{1}{2}(2td) \in K$ by convexity). Conversely, if $x + \mathbb{R}^+ d \subseteq K$, i.e., if

$$d \in \tfrac{K-x}{t} = K - \left\{t^{-1}x\right\} \text{ for all } t > 0,$$

then (K is closed), $d \in K$. In words, a closed convex cone is also the set of directions along which one can go straight to infinity. We now generalize this concept to non-conical sets.

In this subsection, C will always be a nonempty *closed convex* set. For $x \in C$, let

$$C_\infty(x) \triangleq \{d \in \mathbb{R}^n : x + td \in C \; for \; all \; t > 0\}.$$

Due to convexity of C, $C_\infty(x)$ depends only on the behavior of C at "infinity": in fact, $x + td \in C$ implies that $x + \tau d \in C$ for all $\tau \in [0, t]$. Thus $C_\infty(x)$ is just the set of directions from which one can go straight from x to infinity, while staying in C. Another equivalent formulation is:

$$C_\infty(x) = \bigcap_{t>0} \frac{C-x}{t},$$

which clearly shows that $C_\infty(x)$ is a closed convex cone (as $\frac{C-x}{t}$ is closed convex for all t), which of course contains 0. The following Theorem 6.4 is fundamental.

Theorem 6.4. (Proposition 2.2.1) *The closed convex cone $C_\infty(x)$ does not depend on $x \in C$.*

Proof: Take two different points x_1 and x_2 in C. It suffices to prove one inclusion $C_\infty(x_1) \subseteq C_\infty(x_2)$. Let $d \in C_\infty(x)$ and $t > 0$, we have to prove that $x_2 + td \in C$. With $\varepsilon \in (0, 1)$, consider the point

$$\bar{x}_\varepsilon \triangleq x_1 + td + (1-\varepsilon)(x_2 - x_1).$$

Writing as

$$\bar{x}_\varepsilon \triangleq \varepsilon\left(x_1 + \frac{t}{\varepsilon}d\right) + (1-\varepsilon)x_2,$$

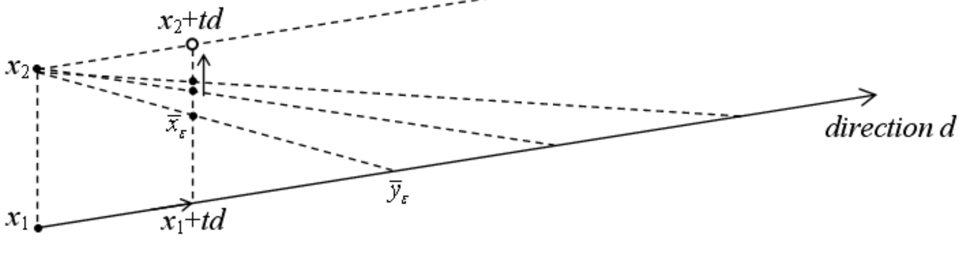

Figure 6.2. Intuitive idea of proof for Theorem 6.4.

we see that $\bar{x}_\varepsilon \in C$ (use the definition of $C_\infty(x_1)$ and of a convex set). On the other hand, $x + td = \lim_{\varepsilon \downarrow 0} \bar{x}_\varepsilon \in clC = C$.

The Figure 6.2 illustrates the idea behind the above math derivation. With ε going from 1 to 0, $\bar{x}_\varepsilon$ will tend to $x_2 + td$ from $x_1 + td$. As each $\bar{y}_\varepsilon$ is in C, so each $\bar{x}_\varepsilon$ is also in C. $\square$

It follows that the notation C_∞ is more appropriate:

Definition 6.5. (Definition 2.2.2) *The asymptotic cone, or recession cone of the closed convex set C is the closed convex cone C_∞ defined above, in which the Theorem 6.4 is exploited.*

The Figure 6.3 gives three examples in $\mathbb{R}^2$.

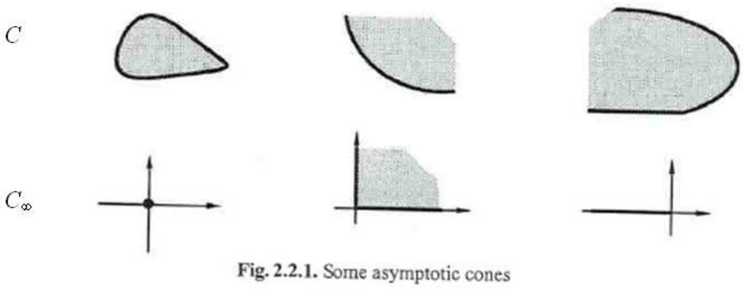

Figure 6.3. Three examples of C_∞ in $\mathbb{R}^2$.

Finally, there is a useful result concerning compactness.

Theorem 6.6. (Proposition 2.2.2) *A closed convex set C is compact if and only if $C_\infty = \{0\}$.*

Proof: If C is bounded, it is clear that C_∞ cannot contain any nonzero direction. Conversely, let $\{x_k\} \subseteq C$ be such that $\|x_k\| \to +\infty$ (we assume that $x_k \neq 0$). As the sequence $\left\{d_k \triangleq x_k/\|x_k\|\right\}$ is bounded, extract a convergent subsequence: $d = \lim_{k \in I} d_k$ where $I \subseteq \mathbb{N}$, $\|d\| = \|d_k\| = 1$. Now given $x \in C$ and $t > 0$, take k so large that $\|x_k\| \geqslant t$.

Then we see that

$$x + td = \lim_{k \in I} \left[\left(1 - \frac{t}{\|x_k\|}\right) x + \frac{t}{\|x_k\|} x_k \right]$$

is in the closed convex set C, hence $d \in C_\infty$, contradicting $C_\infty = \{0\}$. $\square$

Exercise:

Find an open or non-closed set C, such that $\exists x_1, x_2 \in C$ and $C_\infty(x_1) \neq C_\infty(x_2)$.