Graphical Generalization of Power Control in Multiuser Interference Channels

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***Abstract*—In this paper, we present a graphical approach to the power control problem, and show how to find a near optimal solution to maximizing sum rate with individual power and user requirements by searching from the vertices of the power region. By leveraging our two previous results on the quasiconvexity and asymptotic nature of sum signal-to-interference-plus-noise ratio (SINR) and sum rate respectively, we determine an equivalency relationship between sum rate and sum SINR. As a related aside, we show that sum rate is indeed convex with respect to one varying power. Our conclusions are applicable to multi-user interference channels where resources are shared, and received powers vary by an order of magnitude or more, e.g., heterogeneous network whose transmitting powers span a magnitude of ranges.**

***Index Terms*—Power control, sum rate optimization, interference channel, heterogeneous networks.**

I. Introduction

The general multi-user interference channel allows multiple transmitters and receivers to communicate simultaneously, leading to higher spectral efficiencies. However, in networks with high loads or large number of users, devices must share the same resources, leading to high interference scenarios. The most common way of combating these unfavourable reuse conditions is through power control, while at the same time aiming to maximize some system metric and satisfying various requirements. While capacity is ideal from an information theory perspective, the capacity of a multiuser interference channel is still unknown [1]. Alternatively, researchers have focused on sum rate as a metric.

However, even this option has its shortcomings. Although it has been proven for two transmitting sources that sum rate is a convex expression [2], it is known that for more than two sources sum rate is generally non-convex, and thus not easy to solve using standard optimization techniques. Often, non-convex formulations can be transformed into more manageable forms, as is the case with geometric programming [3], [4], but these require certain approximations (e.g., high signal-to-noise-plus-interference (SINR) regime).

An alternative for sum rate for more than two users is to use sum SINR [5], which, due to the absence of a log term, is easier to solve. Although for one or two users this will lead to the same optimal solutions, in general, maximizing sum SINR may not also maximize sum rate. It is known that sum rate is not convex with respect to arbitrary combinations of varying powers (i.e., not jointly convex in all powers), but neither is sum SINR, which is known to be convex with one varying power but not convex in general [6]. Although these facts may be known by researchers, to the best of the authors' knowledge there has been no discussion or mathematical study on the relationship between sum rate and sum SINR and their behaviour with varying powers. A natural question to ask is: *Will the same set of powers that maximize sum SINR also maximize sum rate?*

An early approach to the power control problem was through finding a Pareto optimal solution [7]. This was done by rearranging the constraints and expressing it in matrix form, then solving the system of equations to obtain a set of powers. However, since this is a Pareto optimal solution, it does not necessarily maximize the objective, nor does it always satisfy individual power constraints. Such an approach has been used as an initial feasibility test [8].

In [3], the authors proved that to maximize sum rate with individual power constraints, binary power control, i.e., each power operates either at maximum or minimum levels, is the optimal solution for two users, and a suboptimal solution for more than two users. Further, the authors indirectly suggest that binary power control is the optimal solution to any objective that is convex. A bound on the approximation of sum rate with an alternative expression relying on the arithmetic-geometric mean inequality and the Specht's ratio is also given, but this does not answer the question of whether the same set of powers can maximize both sum rate and sum SINR, or how similar are the powers that do. Further, [3] does not include individual rate constraints.

In this paper we describe graphically the feasible power region under both individual power and rate (or equivalently, SINR) constraints, and show how the vertices or corners of this region provides the finite set of powers which can yield near-optimal solutions to maximize sum rate. Our work illustrates three main mathematical insights:

1) We show that sum SINR is in fact *quasiconvex* in any combinations of varying powers (i.e., also jointly quasiconvex), and hence its maximizing powers will always occur at the vertices of the power region.
2) We show that when one received power dominates others, sum rate and sum SINR exhibit almost identical behaviour with varying powers, implying that global maxima and minima occur at the same powers.
3) We prove that on the edge of the power region, i.e., when only one power is varying, sum rate is indeed a convex function, and hence the vertices on the edges will always maximize the sum rate along those edges.

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II. System Model

Consider a system with N links, each of which has a unique transmitter and receiver. Each receiver treats any interference it receives from the other links as noise. We desire to solve:

$$\text{maximize} \quad R = \sum_{i=1}^{N} \log_2 \left(1 + \frac{h_{i,i}p_i}{\sum_{j \neq i} h_{j,i}p_j + \sigma^2}\right) \quad (1)$$

$$= \log_2 \left(\prod_{i=1}^{N} \left(1 + \frac{h_{i,i}p_i}{\sum_{j \neq i} h_{j,i}p_j + \sigma^2}\right)\right) \quad (2)$$

$$\text{subject to} \quad p_i \leq P_i^{\max}, \quad (3)$$

$$\frac{h_{i,i}p_i}{\sum_{j \neq i} h_{j,i}p_j + \sigma^2} \geq \gamma_i, \ i = 1, \ldots, N. \quad (4)$$

where $h_{j,i}$ is the channel gain from the jth transmitter to the ith receiver, p_i is the transmission power at the ith transmitter, $P_i^{\max}$ is the maximum transmission power at the ith transmitter, γ_i is the SINR threshold corresponding to the minimum rate for the ith user, and σ^2 is the zero mean additive Gaussian white noise (AWGN). Equations (3) and (4) represent the individual transmit power and minimum user rate constraints respectively.

For analytical simplicity, we drop the channel gains $h_{j,i}$, as fading characteristics become less significant compared to differences in magnitudes of transmit powers. Thus, p_i represent the received powers from each transmitter. For example, a macro station may transmit at 43 dB compared to 23 dB for a femtocell, where the 20 dB difference in magnitude will mostly dominate fading effects. Our conclusions are therefore more accurate for Gaussian channels. We also let $a_i = \sum_{j \neq i} p_j + \sigma^2$ to represent the interference plus noise at the ith receiver.

III. Power Region in N-dimensions

Plotting individual power constraints on their own orthogonal axis in N-dimensional space $\mathbb{R}^N$, the feasible power region can be described as a hypercube, the interior and boundary of which contains all possible transmit powers. The corners or vertices of the hypercube are the points with coordinates $(p_1, \ldots, p_N)$ either $p_i = 0$ or $p_i = P_i^{\max}$. For different maximum powers, e.g., in a downlink heterogeneous network, the hypercube will have different side lengths.

In addition to individual power constraints, the feasible region can be formed by minimum user rate constraints. By rearranging the minimum rate constraints in (2), we can obtain N inequalities of the form

$$p_i - \gamma_i \left(\sum_{j \neq i} p_j \right) \geq \gamma_i \sigma^2, \forall i \in 1, \ldots, N. \quad (5)$$

Geometrically, with equality the above is the equation of a hyperplane in $\mathbb{R}^N$, while with inequality it is the region above[1] the hyperplane. Thus, *the power constraints form the hypercube, while the minimum user rate constraints further bound the power region into a polytope*. Increasing the number of powers increases the dimensionality of the region, while increasing the number of users increases the number of hyperplanes and further restricts the polytope.

The intersection of all the SINR inequalities, denoted as the point Q, can be found by solving for their equality expressions simultaneously, which can be done using methods such as Cramer's Rule. The final region bounded by the boundaries of the hypercube and the hyperplanes form the feasible power region. Possible regions for two and three transmitting powers are illustrated in Figs. 1 and 2.

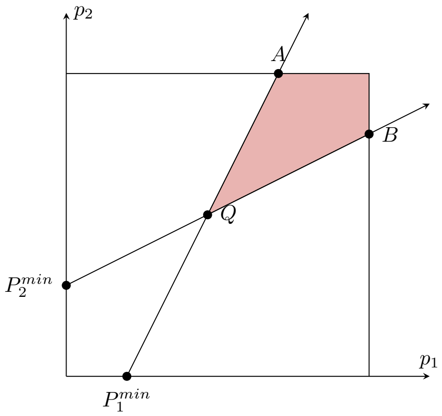


Fig. 1: Power region for two transmitters bound by edges of the rectangle (power constraint) and lines (minimum rate constraint).

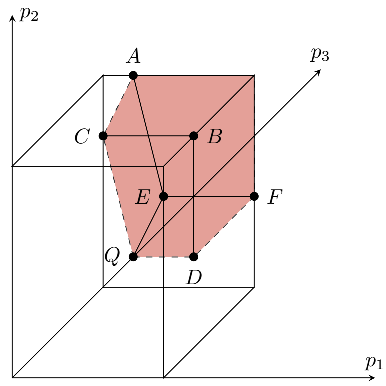


Fig. 2: Power region for three transmitters bound by edges of the cube (power constraint) and planes (minimum rate constraint, not shown for clarity).

IV. Sum SINR approximation

From [3], we know that the optimal powers that maximizes sum rate occur at the boundary of the power region. Further, if the function to be maximized is convex, the vertices of the power region, with the exception of the point Q, form the finite set of points that contain the optimal powers. The coordinates of these vertices are powers that are either maximum powers defined by the power constraints, or minimum powers allowed by other users' constraints. However, although convexity is a more common property to prove, it is in fact *quasiconvexity* which states that for a given domain, the maximum of a function lies on the endpoints. Of course, convex functions are also quasiconvex and share this property.

Previously, other works have claimed that since SINRs are convex expressions in individual powers, which implies that the optimal powers lie on the vertices. However, this only

[1]Here, 'above' refers to the region satisfying the inequality, and may not always be 'above' in the everyday sense.

applies if the vertices also lie on the hypercube edges of the power region, since on the edges only one power is varying. For N powers, the hypercube will have vertices that lie on other types of boundaries, e.g. a face, which represent the condition that there is more than one varying power. Hence, on these boundaries where more than one power is varying, it is also required that the function is also quasiconvex with respect to more than one varying power in order to justify searching only vertices for optimal powers.

To illustrate, consider the simple case of a two user system in Fig. 1. Without minimum user constraints, binary power control tells us the powers which will maximize sum rate will either be $p_i = P_i^{\max}$ or $p_i = 0$. However, with the minimum user constraints, the power region is now also bounded by the lines, and thus the optimal set of powers also include the points A and B, whose coordinates can be found by using (5). A similar approach can be used to determine the set of points for more dimensions, although they become increasingly more difficult to visualize.

A. Quasiconvexity of Sum SINR for arbitrary number of varying powers

Although it is commonly stated that sum SINR, i.e.,

$$S = \sum_{i=1}^{N} \frac{p_i}{a_i} \tag{6}$$

is convex in each individual power (since each SINR is convex, and the sum of convex functions is also convex), sum SINR is not *jointly* convex, nor is it convex with respect to arbitrary combinations of powers, i.e., treat a set of powers as varying and the rest as constants. This can be easily observed by evaluating the Hessian matrix for a generic set $\{p_1, \ldots, p_N\}$, which is not positive semidefinite. However, in order to justify searching only the vertices for powers that maximize sum SINR, we require the sum SINR function to be quasiconvex in any combinations of varying powers. The following proposition proves this property:

Proposition 1. *Sum SINR is a quasiconvex function for any combination of varying powers. Hence, it is also jointly quasiconvex in all powers.*

Proof. See Appendix A in [9] for details. The proof involves showing that (6) satisfies the quasiconvexity condition in [10].

Remark 1. The joint quasiconvexity of sum SINR ensures that the optimal powers will be a subset of the vertices of the power region.

B. Sum SINR as a close approximation of sum rate

Since sum rate is not convex (or quasiconvex) in general, the vertices of the power region may not give the maximum sum rate. At first it may seem that sum SINR can be a good approximation to sum rate since the logarithm is a monotonically increasing function, and thus any set of powers which maximize SINR will also maximize sum rate. In the low SINR regime, this is true, since for low SINR $\log_2(1 + SINR) \approx \frac{SINR}{\ln 2}$ [3]. However, this is not the case in general for more than three SINR terms, since the product of the $(1 + SINR)$ terms in (2) is not convex in the powers.

Although a direct match between sum rate and sum SINR is not immediately clear, we can show that under asymptotic conditions, i.e., when one receive power dominates over others, sum rate and sum SINR have very similar derivatives.

Proposition 2. *When one receive power dominates, e.g., an order of magnitude larger than others, maxima and minima of sum rate and sum SINR occur at the same set of powers.*

Proof. See Appendix B in [9]. The proof involves taking derivatives of both the $\log_2$ expression (2) and (6) with respect to an arbitrary p_i, then showing their equivalence as $p_i \to \infty$. Since monotonic functions such as logarithms preserve order, having the same derivatives indicates that maxima and minima occur at the same locations.

Remark 2. We can conclude from this that since the two derivatives are equal, when one receive power dominates, and logarithm is monotonic and thus doesn't affect the locations of maxima and minima, global maxima and minima for sum rate and sum SINR will occur at almost identical powers.

As a related aside, although sum rate is not convex in general, we make the following proposition in relation to the convexity of sum rate when only one power is varying:

Proposition 3. *For any number of transmitting powers, sum rate in individual powers, i.e., one power varying and the others constant, is always convex.*

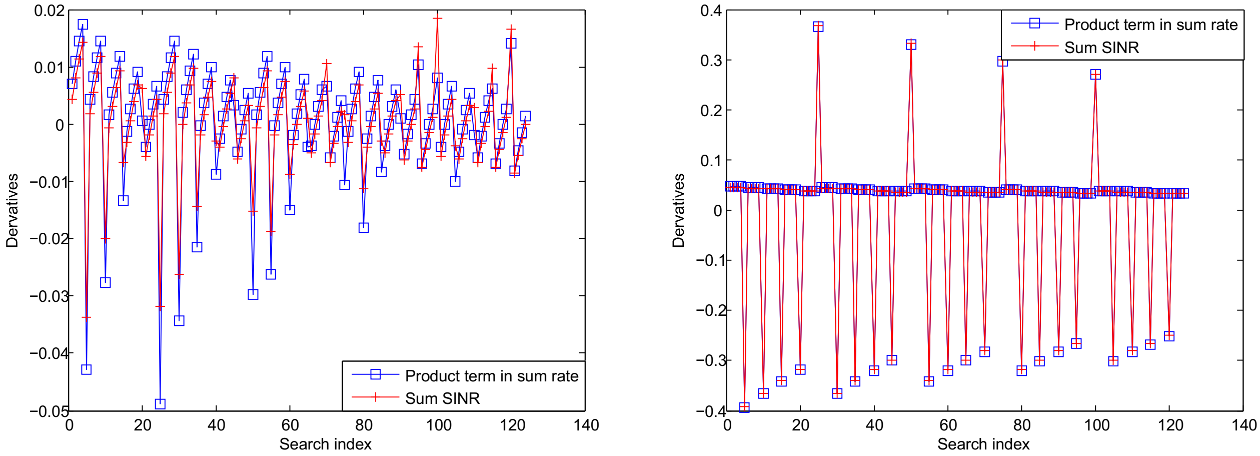
Proof. See Appendix A.

V. Simulation Results

In our simulations we considered three and four transmitting powers, and tested all combinations to illustrate the validity of Proposition 2. All powers were normalized with respect to the noise power. We set a power range for each transmitter, and tested all combinations of powers with step sizes chosen such that there were five powers in each transmitting set. All possible combinations of powers were searched through, with each combination labelled with a 'search index.'

For three transmitters, Fig. 3a shows the derivatives of sum SINR and the product term in (2) when received powers are of the same order of magnitude around 10 dB with respect to the noise power, while Fig. 3b shows the derivatives when one power is an order of magnitude larger than others. It is clear that when there is one dominating power, the derivatives coincide almost perfectly, indicating that the log term in sum rate and sum SINR, 'follow' one another and thus have their maxima and minima occur at the same locations. We observe the same trend when there are four transmitters as shown in Fig. 4a and Fig. 4b.

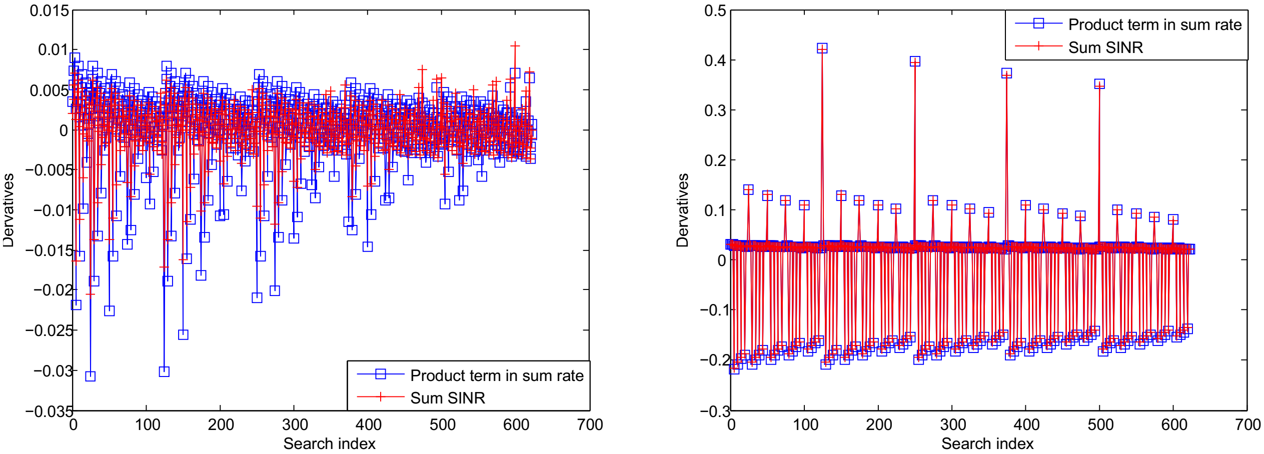
When considering the actual sum rate, i.e., taking the logarithm, we find that the global maxima and minima indeed still occur at the same set of powers as expected when a received power is an order of magnitude larger, as shown in Fig. 5 for three transmitters. In other words, while the logarithm does change the actual asymptotic derivative values of sum rate and sum SINR, its monotonicity ensures that the



(a) Powers the same order of magnitude. There is a mismatch of derivative values with no consistency.

(b) One power an order of magnitude larger. Derivative values match almost perfectly.

Fig. 3: Derivatives of rate and SINR with 3 transmitting powers.



(a) Powers the same order of magnitude. There is a mismatch of derivative values with no consistency.

(b) One power an order of magnitude larger. Derivative values match almost perfectly.

Fig. 4: Derivatives of rate and SINR with 4 transmitting powers.

locations of maxima and minima remain the same. In the case of the chosen powers, there is one global maximum each for sum rate and sum SINR, with both occurring at the same location at search index 5.

Our simulated scenarios can exist in high load downlink HetNets, e.g., when a macro receiver receives much more power than a femto user. As shown in [9], searching the vertices to maximize sum rate is much less computationally extensive for small number of users compared to conventional methods such as geometric programming, and produces near-optimal solutions. Thus, using power region vertices is a suitable near-optimal method for sum rate maximization.

VI. Conclusion

We have provided a graphical and geometric description of the feasible power region for multiuser interference channels for arbitrary number of users subject to individual power and minimum user rate constraints. We have shown that sum SINR is quasiconvex with respect to any number of varying powers, and that sum SINR is an almost equivalent objective to maximize as sum rate when transmit powers are orders of magnitude apart, or when one power dominates the others. Through our findings, we confidently conclude that for multi-user interference scenarios where received powers can vary by an order of magnitude, searching for the vertices of the power region is a suitable near-optimal approach to maximizing sum rate.

Appendix A
Convexity of Sum Rate with Respect to One Varying Power

Consider the expression within the $\log_2$ in (2), i.e.,

$$\prod_{i=1}^{N}\left(1+\frac{p_i}{a_i}\right) = \frac{(\sum_{i=1}^{N} p_i + \sigma^2)^N}{\prod_{i=1}^{N} a_i}$$
$$= \frac{(x+a_i)^N}{a_i \prod_{k\neq i}^{N-1}(x+a_{i,k})} \triangleq f(x) \quad (7)$$

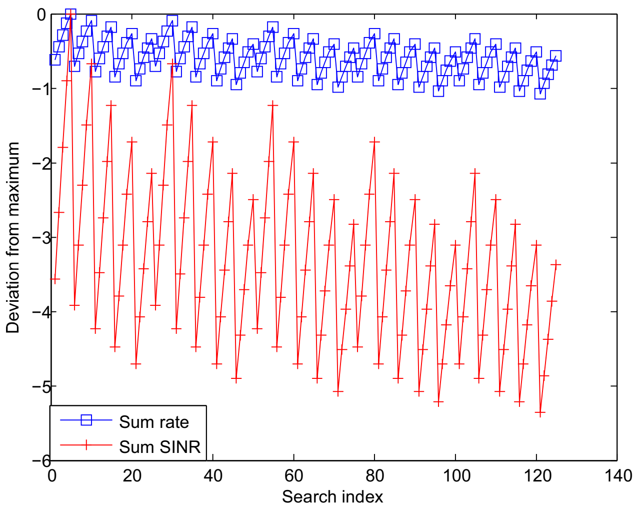


Fig. 5: Derivatives of sum rate and sum SINR with 3 transmitters including one larger power. Maxima and minima occur at the same locations, despite there being a mismatch in magnitude.

where $x = p_i$, $a_i = \sum_{j \neq i} p_j + \sigma^2$ and $a_{i,k} = \sum_{j \neq i,k} p_j + \sigma^2$. The N roots of $f(x)$ are at $x = -a_i$, while the asymptotes are at $x = -a_{i,k}$ for $k \neq i$. Since $a_i = a_{i,k} + p_k$, $a_i > a_{i,k}$, meaning that the roots occur to the left of all the asymptotes. To show that $f(x)$ is convex for $x > 0$, we can take derivatives and use the precise definition of convexity, but this is tedious to do with so many products. Instead, we adopt a graphical approach.

In general, since $f(x)$ is a function with polynomial numerators and denominators, basic curve sketching techniques can be employed to determine its generic shape.

1) Consider the case when N is even (Fig. 6). The smallest, i.e., left most critical point is the root at $x = -a_i$. If N is even $f(x)$ must have either a maximum or minimum turning point at that root. It is easy to see that since for x to the left of the first vertical asymptote, $f(x) \leq 0$, the function must have a maximum turning point at $x = -a_i$. The behaviour and shape of $f(x)$ then alternates between convex positive and concave negative graphs between each set of asymptotes. Since there are an even number of such graphs, the right most one corresponding to when $x > 0$ will always be positive and convex.
2) Consider the case when N is odd (Fig. 7). At the root, the function has a point of inflexion due to the odd power, while it is easy to see that $f(x)$ will be negative between when x is between the two left-most vertical asymptotes. Following the same pattern as the even case, the function will alternate between convex positive and concave negative graphs between each set of asymptote, and again will end up being positively convex for $x > 0$.

Since $\log_2$ is a monotonically increasing function, and the relevant branches are decreasing with second derivatives less than 0, the sum rate over those ranges will remain convex.

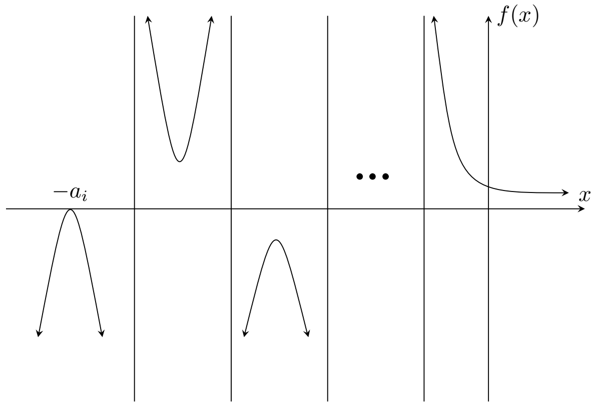


Fig. 6: General curve behaviour of sum rate with respect to one power for even number of powers.

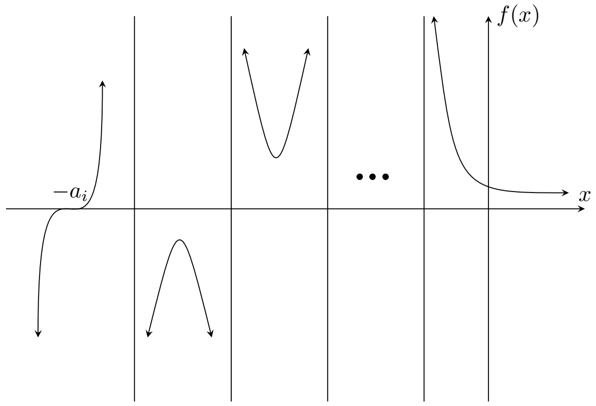


Fig. 7: General curve behaviour of sum rate with respect to one power for odd number of powers.

REFERENCES

[1] K. Illanko, A. Anpalagan, E. Hossain, and D. Androutsos, "On the power allocation problem in the gaussian interference channel with proportional rate constraints," *IEEE Trans. Wireless Commun.*, vol. 13, no. 2, pp. 1101–1115, Feb. 2014.

[2] K. Illanko, A. Anpalagan, and D. Androutsos, "Convex structure of the sum rate on the boundary of the feasible set for coexisting radios," in *Proc. IEEE ICC*, June 2011, pp. 1–6.

[3] A. Gjendemsj, D. Gesbert, G. E. Oien, and S. G. Kiani, "Binary power control for sum rate maximization over multiple interfering links," vol. 7, no. 8, pp. 3164–3173, Aug. 2008.

[4] M. Chiang, C. W. Tan, D. Palomar, D. O'Neill, and D. Julian, "Power control by geometric programming," *IEEE Trans. Wireless Commun.*, vol. 6, no. 7, pp. 2640–2651, July 2007.

[5] S. Jafar and A. Goldsmith, "Adaptive multirate cdma for uplink throughput maximization," *IEEE Trans. Wireless Commun.*, vol. 2, no. 2, pp. 218–228, Mar. 2003.

[6] H. Boche, S. Naik, and T. Alpcan, "Characterization of convex and concave resource allocation problems in interference coupled wireless systems," *IEEE Trans. Signal Process.*, vol. 59, no. 5, pp. 2382–2394, May 2011.

[7] T. Holliday, A. Goldsmith, P. Glynn, and N. Bambos, "Distributed power and admission control for time varying wireless networks," in *Proc. IEEE GLOBECOM*, vol. 2, Nov. 2004, pp. 768–774 Vol.2.

[8] L. P. Qian, Y. J. A. Zhang, and J. Huang, "MAPEL: Achieving global optimality for a non-convex wireless power control problem," *IEEE Trans. Wireless Commun.*, vol. 8, no. 3, pp. 1553–1563, Mar. 2009.

[9] Y. Huang, A. A. Nasir, S. Durrani, and X. Zhou, "Mode selection, resource allocation and power control for d2d-enabled two-tier cellular network," *arXiv*, Sep. 2015. [Online]. Available: http://arxiv.org/abs/1509.00123

[10] S. Boyd and L. Vandenberghe, *Convex Optimization*. New York, NY, USA: Cambridge University Press, 2004.