The Uncertainty Principle

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The uncertainty principle shows that one cannot jointly localize a signal in time and frequency arbitrarily well; either one has poor frequency localization or poor time localization. The degree of localization is measured in the theorem below by the quantities $d$ and $D$; these are like the standard deviation of a probability distribution.

The proof we present was originally due to Weyl [H. Weyl, Theory of groups and Quantum Mechanics, Dover, NY, (1950); Appendix A]

**Theorem 1 (Uncertainty Principle)** Suppose $f(t)$ is a finite energy signal with Fourier transform $F(\omega)$. Let

$$E := \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

$$d^2 := \frac{1}{E} \int_{-\infty}^{\infty} t^2 |f(t)|^2 dt$$

$$D^2 := \frac{1}{2\pi E} \int_{-\infty}^{\infty} \omega^2 |F(\omega)|^2 d\omega$$

If $\sqrt{|t|} |f(t)| \to 0$ as $|t| \to \infty$, then

$$Dd \geq \frac{1}{2}$$

and equality holds only if $f(t)$ has the form

$$f(t) = Ce^{-\alpha t^2}.$$  

In order to prove the theorem we need the following Lemma which is very useful in many other situations.

**Lemma 2 (Cauchy-Schwarz Inequality)** For any square integrable functions $z(x)$ and $w(x)$ defined on the interval $[a, b]$,

$$\left| \int_{a}^{b} z(x)w(x)dx \right|^2 \leq \int_{a}^{b} |z(x)|^2 dx \int_{a}^{b} |w(x)|^2 dx$$  \hspace{1cm} (1)

and equality holds if and only if $z(x)$ is proportional to $w^*(x)$ (almost everywhere on $[a, b]$).

**Proof** Assume $z(x)$ and $w(x)$ are real (the extension to complex-valued functions is straight-forward). Let

$$I(y) = \int_{a}^{b} [z(x) - yw(x)]^2 dx$$

$$= \int_{a}^{b} z^2(x)dx - 2y \int_{a}^{b} z(x)w(x)dx + y^2 \int_{a}^{b} w^2(x)dx$$

$$= A - 2yB + y^2C$$
Clearly \( I(y) \geq 0 \) for all \( y \in \mathbb{R} \). But if \( I(y) = A - 2yB + y^2C \geq 0 \) for all \( y \in \mathbb{R} \) then \( B^2 - AC \leq 0 \). If \( B^2 - AC = 0 \), then \( I(y) \) has a double real root: \( \exists k: I(k) = 0 \) for \( y = k \). Thus (1) holds and if it is an equality, then \( I(y) \) has a real root which implies \( \exists k: I(k) = \int_a^b [z(x) - kw(x)]^2 dx = 0 \). But this can only occur if the integrand is identically zero; thus \( z(x) = kw(x) \) for all \( x \).

**Proof (Theorem)** Assume \( f(t) \) is real. Lemma 2 implies

\[
\left| \int_{-\infty}^{\infty} t f \frac{df}{dt} dt \right|^2 \leq \int_{-\infty}^{\infty} t^2 f^2 dt \int_{-\infty}^{\infty} \left| \frac{df}{dt} \right|^2 dt.
\]

(2)

Let

\[
A := \int_{-\infty}^{\infty} t f \frac{df}{dt} dt = \int t \frac{d(f^2/2)}{dt} dt
\]

(by the chain rule for differentiation)

\[
= t \frac{f^2}{2} \bigg|^{\infty}_{-\infty} - \int_{-\infty}^{\infty} \frac{f^2}{2} dt
\]

using integration by parts. By assumption \( \sqrt{|t|} f \to 0 \Rightarrow |t|f^2 \to 0 \Rightarrow tf^2 \to 0 \). Thus \( \alpha = 0 \). Furthermore \( \beta = E/2 \) and so

\[
A = -\frac{E}{2}
\]

(3)

Recalling that \( \frac{d}{dt} f(t) \leftrightarrow j\omega F(\omega) \), by Parseval’s theorem we have

\[
\int_{-\infty}^{\infty} \left| \frac{df}{dt} \right|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 |F(\omega)|^2 d\omega
\]

(4)

Substituting (3) and (4) into (2) we obtain

\[
\left| -\frac{E}{2} \right|^2 = \left| \int_{-\infty}^{\infty} t f \frac{df}{dt} dt \right|^2 \leq \int_{-\infty}^{\infty} t^2 f^2 dt \times \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 |F(\omega)|^2 d\omega
\]

(5)

\[
\Rightarrow dD \geq \frac{1}{2}
\]

(6)

If (6) is an equality, then (2) must be also which is possible only if

\[
\frac{d}{dt} f(t) = kf(t) \Rightarrow f(t) = Ce^{-\alpha t^2}
\]

(note \( e^{-\pi(bt)^2} \leftrightarrow e^{-\pi(f/b)^2} \) — the Fourier transform of a Gaussian is a Gaussian).