The Uncertainty Principle

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The uncertainty principle shows that one can not jointly localize a signal in time and frequency arbitrarily well; either one has poor frequency localization or poor time localization. The degree of localization is measured in the theorem below by the quantitities d and D; these are like the standard deviation of a probability distribution.

The proof we present was originally due to Weyl [H. Weyl, *Theory of groups and Quantum Mechanics*, Dover, NY, (1950); Appendix A]

Theorem 1 (Uncertainty Principle) *Suppose $f(t)$ is a finite energy signal with Fourier transform $F(\omega)$. Let*

$$\begin{aligned} E &:= \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega \\ d^2 &:= \frac{1}{E}\int_{-\infty}^{\infty} t^2 |f(t)|^2 dt \\ D^2 &:= \frac{1}{2\pi E}\int_{-\infty}^{\infty} \omega^2 |F(\omega)|^2 d\omega \end{aligned}$$

If $\sqrt{|t|}f(t) \to 0$ as $|t| \to \infty$, then

$$Dd \geq \frac{1}{2}$$

and equality holds only if $f(t)$ has the form

$$f(t) = Ce^{-\alpha t^2}.$$

In order to prove the theorem we need the following Lemma which is very useful in many other situations.

Lemma 2 (Cauchy-Schwarz Inequality) *For any square integrable functions $z(x)$ and $w(x)$ defined on the interval $[a, b]$,*

$$\left| \int_a^b z(x)w(x)dx \right|^2 \leq \int_a^b |z(x)|^2 dx \int_a^b |w(x)|^2 dx \tag{1}$$

and equality holds if and only if $z(x)$ is proportional to $w^*(x)$ (almost everywhere on $[a, b]$).

Proof Assume $z(x)$ and $w(x)$ are real (the extension to complex-valued functions is straight-forward). Let

$$\begin{aligned} I(y) &= \int_a^b [z(x) - yw(x)]^2\, dx \\ &= \underbrace{\int_a^b z^2(x)dx}_{A} - 2y \underbrace{\int_a^b z(x)w(x)dx}_{B} + y^2 \underbrace{\int_a^b w^2(x)dx}_{C} \\ &= A - 2yB + y^2 C \end{aligned}$$

Clearly $I(y) \geq 0$ for all $y \in \mathbb{R}$. But if $I(y) = A - 2yB + y^C \geq 0$ for all $y \in \mathbb{R}$ then $B^2 - Ac \leq 0$. If $B^2 - AC = 0$, then $I(y)$ has a double real root: $\exists k\colon I(k) = 0$ for $y = k$. Thus (1) holds and if it is an equality, then $I(y)$ has a real root which implies $\exists k\colon I(k) = \int_a^b [z(x) - kw(x)]^2 dx = 0$. But this can only occur if the integrand is identically zero; thus $z(x) = kw(x)$ for all x. ∎

Proof (Theorem) Assume $f(t)$ is real. Lemma 2 implies

$$\left| \int_{-\infty}^{\infty} tf \frac{df}{dt} dt \right|^2 \leq \int_{-\infty}^{\infty} t^2 f^2 dt \int_{-\infty}^{\infty} \left| \frac{df}{dt} \right|^2 dt. \tag{2}$$

Let

$$\begin{aligned} A &:= \int_{-\infty}^{\infty} tf \frac{df}{dt} dt \\ &= \int t \frac{d(f^2/2)}{dt} dt \end{aligned}$$

(by the chain rule for differentiation)

$$= \underbrace{t \frac{f^2}{2} \bigg|_{-\infty}^{\infty}}_{\alpha} - \underbrace{\int_{-\infty}^{\infty} \frac{f^2}{2} dt}_{\beta}$$

using integration by parts. By assuption $\sqrt{|t|} f \to 0 \Rightarrow |t| f^2 \to 0 \Rightarrow tf^2 \to 0$. Thus $\alpha = 0$. Furthermore $\beta = E/2$ and so

$$A = -\frac{E}{2} \tag{3}$$

Recalling that $\frac{d}{dt} f(t) \leftrightarrow j\omega F(\omega)$, by Parseval's theorem we have

$$\int_{-\infty}^{\infty} \left| \frac{df}{dt} \right|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 |F(\omega)| d\omega \tag{4}$$

Substituting (3) and (4) into (2) we obtain

$$\left| -\frac{E}{2} \right|^2 = \left| \int_{-\infty}^{\infty} tf \frac{df}{dt} dt \right|^2 \leq \underbrace{\int_{-\infty}^{\infty} t^2 f^2 dt}_{Ed^2} \times \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 |F(\omega)|^2 d\omega}_{ED^2} \tag{5}$$

$$\Rightarrow \quad dD \geq \frac{1}{2} \tag{6}$$

If (6) is an equality, then (2) must be also which is possible only if

$$\begin{aligned} & \frac{d}{dt} f(t) = ktf(t) \\ \Rightarrow \qquad & f(t) = Ce^{-\alpha t^2} \end{aligned}$$

(note $e^{-\pi(bt)^2} \leftrightarrow e^{-\pi(f/b)^2}$ — the Fourier transform of a Gaussian is a Gaussian). ∎