Theory and design of broadband sensor arrays with frequency invariant far-field beam patterns

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Abstract

The theory and design of a broadband array of sensors with a frequency invariant far-field beam pattern over an arbitrarily wide design bandwidth is presented. The frequency invariant beam pattern property is defined in terms of a continuously distributed sensor, and the problem of designing a practical sensor array is then treated as an approximation to this continuous sensor using a discrete set of filtered broadband omni-directional array elements. The design methodology is suitable for one, two, and three dimensional sensor arrays; it imposes no restrictions on the desired aperture distribution (beam shape), and can cope with arbitrarily wide bandwidths. An important consequence of our results is that the frequency response of the filter applied to the output of each sensor can be factored into two components: one component is related to a slice of the desired aperture distribution, and the other is sensor independent. The results also indicate that the locations of the sensors are not a crucial design consideration, although we show that nonuniform spacings simultaneously avoid spatial aliasing and minimize the number of sensors. An example design which covers a 10:1 frequency range (which is suitable for speech acquisition using a microphone array) illustrates the utility of our method. Finally, the theory is generalized to cover a parameterized class of arrays in which the frequency dependence of the beam pattern can be controlled in a continuous manner from a classical single frequency design to a frequency invariant design.

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1   Introduction

The problem of designing a uniformly spaced array of sensors for far-field operation at a single frequency (or within a narrow band of frequencies) is well understood from general array theory [1, 2]. However when it is desired to receive signals over a wide band of frequencies the problem of broadbanding a sensor array arises. We will now review several approaches to solving this problem.

One approach to broadband design is to use a frequency domain beamformer [3]. Since narrowband beamforming is conceptually simpler than broadband beamforming, the beamformer is implemented by a narrowband decomposition structure, whereby the signal received at each sensor is transformed into the frequency domain using a fast Fourier transform, and each narrow band of frequencies is treated as an independent narrowband beamformer. This is very much a brute force approach which is computationally excessive.

Adaptive beamformers, in which each sensor feeds a transversal filter (tapped delay line) and the filter outputs are summed to produce the overall output, can be used for broadband beamforming (see references [4, 5, 6, 7] for a review). An adaptive array with K sensors can produce K constraints on the beam pattern of the array at a single frequency. If each sensor feeds an L-tap transversal filter, then the same constraints can be applied at L different frequencies. For example, a linearly constrained algorithm has been reported [8] which maintains the peak array response in the look direction at L different frequencies, whilst minimizing the non-look direction noise power. Although these adaptive methods can keep the peak array response relatively constant and produce nulls in given directions at a finite number of frequencies, they are unable to produce an identical beam pattern over a continuous range of frequencies (without resorting to a prohibitive number of sensors and taps).

Another approach to the design of broadband sensor arrays is to treat the problem of determining sensor gains and inter-sensor spacings as a multi-dimensional optimization problem [9, 10]. These methods do not use frequency dependent sensor gains, but instead attempt to find optimal sensor spacings and (fixed) gains by minimizing the array power spectral density over a given frequency band. Because the sensor gains are frequency independent, the resulting array structure allows a very simple implementation. However, it is impossible to achieve a frequency invariant beam pattern using these optimization methods. In addition, these methods are very computationally intensive. Note that "optimum" array aperture designs (which optimize the compromise between beam width and sidelobe level [11, 12]) can be easily incorporated into our broadband design method, since the aperture distribution is totally arbitrary for our theory and design methodology.

Yet another approach, typically used by researchers interested in designing microphone arrays for speech acquisition, is harmonic nesting [13, 14, 15, 16], whereby the array is composed of a set of nested equally spaced subarrays, each of which is a single frequency design. The outputs of the subarrays are then combined via appropriate bandpass filtering. For example, if the sensor spacing used at a frequency f is d, then at a frequency $f/2$, the spacing used will be $2d$, etc. This produces an array which has an identical beam pattern at frequencies f, $f/2$, $f/4$, etc, but which varies at intermediate frequencies. The effect of harmonic nesting is to reduce the extent of bandwidth variation to that which occurs within a single octave. Frequency dependent sensor gains can be used to interpolate to frequencies in between the subarray design frequencies [17, 18], but this requires additional complicated filtering. Another problem with arrays based upon harmonic nesting is that only a very limited set of band ratios is possible, whereas our method is applicable for any frequency design band.

For the purposes of this paper, we will consider broadband arrays in which there is little or no frequency variation in the far-field array beam pattern over an arbitrarily wide desired bandwidth. A method has been proposed [19] in which the array beam pattern has little or no frequency dependence. The asymptotic theory of unequally spaced arrays [20, 21] is used to derive relationships between beam pattern properties (such as peak response, main lobe width, plateau sidelobe level, and clean sweep width) and array design. These relationships are then used to translate beam pattern requirements into functional requirements on the sensor spacings and weightings, thereby deriving a broadband design. This results in a space tapered array with frequency dependent sensor weightings; at each frequency in the design band the nonzero sensor weights identify a subarray having total length and largest spacing which are appropriate to that frequency. Although this method provides a frequency invariant beam pattern over a specified frequency design band, it is based on a single-sided uniform aperture distribution and a linear array. No insight is given into the problem of designing double-sided or higher dimensional arrays, or arrays with arbitrary aperture distributions in both

magnitude and phase (and thus arbitrary beam patterns).

The purpose of this paper is to provide a very general theory and design method for a truly broadbanded array. Our approach to the broadbanding problem is to develop a frequency invariant (FI) beam pattern property for a theoretical continuous sensor, and then to approximate this continuous sensor by an array of discrete sensors. The problem of designing a broadband array is then reduced to one of providing an approximation to a theoretically continuous sensor. We later show that FI arrays are a subset of a more general class of arrays in which the frequency variation of the beam pattern can be controlled. An important consequence of our development is that there are specific simple structural properties that a FI array must have; such structural properties reduce the number of free variables which have to be chosen in designing the array.

2 Theory

2.1 Background

Throughout this paper we are only concerned with reception of planar waves and will no longer specifically state far-field operation. We define the notion of a broadband FI array in terms of the array beam pattern: the beam pattern must be frequency independent. To obtain an identical beam pattern at k different frequencies would require a compound array of k subarrays. These k subarrays would be identical if the spatial coordinate was expressed in wavelengths. Thus, to produce an identical beam pattern over a continuous range of frequencies requires an infinite number of subarrays. We must thus acknowledge that it is not possible to produce a strictly frequency invariant beam pattern from a finite number of discrete sensors (although we will show in later sections how a frequency invariant beam pattern can be approximated from a finite array of discrete sensors). It is thus necessary to initially consider the concept of a *continuous sensor* to develop a FI broadband theory. From this vantage point we will see that a discrete array which exhibits an approximate FI broadband character (that can be made to arbitrarily closely approximate the ideal frequency invariance uniformly over the design bandwidth) is readily derived from the continuous sensor theory.

2.2 One Dimensional Sensor

Let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers respectively. Consider a one dimensional

(linear) continuous sensor aligned with the x axis. The output of this continuous sensor is

$$Z_f = \int_{-\infty}^{\infty} S(x,f)\,\rho(x,f)\,dx, \quad f > 0 \qquad (1)$$

where $S : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{C}$ is the signal received at a point x on the sensor due to a signal of frequency f (and zero phase offset), and $\rho : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{C}$ defines the *sensitivity distribution* or gain of the sensor at a point x and for a frequency f. The function $\rho(x,f)$ can also be referred to as the aperture distribution, but we reserve this term for a slightly different concept later. Here we assume that the sensitivity distribution is absolutely integrable to ensure that the integral in (1) exists for finite power signals. It should be noted that we have indicated the limits on the integral as doubly infinite, which means that in the case of a practical finite-aperture continuous sensor the function $\rho(x,f)$ should have finite support.

Consider the output of the sensor when subject to plane waves arriving from an angle θ measured relative to broadside. In this case the signal received at a point on the sensor is given by

$$S(x,f) = e^{-j2\pi c^{-1} f x \sin\theta}$$

where c is the speed of wave propagation. With $S(x,f)$ thus defined, the output of the sensor (1) is implicitly a function of θ, lending its interpretation as the sensor beam pattern (at frequency f) as follows:

$$b_f(\theta) = \int_{-\infty}^{\infty} e^{-j2\pi c^{-1} f x \sin\theta}\,\rho(x,f)\,dx. \qquad (2)$$

Note that the sensor beam pattern will have both magnitude and phase components, although often only the magnitude is considered. In this work we prefer to keep the phase information. We are now in a position to formally define the notion of a broadband FI beam pattern.

Definition: A *broadband frequency invariant (FI) sensor* is one in which the far-field beam pattern is frequency invariant, i.e., $b_f(\theta) = b(\theta)$, $\forall f > 0$.

We now come to our first result.

Theorem 1 (Frequency Invariant Beam Pattern)
Suppose the sensitivity distribution of a one dimensional sensor, which is a function of distance x along the sensor and frequency f, is given by

$$\rho(x,f) = f\,G(xf), \quad \forall f > 0 \qquad (3)$$

where $G : \mathbb{R} \to \mathbb{C}$ is an arbitrary absolutely integrable complex function of a single real variable. Then the

far-field beam pattern, $b_f(\theta)$, which is a function of the angle θ measured relative to broadside and frequency f, will be frequency invariant, i.e.,

$$b_f(\theta) = b(\theta) = \int_{-\infty}^{\infty} e^{-\jmath 2\pi c^{-1}\xi \sin\theta} G(\xi) \, d\xi.$$

Proof: Substituting $\rho(x,f) = f\, G(xf)$ into the expression for the sensor beam pattern (2), yields

$$b_f(\theta) = \int_{-\infty}^{\infty} e^{-\jmath 2\pi c^{-1}fx \sin\theta} fG(xf) \, dx, \quad f > 0$$

$$= \int_{-\infty}^{\infty} e^{-\jmath 2\pi c^{-1}\xi \sin\theta} G(\xi) \, d\xi \stackrel{\triangle}{=} b(\theta)$$

where we have changed variables $\xi = xf$.

Comments

- The theorem provides a sufficient condition on the sensitivity distribution to imply an infinite bandwidth FI broadband beam pattern. The result is trivially modified to cater for finite bandwidths, e.g., for frequencies from f_L to f_U (say), which is more relevant to practical designs.

- The theorem expresses the known property that the sensitivity distribution, $\rho(x,f)$, scales with wavelength or inversely with frequency to attain the same beam shape (ignoring the gain). Equivalently, apart from the gain, the sensitivity distribution is a fixed function when the spatial coordinate is expressed in wavelengths.

- The multiplicative f factor in (3) can be interpreted as normalizing the beam pattern. It has no effect on the beam shape.

- The functions $G(\xi)$ and $b(\theta)$ form a Fourier transform pair (modulo various constants and the $\sin\theta$ distortion). This Fourier pair relation is explicated in [12]. Hence it is straightforward to take any beam shape specification and translate that to a specification on the aperture distribution to achieve a broadband FI result. These specifications can be expressed in both the magnitude and phase.

The following theorem is a converse to Theorem 1. (See Appendix A for the proof.)

Theorem 2 (Sensitivity Distribution) *Let $b(\theta)$ be an arbitrary continuous square-integrable frequency invariant far-field beam pattern, which is specified for $\theta \in (-\pi/2, \pi/2)$. Then the sensitivity*

distribution, $\rho(x,f)$, of a linear sensor which realizes this beam pattern must satisfy the following conditions:

1. *$\rho(x,f) = fG(xf)$ for some function G.*

2. *G has a Fourier transform Γ satisfying*

   (a) *$\Gamma(s) = B(s) = b\left(\sin^{-1}(sc)\right), \quad s \in (-1/c, 1/c)$*

   (b) *$\Gamma(s) = A(s), \quad s \notin (-1/c, 1/c)$*

   where c is the speed of wave propagation, and $A(\cdot)$ is an arbitrary square integrable function such that

   $$A\left((-1)^i/c\right) = \lim_{s \to \frac{(-1)^i}{c}} B(s)$$

   for $i = 0, 1$.

Thus the only freedom in choosing $\rho(x,f)$ for a desired FI beam pattern is in the sufficiently high "spatial frequency" behaviour of G. Apart from that, $b(\theta)$ for $\theta \in (-\pi/2, \pi/2)$ determines $\rho(x,f)$ uniquely.

2.3 Two and Three Dimensional Sensors

Having demonstrated a sufficient property for a one dimensional sensor to be FI, we will now consider the same problem for two and three dimensional continuous sensors. The results extend in a simple manner.

The signal received by a continuous two dimensional (planar) sensor is

$$S(\mathbf{x}, f) = e^{-\jmath 2\pi c^{-1}f(x_1 \sin\theta \cos\phi + x_2 \sin\theta \sin\phi)}$$

where $\mathbf{x} = [x_1 \ x_2]$ is a two dimensional vector denoting a point on the sensor, and θ (elevation) and ϕ (azimuth) define the direction of arrival of the plane waves as shown by Fig. 1.

The beam pattern produced by the sensor is given by

$$b_f(\theta, \phi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\jmath 2\pi c^{-1}f(x_1 \sin\theta \cos\phi + x_2 \sin\theta \sin\phi)}$$
$$\times \rho(\mathbf{x}, f) \, dx_1 \, dx_2.$$

Let $\rho(\mathbf{x}, f) = f^2 G(x_1 f, x_2 f), \ \forall f > 0$, where G is defined analogously to the one dimensional case (3).
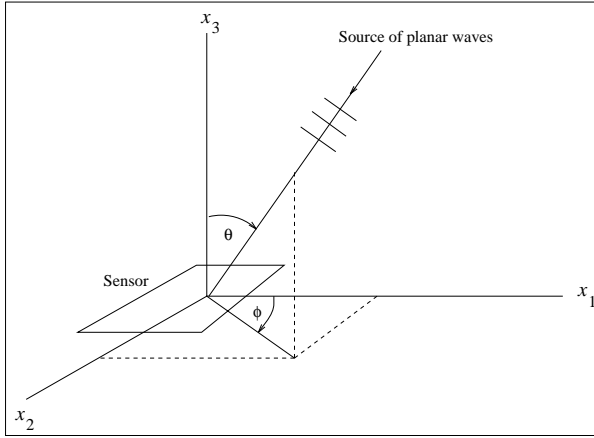
Figure 1: Geometry for a two dimensional sensor located in the $x_1 x_2$ plane subject to planar waves from direction (θ, ϕ).

The beam pattern can be written

$$
\begin{aligned}
b_f(\theta, \phi) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j 2\pi c^{-1}(f x_1 \sin\theta \cos\phi + f x_2 \sin\theta \sin\phi)} \\
&\quad \times f^2 G(x_1 f, x_2 f) \, dx_1 \, dx_2 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j 2\pi c^{-1}(\xi_1 \sin\theta \cos\phi + \xi_2 \sin\theta \sin\phi)} \\
&\quad \times G(\xi_1, \xi_2) \, d\xi_1 \, d\xi_2, \quad \xi_1 = x_1 f, \ \xi_2 = x_2 f \\
&\stackrel{\triangle}{=} b(\theta, \phi), \quad \forall f > 0
\end{aligned}
$$

which implies a FI beam pattern.

Similarly, for a three dimensional sensor exposed to planar waves arriving from the direction (θ, ϕ) the signal received is

$$
S(\mathbf{x}, f) = e^{-j 2\pi c^{-1} f (x_1 \sin\theta \cos\phi + x_2 \sin\theta \sin\phi + x_3 \cos\phi)}
$$

where $\mathbf{x} = [x_1 \ x_2 \ x_3]$ denotes a point on the sensor. In an analogous fashion it is easily shown that $b_f(\theta, \phi) = b(\theta, \phi)$, $\forall f > 0$ if

$$
\rho(\mathbf{x}, f) = f^3 G(x_1 f, x_2 f, x_3 f), \quad \forall f > 0.
$$

Only the sufficient condition for a FI beam pattern is considered for higher dimensional sensors, and hence the higher dimensional equivalent of Theorem 2 is not given.

2.4 General Broadband Condition

Summarizing the results of the previous subsections we state a general result using vector notation which gives sufficient conditions on a D dimensional array to exhibit a broadband FI beam pattern. The result is of practical relevance for $D \in \{1, 2, 3\}$.

Theorem 3 (General Broadband Condition)
Let the output of a D dimensional continuous sensor be given by

$$
Z_f = \int_{\mathbb{R}^D} S(\mathbf{x}, f) \rho(\mathbf{x}, f) \, d\mathbf{x}
$$

where $D \in \{1, 2, 3\}$, $S : \mathbb{R}^D \times \mathbb{R}^+ \to \mathbb{C}$ is the signal received at a point $\mathbf{x}$ on the sensor for a frequency f, and $\rho : \mathbb{R}^D \times \mathbb{R}^+ \to \mathbb{C}$ is the sensitivity distribution. The sensor has a frequency invariant far-field beam pattern if

$$
\rho(\mathbf{x}, f) = f^D G(\mathbf{x} f), \quad \forall f > 0 \tag{4}
$$

where $G : \mathbb{R}^D \to \mathbb{C}$ is an arbitrary absolutely integrable complex valued function.

2.5 Representations of the Sensitivity Distribution

As an aid to interpretation of the broadband condition, we will express $G(\mathbf{x} f)$, which appears in the expression for the sensitivity distribution function (4), in two equivalent representations:

$$
G(\mathbf{x} f) = A_f(\mathbf{x}) = H_\mathbf{x}(f), \quad \forall \mathbf{x}, \ f > 0 \tag{5}
$$

where $A_f : \mathbb{R}^D \to \mathbb{C}$ defines the *aperture distribution* at a nominally fixed frequency, f, and $H_\mathbf{x} : \mathbb{R}^+ \to \mathbb{C}$ defines the primary frequency response or *primary filter* at a single point, $\mathbf{x}$, on the sensor. Note that from the expression for the broadband sensitivity distribution (4), and using (5), we can express the total filtering required at a fixed point $\mathbf{x}$ as

$$
\rho(\mathbf{x}, f) = f^D H_\mathbf{x}(f).
$$

We refer to the f^D component as the *secondary filter*. Note that the secondary filter is independent of the sensor spatial vector $\mathbf{x}$ and a function of the sensor dimension D only. This sensor invariance property of the secondary filter is of practical significance as we will see later.

We now demonstrate an important result regarding the aperture distribution and the primary filter response as a consequence of (5). We briefly consider the one dimensional case for motivation. Note that in the scalar version of (5), $G(x f)$ is a symmetric function of spatial variable x and of the frequency variable f. This implies that f and x can be interchanged without affecting the value of the function. This can be interpreted as saying that the $G(x f)$ function, which appears in the sensitivity function (3), looks the same if we vary f whilst holding x fixed or vary x whilst holding f fixed. In

other words, the primary filter response takes the same shape as the aperture distribution. Next, we make this more precise and present a more general result for the D dimensional sensor. Note that we cannot freely interchange $f \in \mathbb{R}^+$ and $\mathbf{x} \in \mathbb{R}^D$ even in the scalar case since f must be positive, so this must be taken into account.

Define a unit vector in the direction of $\mathbf{x}$ as follows:

$$\widehat{\mathbf{x}} = \frac{\mathbf{x}}{\|\mathbf{x}\|}, \quad \mathbf{x} \in \mathbb{R}^D$$

where $\| \cdot \|$ denotes Euclidean distance. Then we have the following result:

Theorem 4 (Filter Shape) *If $H_{\mathbf{x}}(f)$ denotes the frequency response of the primary filter at point $\mathbf{x}$ and $A_f(\mathbf{x})$ denotes the aperture distribution for a given frequency $f > 0$, then for a frequency invariant broadband D dimensional sensor*

$$H_{\mathbf{x}}(f) = A_{\|\mathbf{x}\|}(f\widehat{\mathbf{x}}), \quad \mathbf{x} \in \mathbb{R}^D, \ f \in \mathbb{R}^+,$$
$$D \in \{1, 2, 3\}.$$

Proof: The proof follows from the following straightforward manipulation:

$$H_{\mathbf{x}}(f) = G(\mathbf{x}f)$$
$$= G(f\,\widehat{\mathbf{x}}\,\|\mathbf{x}\|)$$
$$= A_{\|\mathbf{x}\|}(f\,\widehat{\mathbf{x}}).$$

Comments

- In words, this result says that the primary filter response required at point $\mathbf{x}$ can be obtained by taking a slice through the aperture distribution from the origin in the direction of $\mathbf{x}$. The aperture distribution can be determined from the desired beam pattern and vice versa.

- The correspondence between aperture distribution and primary filter response is for both magnitude and phase.

- In the one dimensional case the result reduces to

$$H_x(f) = \begin{cases} A_x(f), & \text{if } x > 0 \\ A_{-x}(-f), & \text{if } x < 0. \end{cases}$$

  Note that the subscript on the aperture function needs to be positive since it denotes the frequency of interest.

The $G(\mathbf{x}f)$ function possesses an additional highly desirable property:

Theorem 5 (Filter Dilation) *All primary filter responses in a D dimensional frequency invariant broadband sensor for a given $\widehat{\mathbf{x}}$ are identical up to a frequency dilation.*

Proof: Let $H_{\mathbf{x}}(f)$ represent the filter response at an arbitrary point $\mathbf{x}$ on a frequency invariant broadband sensor and consider the filter response at a point $\gamma\mathbf{x}$ where $\gamma > 0$, i.e.,

$$H_{\gamma\mathbf{x}}(f), \quad \gamma > 0,$$

which lies on the radial line from the origin through $\mathbf{x}$, and implies $\widehat{(\gamma\mathbf{x})} = \widehat{\mathbf{x}}$. Then

$$H_{\gamma\mathbf{x}}(f) = G(f\gamma\mathbf{x})$$
$$= H_{\mathbf{x}}(\gamma f),$$

which is a dilation property.

Comments

In the following comments we are referring always to a broadband FI sensor.

- Not only do the primary filter responses relate to the aperture distribution but they also relate to each other by a frequency scaling whenever the filters lie on a common radial line through the origin.

- In the one dimensional case the sensor (and hence each filter) always lies on a line through the origin. So, for example, if $H_{x_1}(f)$ represents the filter response at a point x_1 on the sensor, and $H_{x_2}(f)$ represents the filter response at a point x_2 on the sensor, and $x_1 x_2 > 0$ then

$$H_{x_2}(f) = G(x_2 f) = G(x_1 x_2 / x_1 \ f)$$
$$= H_{x_1}(x_2 / x_1 \ f), \quad x_1 x_2 > 0.$$

- In the one dimensional case, for the above example, if $x_1 x_2 < 0$ then the filter responses H_{x_1} and H_{x_2} need not be related via a dilation since the final equality above is not valid. So there are just two primary filter shapes to consider depending on the sign of the x coordinate. An example later will make this clearer.

3 Broadband Array Design

3.1 Overview

Having developed the theory of a broadband FI continuous sensor, we will now describe the implementation of a broadband FI *array*, where an array is

defined as a practical structure that uses a finite set of identical, discrete, omni-directional broadband sensors. Without loss of generality we will initially concentrate on single-sided one dimensional array apertures with the first element located at $x = 0$, since this will form a major component of a practical design. Implementation issues for higher dimensional and double-sided arrays will be discussed later.

3.2 Approximation to a Continuous Sensor

An array of sensors can only approximate the ideal broadband continuous sensor. In our formulation this reduces to a numerical approximation uniformly in f (using classical techniques) to the following integral representing the output of the ideal continuous sensor for an arbitrary signal $S(x, f)$:

$$Z_f = \int_{-\infty}^{\infty} S(x, f)\, f G(xf)\, dx, \quad f > 0. \quad (6)$$

To obtain an approximation, let $\{x_i\}_{i=0}^{N-1}$ denote a finite set of N (possibly nonuniformly-spaced) discrete sensor locations. To a large extent this set is arbitrary, but sensibly it should satisfy certain physical constraints described later. Further, because only a finite number of sensors is practical, we limit the range of frequency to the interval $[f_L, f_U]$.

In approximating the family of integrals in (6), parameterized by f, we can consider the following simple class:

$$\widetilde{Z}_f = f \sum_{i=0}^{N-1} g_i\, S(x_i, f)\, G(x_i f), \quad \forall f \in [f_L, f_U]. \quad (7)$$

(In the next subsection we will show that a trapezoid numerical integration rule fits into this class.) Note that $S(x_i, f)$ is the complex signal received at a point x_i on the sensor for a frequency f, $G(x_i f)$ is the sampled value of $G(xf)$ at $x = x_i$, and g_i is a frequency independent weighting function to compensate for the possibly nonuniform sensor locations. An important aspect of our broadband array design is that the array design comes from approximating an integral describing a broadband FI continuous sensor.

3.3 Trapezoid Rule

We will illustrate the use of (7) for a special case corresponding to the well-known trapezoid integration method. Using (5) write the output of the primary filter attached to the ith sensor as

$$y_i(f) = H_{x_i}(f)\, S(x_i, f), \quad i \in \{0, 1, \cdots, N-1\}.$$

Equivalently, because of Theorem 5, this can be written

$$y_i(f) = H_{x_1}(x_i/x_1\, f)\, S(x_i, f), \quad i \in \{0, 1, \cdots, N-1\},$$

which emphasizes that only one primary filter shape is required in the numerical integration approximation. The trapezoid approximation to (6) can now be written as

$$\widetilde{Z}_f = f\, \mathbf{y}(f)'\mathbf{T}\mathbf{x}$$

where

$$\mathbf{y}(f) = [y_0(f),\ y_1(f),\ \cdots,\ y_{N-1}(f)]',$$
$$\mathbf{x} = [x_0,\ x_1,\ \cdots,\ x_{N-1}]',$$

and

$$\mathbf{T} = \begin{bmatrix} -0.5 & 0.5 & & & & \\ -0.5 & 0 & 0.5 & & \bigcirc & \\ & -0.5 & 0 & \ddots & & \\ & & -0.5 & \ddots & 0.5 & \\ & \bigcirc & & \ddots & 0 & 0.5 \\ & & & & -0.5 & 0.5 \end{bmatrix}.$$

In comparing the trapezoid rule above with the more general form of integration approximation in (7), the weighting functions g_i can be seen to relate to $\mathbf{T}\mathbf{x}$ via an unilluminating formula. However, we do emphasize that the weighting functions can be a function of one or more discrete sensor locations but (more importantly) are independent of frequency. This means that we have the capability to approximate the family of integrals for the desired frequency range.

In the remainder of this paper we will assume that the aperture distribution is a slowly varying function with respect to x compared to the exponential term in (2). If this is not the case, the array can be more densely filled, a more complex integration method can be applied, or alternate methods of sampling the continuous aperture [22] can be considered.

With the output of the single-sided one dimensional broadband array thus defined we are led to a particularly simple form of block diagram shown in Fig. 2. This diagram shows a number of important features that we have demonstrated: (i) the primary filters are simple dilations of a single frequency response, $H(f) \triangleq H_{x_1}(f)$; (ii) implicitly, this primary filter frequency response shape, $H(f)$, is identical to the sought after continuous aperture distribution shape both in magnitude and phase; (iii) the primary filter outputs can be combined via frequency
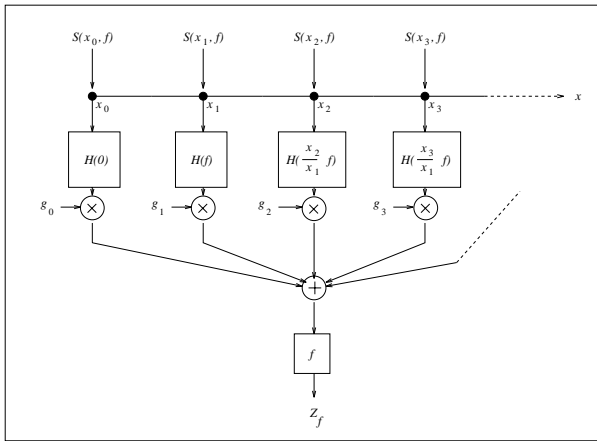
Figure 2: Block diagram of a general single-sided one dimensional broadband FI array with the array origin at $x = 0$. $H(\cdot)$ represent the primary filters (which are dilations of a single frequency response, $H(f) \triangleq H_{x_1}(f)$), and g_i represent frequency independent weights.

independent weights, g_i, that depend only on the sensor locations generating a scalar output; and finally (iv) all sensors share a common secondary filtering response, f, to generate the final output.

The structure shown in Fig. 2 falls short of providing complete guidelines for a practical realization. For example, the choice of discrete sensor locations needs addressing, along with the differences which arise when two-sided or higher dimensional arrays are employed. These and other points form the subject of the following subsections.

3.4   Sensor Locations

In determining the sensor locations for the broadband array implementation, it is desirable to minimize the number of sensors required while maintaining performance. The major factor determining the minimum number of sensors possible is *spatial aliasing*. We will develop the optimum sensor locations (with respect to minimizing the number of sensors required) which will avoid spatial aliasing. This sensor location function will be seen to be exponential (linearly increasing inter-sensor spacing) except at the upper design frequency where it is linear (constant inter-sensor spacing).

From the theory of linear uniformly-spaced arrays [2, page 7] it is well known that grating lobes (i.e., periodic repetitions of the main beam) are introduced into the array beam pattern of a broadside array if the spacing of array elements is greater than the wavelength of operation, λ. This is referred to as spatial aliasing. If delay beam steering is to be

applied to the array, the constraint reduces to a maximum spacing of $\lambda/2$ [23]. It is straightforward to see that delay beam steering can be used on a broadband array of the type that we describe in the same fashion that it can be employed for single frequency array design (as long as true time delays are used). Because of the applications we have in mind, we will use the spacing based on $\lambda/2$ in this work.

Since the broadband aperture size scales with frequency, we know that the aperture size is constant if defined in terms of wavelength. We assume the aperture size is finite and thereby define the aperture size as being P half-wavelengths at all frequencies, where, without loss of generality, we restrict P to be an integer. This highlights two related points: (i) since the aperture shape determines the primary filter shape then this implies the primary filter must be strictly bandlimited; and (ii) for all frequencies except at the lowest design frequency some of the sensors are not used. When the response of a sensor is used, i.e., the frequency of the signal lies in the primary filter passband, we will say the sensor is *active* at that frequency. In the following discussion we are referring only to active sensors. The locations of inactive sensors for a given frequency, despite the potential property that they violate a $\lambda/2$ spacing requirement, are completely irrelevant.

Assume the desired frequency range is $[f_L, f_U]$ where f_L is the lower design frequency and f_U is the upper design frequency. As before, the first sensor is located at $x = 0$. The finite aperture constraint implies a sensor positioning constraint

$$x_i = P \frac{\lambda_i}{2}$$

where i is the index of the active sensor of greatest distance from the origin, and λ_i is the wavelength corresponding to the bandwidth of the ith primary filter (or the highest frequency at which the ith sensor remains active).

The condition for a maximum spacing of $\lambda/2$ for all active sensors defines a second sensor positioning constraint:

$$x_i = x_{i-1} + \frac{\lambda_i}{2}, \quad \text{for } i > 0$$

where i corresponds to the same condition as for the first sensor positioning constraint.

Combining these two constraints gives

$$x_i = \left(\frac{P}{P-1} \right) x_{i-1} \tag{8}$$

whenever $x_{i-1} > 0$, where, recall, P is the aperture size measured in half-wavelengths. This con-

straint must be maintained within the desired frequency range to avoid spatial aliasing. Since spacings less than $\lambda_U/2$ will not cause spatial aliasing at any frequency within the design band, it follows that the spacing within the densest portion of the array should be $\lambda_U/2$ to minimize the number of sensors. This densely packed portion of the array should have a total size of $P\lambda_U/2$ and will contain a minimum of $P+1$ sensors. Hence, the maximum spacing to avoid spatial aliasing can be summarized as

$$x_i = \begin{cases} (\lambda_U/2)i, & \text{for } 0 \le i \le P \\ P(\lambda_U/2)\left(\frac{P}{P-1}\right)^{i-P}, & \text{for } P < i < N-1 \\ P(\lambda_L/2), & \text{for } i = N-1 \end{cases}$$

$$(9)$$

where λ_L and λ_U are the wavelengths corresponding to the lower and upper design frequencies respectively, P is the aperture size measured in half-wavelengths, and N is the number of array elements. The maximum allowable spacing to avoid spatial aliasing, as defined by (9), is illustrated in Fig. 3. In the sense of producing an approximate
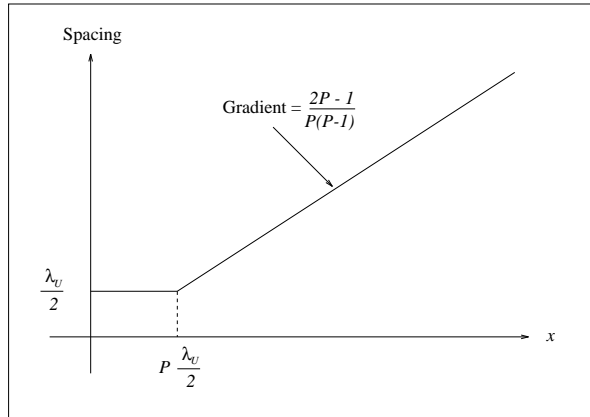

Figure 3: Maximum permissible spacing of a single-sided one dimensional array to avoid spatial aliasing.

broadband array which avoids spatial aliasing, this spacing relation represents the optimal sensor positioning function.

Using this optimal spacing relation, the minimum number of sensors required to implement a broadband array over a desired frequency range is

$$N = (P+1) + \left\lceil \frac{\log\left(\frac{f_U}{f_L}\right)}{\log\left(\frac{P}{P-1}\right)} \right\rceil \tag{10}$$

where $\lceil \cdot \rceil$ denotes the ceiling function.

A similar spacing function was developed in [19], although fewer sensors were required by allowing

controlled grating lobes to appear in the array beam pattern (by adding a constraint on the aperture distribution). By using slightly more elements, our spacing function avoids any effects due to spatial aliasing, does not add any constraint to the aperture distribution, and provides only a maximum constraint on the spacings.

A final set of remarks is in order. The same guidelines apply for our broadband design as in the case of a single frequency array design: (i) P, the aperture size in half-wavelengths, is chosen to be sufficiently large to achieve the desired beam shape properties usually expressed in terms of the main beam width; and (ii) the aperture distribution or sensitivity distribution is a slowly varying function of distance along the array. This latter condition is compatible with some assumptions we made earlier regarding the use of numerical integration to approximate the ideal broadband continuous FI sensor response.

3.5 Filter Implementation

The broadband theory we have developed has assumed positive frequency only. Conventionally in filter theory frequency is represented by both positive and negative frequencies. In this case, to implement a filter with a real time domain impulse response (i.e., a real filter) requires that the frequency response of the filter be Hermitian symmetric. Let the real filter frequency response used to implement $H_x(f)$ be denoted by $\widetilde{H}_x(f)$ and be defined by

$$\widetilde{H}_x(f) = \begin{cases} \frac{1}{2}H_x(f), & \text{for } f > 0 \\ \frac{1}{2}H_x^*(-f), & \text{for } f < 0 \end{cases} \tag{11}$$

where $\widetilde{H}_x : \mathbb{R} \to \mathbb{C}$, and H_x^* denotes the complex conjugate of H_x. The relation between an example single-sided aperture distribution and the required real filter response is shown in Fig. 4.

3.6 Double-sided Aperture Distributions

A double-sided aperture distribution requires two distinct primary filter responses as can be readily gleaned from Theorem 5. The real filter responses used to implement these primary filters are given by applying (11). An example of a double sided aperture distribution and the corresponding real filter responses are shown in Fig. 5. Note that this aperture will give the same beam pattern as the equivalent single-sided aperture shown in Fig. 4. As can be seen from this figure, although the choice of array origin has no effect on the array beam pattern, the
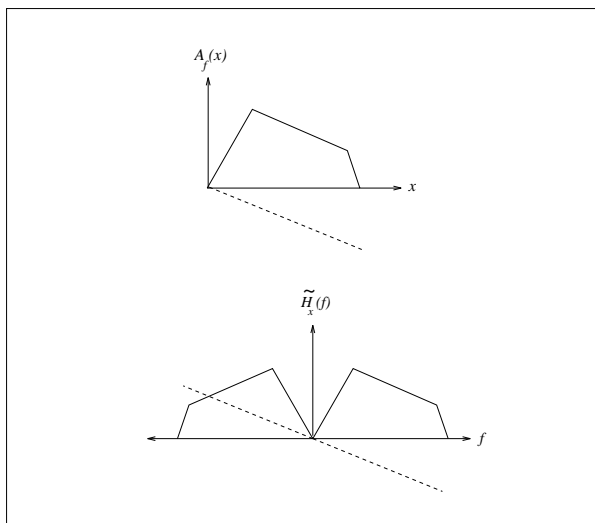
Figure 4: Single-sided aperture distribution $A_f(x)$ and corresponding real filter response $\widetilde{H}_x(f)$. The solid line is the magnitude, the dashed line indicates the phase.
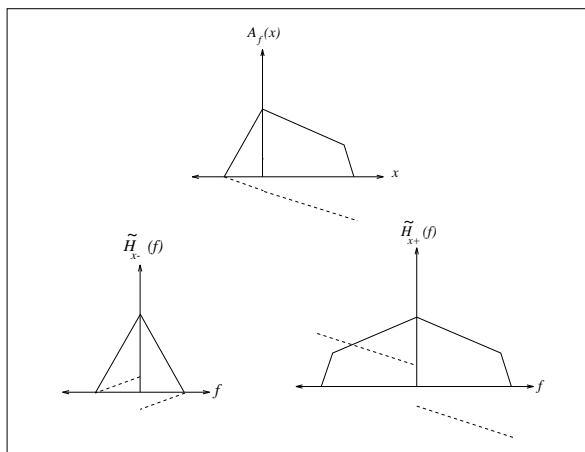

Figure 5: Double-sided aperture distribution and corresponding real filter responses. The solid line is the magnitude, the dashed line indicates the phase. $\widetilde{H}_{x+}(f)$ is the real filter response for sensors located at $x > 0$, and $\widetilde{H}_{x-}(f)$ is the real filter response for sensors located at $x < 0$.

position of the origin can have a significant effect on the complexity of filter implementation.

3.7 Implementation of a Two Dimensional Array

Theorem 5 (the filter dilation theorem) gives no guarantee that the primary filter responses for two and three dimensional arrays will exhibit a dilation property. This is not a restriction on being able to build a broadband array, it simply restricts the appearance of self-similarities which may be exploited to simplify the array design. Thus generally a two dimensional array corresponds to approximating a double integral in the spirit of (7) for the one dimensional case. However, there are at least two special cases which will produce primary filters which have the same frequency response at more than one position within the array. (These cases are discussed for the two dimensional case and are easily extended to the three dimensional case.) These special cases are illustrated in Fig. 6.

Separable Aperture Distributions

If the aperture distribution is separable into the product of two one dimensional aperture distributions, i.e.,

$$A_f(x_1, x_2) = A'_f(x_1)A''_f(x_2)$$

then the primary filter responses are also separable meaning that at any point $[x_1\ x_2]$

$$H_{x_1, x_2}(f) = H'_{x_1}(f)H''_{x_2}(f).$$

Hence at least two, and at most four, different filter responses are required (depending on whether the component one dimensional arrays are one or two sided). Note that this class of aperture sensitivities requires that

$$G(x_1 f, x_2 f) = G_1(x_1 f)G_2(x_2 f).$$

Discrete Sensor Radial Pattern

If the array elements are arranged in radial patterns from the origin, then each of these radial lines is equivalent to a linear one dimensional array, and thus each of the primary filters on the radial line is given by a dilation of the same function. For an array with N elements which is arranged into $k < (N-1)$ different radial lines, there will be only k distinct filter responses, as opposed to $(N-1)$. This is true for any arbitrary two dimensional aperture distribution. In the sense that this does not restrict the aperture distribution (and thus the desired beam pattern) then this type of sensor location pattern is recommended for any design. Naturally if the desired aperture distribution further satisfies a radially symmetric pattern the design is further simplified and only a single filter shape is required and the discrete sensors need not be restricted to radial lines.

4 Design Example

As a demonstration of our broadband theory we will introduce an example of a typical practical broad-
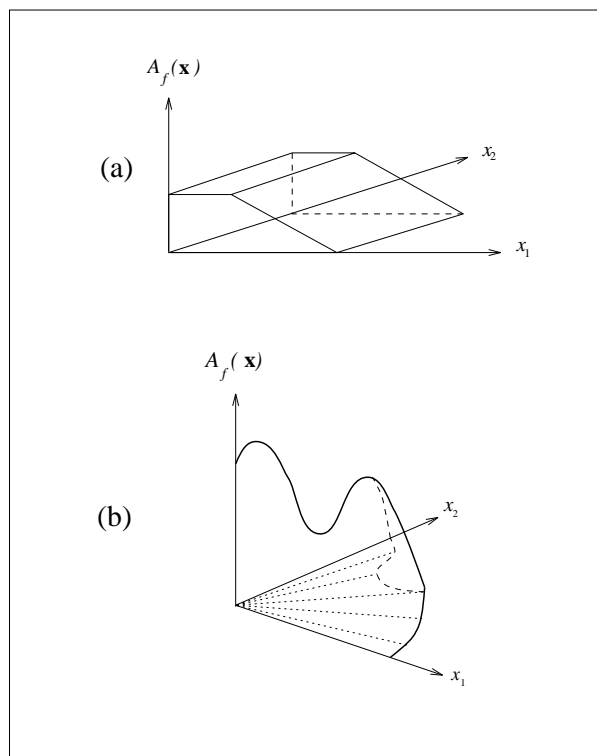
Figure 6: Examples of two dimensional apertures which produce primary filters having the same response at more than one location. a) Aperture is separable into two one dimensional aperture distributions, i.e., $A_f(x_1, x_2) = A'_f(x_1)A''_f(x_2)$. It requires only two different filters and allows arbitrary sensor locations. b) Sensors are positioned in radial lines (denoted by dotted lines). It requires a different primary filter response for each radial line of sensors. Any arbitrary aperture distribution function may be used.

band sensor array design. We will consider the design of an array with a single-sided uniform aperture, although we stress that our design method is not restricted to uniform apertures. The aperture size is $P = 5$ half-wavelengths and the array is intended to have a FI beam pattern over a 10:1 frequency range. It is not necessary to choose numerical values for the frequency range, but rather, we will introduce nondimensional variables by scaling all array dimensions by λ_U.

From (10) it follows that a minimum of $N = 17$ sensors are required to avoid spatial aliasing. The maximum spacing relation (9) yields the sensor locations given in Table I. These sensor locations have been made dimensionless by expressing them in terms of λ_U. (Using a bandwidth suitable for speech with f_L=300 Hz and f_U=3000 Hz results in an array that is approximately 2.7 metres long.)

For a uniform aperture distribution, Theorem 3 implies the use of primary filters having ideal low-pass filter characteristics. In order to demonstrate a practical design we have chosen to implement the primary filters with causal 8th order Butterworth low-pass filters (possessing both magnitude and phase components). This will result in an aperture distribution having the same Butterworth shape. The magnitude and phase of the practical aperture distribution are shown in Fig. 7 (solid curves) along with the ideal zero-phase uniform aperture distribution (dashed curves); the spatial variable is expressed in terms of half wavelength. Since a Butterworth low-pass filter is not strictly bandlimited, it follows from Theorem 3 that the resultant aperture distribution will not have strictly finite support; the significance of this statement is made apparent later.
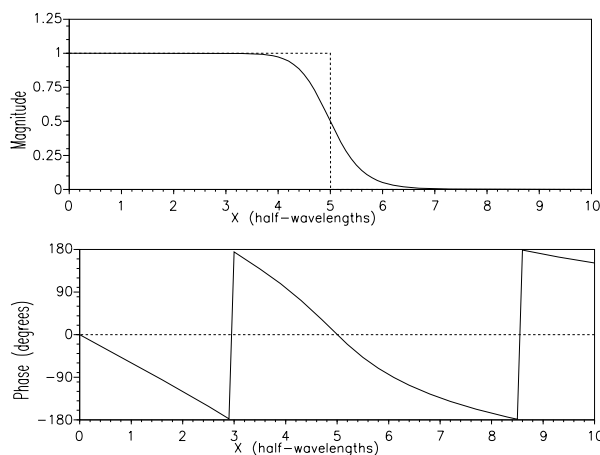

Figure 7: Aperture distribution used in the example FI array (solid curve) and an ideal uniform aperture distribution (dashed curve). The spatial variable has been normalized in terms of half-wavelength.

The array response produced by the given aperture distribution is shown in Fig. 8 along with the pattern that would be produced by an ideal uniform aperture. The effect of the nonzero phase component of the aperture distribution is apparent in this diagram. The negative slope of the phase is equivalent to delay steering, thus resulting in the main beam being offset from $\theta = 0°$; this effect could be nullified by use of appropriate delays across the array. The asymmetric sidelobes are due to the phase non-linearity.

Applying the trapezoidal approximation method described in Section 3.3 results in the frequency invariant beam pattern shown in Fig. 9 in which the array spatial response is displayed as a function of frequency over the entire design frequency band.

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
x_i/λ_U	0	0.5	1	1.5	2	2.5	3.1	3.9	4.9	6.1	7.6	9.5	11.9	14.9	18.6	23.3	25

Table I: Sensor locations for example FI array (given in terms of the upper design wavelength λ_U).
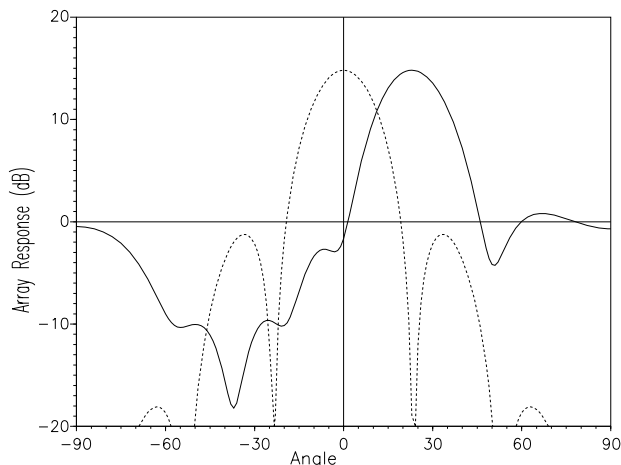

Figure 8: Array responses produced by the aperture distribution used in the example FI array (solid curve) and an ideal uniform aperture distribution (dashed curve). The patterns are calculated at $f = f_U$.

Figure 9: Array response of example FI array over the entire design frequency range. Frequencies have been normalized and are expressed in terms of f_L.

Frequency has been expressed as multiples of f_L. The array beam pattern is remarkably close to being frequency independent with negligible variation in main beam magnitude or beam width. Slight ripple is evident in the sidelobes: it can be seen that the peaks of the sidelobe ripple correspond to the cutoff frequencies of the sensors given by

$$f_i = \frac{Pc}{2x_i}, \quad i \in \{0, 1, \cdots, N-1\}.$$

The peak response of the array as a function of frequency is shown in Fig. 10. The variation in peak response at frequencies close to f_L is due to the primary filters not being strictly bandlimited, thus not placing a finite support constraint on the aperture distribution. Because of the finite size of the array, a portion of the aperture distribution is not realized. This effect is most pronounced at frequencies close to f_L where a significant portion of the aperture distribution is discarded, resulting in a slight difference in beam pattern in the lowest portion of the design frequency band. There are several methods which could be used to alleviate this inconsistency in the beam pattern at low frequencies.

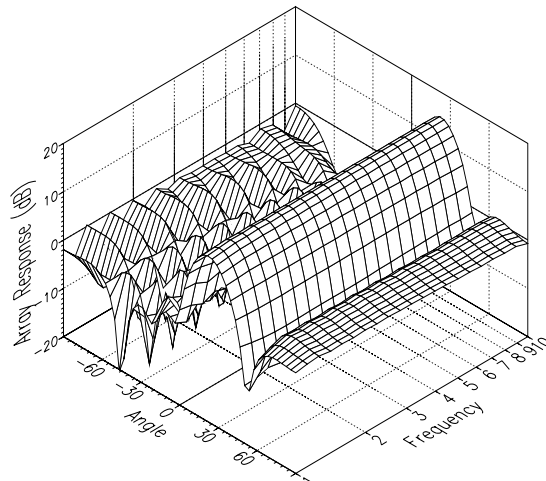- The primary filters could be made to be strictly bandlimited, thus producing an aperture distribution which has finite support. (This is not physically realizable).

- The cutoff frequencies of the primary filters could be reduced so that a negligible portion of the aperture distribution was discarded at frequencies close to f_L. This is equivalent to lengthening the array to produce the same result.

- The secondary filter, which depends only on frequency, could be modified such that the peak main beam level was equalized. This method attempts to compensate for the loss of a portion of the aperture distribution at low frequencies by weighting the remainder of the aperture more strongly. This demonstrates an important practical consideration of our design method: a simple filter can be used for each of the N primary filters, and any ripple on the main beam level can then be removed by modification of the single secondary filter response.
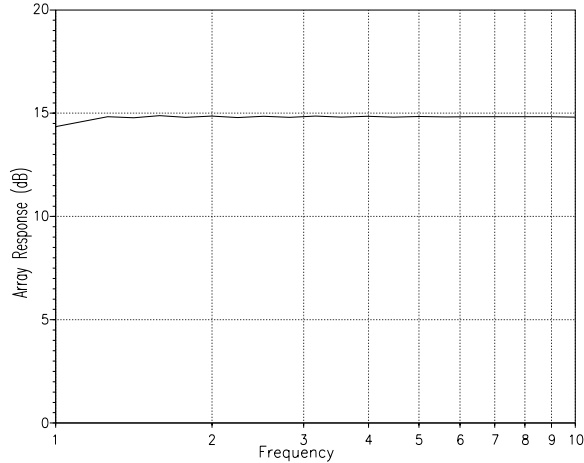
Figure 10: Peak array response of example FI array as a function of frequency. Frequencies have been normalized and are expressed in terms of f_L.

5   Frequency Variant Arrays

5.1   Theory

The theory for broadband arrays having frequency invariant beam patterns has been developed. We will now show that these arrays are only a subset of a more general class of arrays which we shall refer to as *alpha arrays*. The frequency variation of the beam pattern of an alpha array can be controlled, and the beam pattern is found to be a function of $f^{1-\alpha}$, where $\alpha \in [0,1]$. Thus, for $\alpha = 0$ the beam pattern varies directly with frequency, corresponding to a conventional single frequency array operated over a range of frequencies; for $\alpha = 1$ the beam pattern is frequency invariant.

Theorem 6 (Alpha Array) *Let the output of a D dimensional continuous sensor be given by*

$$Z_f = \int_{\mathbb{R}^D} S(\mathbf{x}, f) \rho(\mathbf{x}, f) \, d\mathbf{x}$$

where $D \in \{1, 2, 3\}$, $S : \mathbb{R}^D \times \mathbb{R}^+ \to \mathbb{C}$ is the signal received at a point $\mathbf{x}$ on the sensor for a frequency f, and $\rho : \mathbb{R}^D \times \mathbb{R}^+$ is the sensitivity distribution. The far-field beam pattern of the sensor is a function of $f^{1-\alpha}$ if

$$\rho(\mathbf{x}, f) = f^{D\alpha} G(\mathbf{x} f^\alpha), \quad \forall f > 0$$

where $\alpha \in [0,1]$ and $G : \mathbb{R}^D \to \mathbb{C}$ is an arbitrary absolutely integrable complex valued function.

This theorem can be easily proven using similar arguments to those used in Sections 2.2 and 2.3.

5.2   Properties of Alpha Arrays

Without loss of generality, properties of alpha arrays will only be given for single-sided one dimensional sensors with the array origin at $x = 0$.

1. Let L_1 be the length of the active array at frequency f_1, and similarly for L_2 and f_2. The ratio of active array lengths is given by

$$\frac{L_1}{L_2} = \left(\frac{f_2}{f_1} \right)^\alpha. \tag{12}$$

2. Let the active aperture size be P_1 half-wavelengths at frequency f_1, and similarly for P_2 and f_2. The ratio of active aperture sizes is given by

$$\frac{P_1}{P_2} = \left(\frac{f_2}{f_1} \right)^{\alpha - 1}. \tag{13}$$

3. If we represent $G(xf^\alpha)$ by $A_f(x)$ and $H_x(f)$ (as in the case of FI arrays), then Theorem 5 becomes

$$H_{\gamma x}(f) = H_x(\gamma^{1/\alpha} f) \tag{14}$$

and a similar relation for the aperture distribution is

$$A_{\gamma f}(x) = A_f(\gamma^\alpha x). \tag{15}$$

Thus the filter responses (and aperture distributions) are related by a dilation property (as was the case for FI arrays), but the dilation is not a linear function of the position of the sensor when $\alpha \neq 1$.

5.3   Design of Alpha Arrays

The method of approximating a continuous alpha sensor is identical to that for a FI sensor (see Sections 3.2 and 3.3); only the spacing function requires further comment. The sensor positioning function for an alpha array with a single-sided aperture and the origin at $x = 0$ is given by

$$x_i = \left(\frac{P_i}{P_i - 1} \right) x_{i-1} \tag{16}$$

where

$$P_i = P_{i-1} \left(\frac{f_{i-1}}{f_i} \right)^{\alpha - 1}$$

$$= P_{i-1} \left(\frac{x_i}{x_{i-1}} \right)^{1 - 1/\alpha}$$

(c.f. (8) for frequency invariant arrays).

It is difficult to solve the above equations analytically, so the following recursive procedure is used to determine the sensor locations.

1. Assume that the upper and lower design frequencies (f_U and f_L), α, and the aperture size in half wavelengths at the upper design frequency (P_U) are given. The total array len''' is given by

$$x_N = \frac{P_L c}{2 f_L}$$

where c is the speed of wave propagation ' P_L is the aperture size in half wavelength; f_L (obtained from (13)).

2. Repeat

$$x_i = x_{i+1} - \frac{c}{2 f_{i+1}}$$

$$f_i = \left(\frac{x_N}{x_i} \right)^{\frac{1}{\alpha}} f_N$$

until $x_i \leq \frac{P_U c}{2 f_U}$.

3. Divide the remainder of the array into P_U ec sections (with spacing $\frac{c}{2 f_U}$).

For $\alpha = 0$ this procedure results in an equi-spaced array designed for operation at f_U, but used over the frequency range f_L to f_U. Hence, the beam pattern will spread out for frequencies less than f_U, as found when a conventional single frequency array is operated at frequencies below the design frequency. For $\alpha = 1$ this procedure results in a FI array with sensor locations similar to those given in (9). (The slight difference in sensor locations occurs because sensors are placed from the high frequency end of the array in the FI design procedure, whereas sensors are placed from the low frequency end in the alpha array design procedure.)

The importance of the alpha array is that by allowing controlled frequency variation into the beam pattern, less sensors are required than for a corresponding FI array. This is made apparent in the following example.

5.4 Alpha Array Example

To demonstrate the use of alpha arrays, a simple design example is presented. The design is for $\alpha = 0.75$, covers a frequency range of 10:1, and has an aperture size of $P_U = 5$ half-wavelengths at the upper design frequency. Again we are using causal 8th order Butterworth filters to approximate an ideal uniform aperture distribution. The beam pattern of this design is shown in Fig. 11. This figure should be compared with Fig. 9 which shows the beam pattern of a FI array (i.e. $\alpha = 1$) with $P = 5$, designed for the same frequency range. The array with $\alpha = 0.75$ has a total length of $14.1\lambda_U$

and uses 12 elements, compared with the FI array (with $\alpha = 1$) which has a total length of $25\lambda_U$ and uses 17 elements.
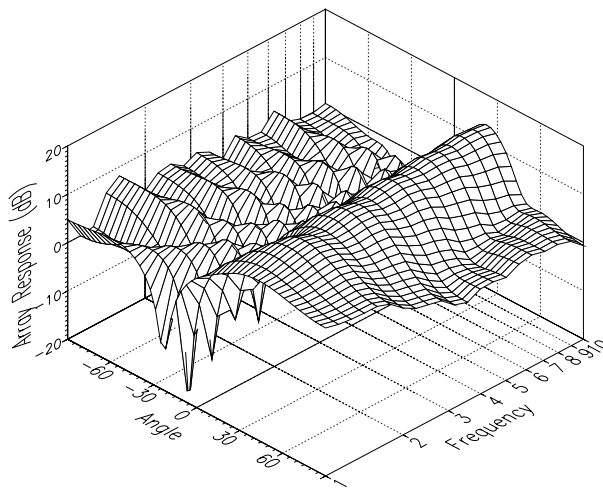

Figure 11: Array response of example alpha array (with $\alpha = 0.75$ and $P_U = 5$) over the entire design frequency range. Frequencies have been normalized and are expressed in terms of f_L.

6   Conclusion

We have presented the theory and design methodology for broadband sensor arrays in which the far-field beam pattern is constant over a desired frequency range. A continuously distributed sensor was used to derive a FI beam pattern property which is valid for one, two, and three dimensional arrays. The array can then be formed by approximating this continuous sensor with a finite set of discrete sensors. The approximation method is arbitrary, although a simple approximation corresponding to the trapezoidal integration method was discussed.

It was shown that the frequency response of the filter applied to the output of each sensor can be factored into two components: (i) a primary filter response which is related (both in magnitude and phase) to a slice of the desired aperture distribution; and (ii) a secondary filter which is independent of the sensor and depends only on the dimension of the array. These results imply that in the case of a linear array (and for suitable sensor geometries in two and three dimensional arrays) the primary filters are related to each other by a frequency dilation.

An example based on 8th order Butterworth filters was given to illustrate that these theoretical in-

vestigations lead to practical and conceptually simple designs.

Finally, the theory for a more general class of arrays in which the frequency dependence of the beam pattern can be controlled was presented. This theory served to show the relationship between our broadband FI arrays and conventional single frequency designs.

Acknowledgments

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Appendix A — Proof of Theorem 2

Assume that a frequency invariant beam pattern $b(\theta), \theta \in (-\pi/2, \pi/2)$ is given. We can rewrite (2) as

$$B(s) = \int_{-\infty}^{\infty} \rho\left(\frac{y}{f}, f\right) e^{-j2\pi sy} \, f^{-1} dy, \quad s \in (-1/c, 1/c) \tag{A.1}$$

with the change of variables $s = c^{-1}\sin\theta$ and $y = xf$.

Since $B(s)$ is frequency invariant, the integrand must also be frequency invariant. Therefore define $G(y) = f^{-1}\rho(y/f, f)$, for some function $G(\cdot)$. Equation (A.1) can now be rewritten as

$$B(s) = \int_{-\infty}^{\infty} G(y) e^{-j2\pi sy} dy = \mathcal{F}\{G(y)\} \tag{A.2}$$

where $\mathcal{F}\{\cdot\}$ represents the Fourier transform. It is now necessary to find to what extent $G(y)$ is determined from $B(s)$, which is only specified for $s \in (-1/c, 1/c)$.

Consider a function $H(f)$ specified only for $f \in (-F, F)$. This has an inverse Fourier transform of

$$h(t) = h_1(t) + h_2(t)$$

where

$$\mathcal{F}\{h_1(t)\} = \begin{cases} H(f), & f \in (-F, F) \\ 0, & \text{otherwise} \end{cases}$$

and

$$\mathcal{F}\{h_2(t)\} = \begin{cases} 0, & f \in (-F, F) \\ A(f), & \text{otherwise} \end{cases}$$

where $A(\cdot)$ is an arbitrary function. Hence, any high frequency perturbation in the function $h(t)$ will not produce any effect on the function $H(f), f \in (-F, F)$.

By analogy, $G(y)$ has a Fourier transform, $\Gamma(s)$, satisfying

$$\Gamma(s) = \mathcal{F}\{G(y)\} = \begin{cases} B(s), & s \in (-1/c, 1/c) \\ A(s), & \text{otherwise.} \end{cases} \tag{A.3}$$

By Plancherel's Theorem [24], the function $G(\cdot)$ is uniquely determined from $\Gamma(\cdot)$ if $B(\cdot)$ and $A(\cdot)$ are both square-integrable functions, and

$$A\left(\frac{(-1)^i}{c}\right) = \lim_{s \to \frac{(-1)^i}{c}} B(s)$$

for $i = 0, 1$.