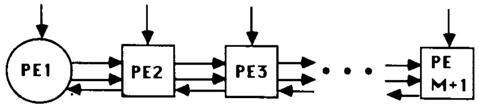

Fig. 5. The linear array for parallel implementations without feedback.

upper triangular matrix R. As long as the first row of R is known, all the subsequent row can be generated recursively, and this is also the basic principle of the proposed fast algorithm.

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The Relationship Between Instantaneous Frequency and Time-Frequency Representations

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***Abstract*—We give the relationship between instantaneous frequency estimation via the derivative of the phase of the analytic signal and the first moment of general time-frequency representations from Cohen's class in both the continuous and discrete-time domains. Many researchers have applied the standard linear definition of first moment to discrete-time time-frequency representations although this leads to biased instantaneous frequency estimators with high variance; we show that periodic (circular) definitions of moments must be used to account for the periodization of the frequency variable due to sampling.**

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I. Introduction

Several authors [3], [9] have investigated the possibility of using the first moments of time-frequency representations with respect to the frequency variable as estimators of instantaneous frequency. This correspondence derives the relationships between instantaneous frequency and the first moments of the general class of time-frequency representations for both continuous and discrete-time signals.

II. Continuous-Time Estimation

Consider a frequency modulated sinusoidal signal of the form $x(t) = a_c(t) \cos \phi(t)$, where a_c represents the envelope function and ϕ is the cumulative phase of the signal. We define the instantaneous frequency of this signal by

$$f_i(t) = \frac{1}{2\pi} \frac{d\phi(t)}{dt}. \tag{1}$$

If x is sufficiently narrow band, a good estimate of the cumulative phase reduced modulo 2π may be obtained from the phase of the analytic signal defined as follows:

Definition 1: Analytic Signal: The analytic signal z associated with the real signal x is defined by $z = A[x]$, where $A[x] = x + jH[x]$ is the operator which forms the analytic signal and $H[\,]$ is the Hilbert transform defined by

$$H[x](t) = \frac{1}{\pi} \lim_{\delta \to 0} \left[\int_{-\infty}^{-\delta} \frac{x(t-\zeta)}{\zeta} d\zeta + \int_{\delta}^{+\infty} \frac{x(t-\zeta)}{\zeta} d\zeta \right].$$

We use the derivative of the phase of the analytic signal to define the following instantaneous frequency estimator.

Definition 2: Analytic Derivative Estimator: Let $z = A[x]$, where x is a real signal. Then the instantaneous frequency of x at time t is estimated by

$$\hat{f}_i^a(t) = \frac{1}{2\pi} \left(\left(\frac{d}{dt} \right) \right)_{2\pi} \arg\ [z(t)]$$
$$= \lim_{\delta \to 0} \frac{1}{4\pi\delta} ((\arg [z(t+\delta)] - \arg [z(t-\delta)]))_{2\pi} \tag{2}$$

where $(())_{2\pi}$ denotes reduction modulo 2π and $((d/dt))_{2\pi}$ denotes the appropriate differentiation of a quantity which is reduced modulo 2π as shown above.

The spectrogram (or magnitude-squared short-time Fourier transform) and time-frequency distributions such as the Wigner-Ville, Born-Jordan-Cohen, Margenau-Hill-Rihaczek, and Choi-Williams exponential distributions can all be examined within the framework of Cohen's general class of time-frequency representations [2].

Definition 3: Cohen's Class of Time-Frequency Representations for Analytic Signals: Each member of this class of *bilinear* representations can be written in the form

$$S(t, f) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{j2\pi(\nu\mu - \tau f - \nu t)} f(\nu, \tau) z\left(\mu + \frac{\tau}{2}\right) \cdot z^*\left(\mu - \frac{\tau}{2}\right) d\mu\, d\tau\, d\nu \quad (3)$$

where $z = A[x]$ and x is a real signal.

It is useful to reformulate Cohen's class as the Fourier transform of a smoothed autocovariance function as follows [4], [5]:

$$S(t, f) = F_{\tau \to f}\left[F^{-1}_{\nu \to t}[f(-\nu, \tau)] \underset{(t)}{*} z\left(t + \frac{\tau}{2}\right) z^*\left(t - \frac{\tau}{2}\right)\right] = F_{\tau \to f}[B(-t, \tau) \underset{(t)}{*} C(t, \tau)] \quad (4)$$

where $F_{\tau \to t}$ denotes Fourier transformation from the lag variable τ to the time variable t, and $\underset{(t)}{*}$ denotes convolution in the time parameter. The bilinear product function C can be considered as an estimate of the autocovariance function of the signal generating process given by

$$C(t, \tau) = z\left(t + \frac{\tau}{2}\right) z^*\left(t - \frac{\tau}{2}\right)$$

and the time-lag kernel function B is given by

$$B(t, \tau) = F^{-1}_{\nu \to t}[f(\nu, \tau)]$$

where f is the same function that appears in (3).

A. Estimation via Representations

The instantaneous frequency estimator, $\hat{f}_i^r(t)$, obtained from the normalized first moment of a time-frequency representation is defined by

$$\hat{f}_i^r(t) = \frac{\int_{-\infty}^{+\infty} f S(t, f)\, df}{\int_{-\infty}^{+\infty} S(t, f)\, df}. \quad (5)$$

While it is well known that $\hat{f}_i^r(t)$ is identical to $\hat{f}_i^a(t)$ for some representations, the following more general theorem appears to be a new result.

Theorem 1: If the time-lag kernel function B is symmetric about $\tau = 0$, then $\hat{f}_i^r(t)$ is related to the analytic derivative estimator $\hat{f}_i^a$ by

$$\hat{f}_i^r(t) = \frac{B(-t, 0) \underset{(t)}{*} |z(t)|^2 \hat{f}_i^a(t)}{B(-t, 0) \underset{(t)}{*} |z(t)|^2}. \quad (6)$$

Proof: The analytic signal z is given by

$$z(t) = A[x](t) = |z(t)| e^{j\arg[z(t)]}. \quad (7)$$

Now use (4)

$$F^{-1}_{f \to \tau}[S(t, f)] = \int_{-\infty}^{+\infty} S(t, f) e^{j2\pi f\tau}\, df = B(-t, \tau) \underset{(t)}{*} C(t, \tau). \quad (8)$$

Evaluating (8) at $\tau = 0$ gives

$$\int_{-\infty}^{+\infty} S(t, f)\, df = B(-t, 0) \underset{(t)}{*} C(t, 0) = B(-t, 0) \underset{(t)}{*} |z(t)|^2 \quad (9)$$

which is the denominator of (6). Similarly

$$\int_{-\infty}^{+\infty} f S(t, f) e^{j2\pi f\tau}\, df = \frac{1}{j2\pi} \int_{-\infty}^{+\infty} j2\pi f S(t, f) e^{j2\pi f\tau}\, df = \frac{1}{j2\pi} \frac{\partial}{\partial \tau} [B(-t, \tau) \underset{(t)}{*} C(t, \tau)] \quad (10)$$

With B symmetric about $\tau = 0$ by assumption, it can be shown [5] that

$$\int_{-\infty}^{+\infty} f S(t, f) e^{j2\pi f\tau}\, df = B(-t, 0) \underset{(t)}{*} |z(t)|^2 \frac{1}{2\pi} \left(\left(\frac{d}{dt}\right)\right)_{2\pi} \arg\,[z(t)] \quad (11)$$

Combining (11), (9), and (2) we have

$$\hat{f}_i^r(t) = \frac{B(-t, 0) \underset{(t)}{*} |z(t)|^2 \frac{1}{2\pi} \left(\left(\frac{d}{dt}\right)\right)_{2\pi} \arg\,[z(t)]}{B(-t, 0) \underset{(t)}{*} |z(t)|^2} = \frac{B(-t, 0) \underset{(t)}{*} |z(t)|^2 \hat{f}_i^a(t)}{B(-t, 0) \underset{(t)}{*} |z(t)|^2} \quad (12)$$

as required. □

Although this result only holds for TFR's with symmetric kernel functions, this class is by far the most interesting since a time-lag kernel function must be real and symmetric about $\tau = 0$ to ensure that the corresponding representation is real and preserves the frequency support of the signal.

Since $B(t, 0) = \delta(t)$ for the Wigner–Ville, Born–Jordan–Cohen, and Choi–Williams exponential distributions, $\hat{f}_i^r(t) = \hat{f}_i^a(t)$ for all of these distributions. The estimator derived from the normalized first moment of more general time-frequency representations, such as the spectrogram, is closely related to the smoothed analytic derivative estimator which we define as follows:

Definition 4: Smoothed Analytic Derivative Estimator: Let $\hat{f}_i^a(t)$ be the analytic derivative estimator calculated from the real signal x and let $h: \mathbb{R} \mapsto \mathbb{R}$, be a smoothing function. Then the smoothed analytic derivative estimator is defined by

$$\hat{f}_i^s(t) = h(t) \underset{(t)}{*} \hat{f}_i^a(t). \quad (13)$$

If we are dealing with a monocomponent, constant amplitude FM signal then Theorem 1 reduces to

$$\hat{f}_i^r(t) = \bar{B}(-t, 0) \underset{(t)}{*} \hat{f}_i^a(t)$$

where

$$\bar{B}(t, 0) = \frac{B(t, 0)}{\int_{-\infty}^{+\infty} B(t, 0)\, dt}$$

and so the estimator yielded by the normalized first moment of a time-frequency representation with respect to the frequency variable is identical to a smoothed analytic derivative estimator; the smoothing function used to calculate the smoothed analytic derivative estimator is given by the time-lag kernel cross section at $\tau = 0$ normalized to have unit area. In the case of the spectrogram with a rectangular data window of duration T, the smoothing window is also rectangular with duration T and $\hat{f}_i^s$ will be a low-pass filtered version of $\hat{f}_i^a$.

III. Discrete-Time Estimation

We use the following definitions for our discrete-time analysis:

Definition 5: Discrete-Time Analytic Signal: The discrete-time analytic signal z associated with the real discrete-time signal x is defined by $z = A[x] = x + jH[x]$, where $A[\,]$ is the linear operator which forms the analytic signal and $H[\,]$ is the discrete-time Hilbert transform defined by

$$H[x](n) = \sum_{\substack{-\infty \\ m \text{ odd}}}^{+\infty} \frac{2x(n-m)}{m\pi}.$$

We use the central finite difference of the phase of z as the analog of the analytic derivative estimator for a discrete time signal.

Definition 6: Central Finite Difference Estimator: Let $z = A[x]$ where x is a real discrete-time signal. Then the instantaneous frequency of x at sample n is estimated by

$$\hat{f}_i^c(n) = \frac{f_s}{4\pi}\left((\arg\,[z(n+1)] - \arg\,[z(n-1)])\right)_{2\pi} \tag{14}$$

where f_s is the sampling frequency.

We reformulate the smoothed covariance form of Cohen's class given by (4) in the discrete time-lag domain as follows [4], [5]:

$$S(n, k) = 2F_{m \to k}\,[B(-n, m) \underset{(n)}{*} C(n, m)] \tag{15}$$

where

$B(n, m) = NF^{-1}_{l \to n}\,[f(l, m)]$ is the discrete-time time-lag kernel function,

$C(n, m) = z(n + m)z^*(n - m)$ is the discrete-time bilinear product, and

$\underset{(n)}{*}$ = represents a linear convolution in the n index.

A. The Circular Nature of Discrete-Time Quantities

To determine the first moment of a discrete-time time-frequency representation, we must use the periodic definition of first moment [5], [7] which we have adapted from Mardia [8]. Claasen and Mecklenbräuker commented on the need for periodic definitions in [1].

Definition 7: Periodic First Moment: Let $\theta(k)$ be a discrete function which is periodic in k with period M. Then the first periodic moment i_1^p is defined by

$$i_1^p = \frac{M}{2\pi}\left(\left(\arg\left[\sum_{k=0}^{M-1} \theta(k)\, e^{j2\pi k/M}\right]\right)\right)_{2\pi} \tag{16}$$

where $j = \sqrt{-1}$.

It is convenient to define the modulo-λ convolution operation from [6], [7] since it arises naturally in the proof of the next theorem.

Definition 8: Modulo-λ Convolution: Let the sequence $\tilde{f}$ be of the form $\tilde{f}(n) = ((f(n)))_\lambda$, $f: \mathbb{Z} \mapsto \mathbb{R}$ and $\lambda \in \mathbb{R}$ be a sequence of circular variables, and $h: \mathbb{Z} \mapsto \mathbb{R}$ be a window function. Then the modulo-λ convolution of $\tilde{f}$ and h is defined by

$$\tilde{f}(n)\ ((*))_\lambda\ h(n) = \frac{\lambda}{2\pi}\left(\left(\arg\left[\sum_{p=-Q}^{Q} h(n-p)\, e^{j2\pi\tilde{f}(p)/\lambda}\right]\right)\right)_{2\pi}. \tag{17}$$

B. Estimation via Discrete-Time Representations

Theorem 2: The estimator yielded by the periodic first moment of a discrete-time time-frequency representation $\hat{f}_i^r$ is related to the central finite difference estimator $\hat{f}_i^c$ by

$$\hat{f}_i^r(n) = B(-n, 1)\ ((*))_{f_s/2}\ |z(n+1)z^*(n-1)|\,\hat{f}_i^c(n) \tag{18}$$

Proof: Using (16) and (15), the instantaneous frequency estimator yielded by the periodic first moment of a member of Cohen's class is given by

$$\begin{aligned}\hat{f}_i^r(n) &= \frac{f_s}{4\pi}\left(\left(\arg\left[\sum_{p=-L}^{L} B(p-n, 1)z(p+1) \cdot z^*(p-1)\right]\right)\right)_{2\pi} \\ &= \frac{f_s}{4\pi}\left(\left(\arg\left[\sum_{p=-L}^{L} B(p-n, 1)|z(p+1) \cdot z^*(p-1)|\,e^{j4\pi\hat{f}_i^c(p)/f_s}\right]\right)\right)_{2\pi}\end{aligned} \tag{19}$$

where $L = (N - 1)/2$. This is just a modulo-$f_s/2$ convolution of the form given by (18).

□

Comparing Theorem 2 with Theorem 1 we see that modulo-λ convolution of the discrete-time frequency estimates corresponds to linear convolution of the continuous-time estimates. This is a sensible result since standard digital filtering with low-pass FIR filters on the central finite difference estimate sequence produces strange effects when estimate values approach the circular domain wrapping points [7]; a problem overcome by applying the filter using the modulo-λ convolution operation instead of conventional linear convolution. Thus it would appear that modulo-λ convolution is the natural form of convolution for circular variables.

It is easily shown that $B(n, 1) = \delta(n)$ for the Wigner–Ville distribution and so $\hat{f}_i^r(n) = \hat{f}_i^c(n)$ in this case as expected. The behavior of the first moment of other discrete-time representations is related to the smoothed central finite difference estimators defined as follows.

Definition 9: Smoothed Central Finite Difference Estimators: Let $\hat{f}_i^c$ be the central finite difference estimator calculated from the real signal x and let $h: \mathbb{Z} \mapsto \mathbb{R}$, be a smoothing function of length P, presumed odd. Then the smoothed central finite difference estimator is defined by

$$\hat{f}_i^s(n) = h(n)\ ((*))_{f_s/2}\ \hat{f}_i^c(n). \tag{20}$$

If we have a monocomponent, constant amplitude FM signal then (18) can be simplified to

$$\hat{f}_i^r(n) = B(-n, 1)\ ((*))_{f_s/2}\ \hat{f}_i^c(n) \tag{21}$$

and so, in this case, the periodic first moment of a discrete-time time-frequency representation is equivalent to a smoothed central finite different estimator. The smoothing function used to calculate the estimator is given by the discrete-time time-lag kernel cross section at lag $m = 1$; for a spectrogram with a rectangular data window of length M, the corresponding smoothing window will also be rectangular and of length $M - 2$. The analysis of the distribution of $\hat{f}_i^s$ is given in [5].

IV. Conclusions

We conclude that there is no point in using the first moment of discrete-time time-frequency representations as instantaneous frequency estimators, since it is much simpler to calculate the corresponding smoothed central finite difference estimator directly from the analytic signal. These results highlight the danger of formulating relationships in the continuous-time domain and then expecting them to hold for discrete-time signals.

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Asymptotic Behavior of Maximum Likelihood Estimates of Superimposed Exponential Signals

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***Abstract*—In this correspondence are derived strong consistency and asymptotic normality of the maximum likelihood estimates (MLE's) of the unknown parameters $(\omega_1, \cdots, \omega_p)$, $(\alpha_1, \cdots, \alpha_p)$, and σ^2 in the superimposed exponential model for signals**

$$Y_t = \sum_{k=1}^{p} \alpha_k \exp(it\omega_k) + e_t, \qquad t = 0, 1, \cdots, n-1$$

where σ^2 is the variance of the complex normal distribution of e_t. As a by-product, we find that the MLE's of the parameters attain the Cramér-Rao lower bound for the asymptotic covariance matrix.

I. Introduction and Main Results

Consider the following model of superimposed exponential signals:

$$Y_t = \sum_{k=1}^{p} \alpha_k \exp(it\omega_k) + e_t, \qquad t = 0, 1, \cdots, n-1 \tag{1.1}$$

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where $i = \sqrt{-1}$, $\omega_k \in [-\pi, \pi)$ is the frequency of the kth signal, α_k is its complex amplitude, and $\{e_t\}$ is a sequence of i.i.d. complex normal variables such that

$$e_t \approx \mathrm{CN}(0, \sigma^2), \qquad 0 < \sigma^2 < \infty. \tag{1.2}$$

Denote by $\mathrm{Re}(e_t)$ and $\mathrm{Im}(e_t)$ the real and imaginary parts of e_t respectively. We mean by (1.2) that both $\mathrm{Re}(e_t)$ and $\mathrm{Im}(e_t)$ obey $N(0, \sigma^2/2)$ and they are independent. In general, $\boldsymbol{\omega} = (\omega_1, \cdots, \omega_p)'$, $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_p)'$, and σ^2 are unknown, $\omega_1, \cdots, \omega_p$ are different, and it is desired to estimate these unknown parameters.

Under the normality assumption (1.2) on $\{e_t\}$, the maximum likelihood estimates (MLE's) of $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are the same as the nonlinear least squares estimates (LSE's) obtained by minimizing

$$D_n(\boldsymbol{\alpha}, \boldsymbol{\theta}) = \sum_{t=0}^{n-1} |Y_t - \sum_{k=1}^{p} a_k e^{it\theta_k}|^2 \tag{1.3}$$

with respect to $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_p)'$ and $\boldsymbol{a} = (a_1, \cdots, a_p)'$. It is easy to show that when $n \geq 2p + 1$, with probability one, the expression (1.3) reaches its minimum value at some $\hat{\boldsymbol{\omega}} = (\hat{\omega}_1, \cdots, \hat{\omega}_p)'$ and $\hat{\boldsymbol{\alpha}} = (\hat{\alpha}_1, \cdots, \hat{\alpha}_p)'$ with $\hat{\omega}_1, \cdots, \hat{\omega}_p$ all different. Hence, the MLE of $\boldsymbol{\omega}$ is obtained by maximizing

$$M_n(\boldsymbol{Y}, \boldsymbol{\theta}) = \boldsymbol{Y}^* V(\boldsymbol{\theta})(V^*(\boldsymbol{\theta}) V(\boldsymbol{\theta}))^{-1} V^*(\boldsymbol{\theta}) \boldsymbol{Y} \tag{1.4}$$

with respect to $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_p)'$, where $\boldsymbol{Y} = (Y_0, \cdots, Y_{n-1})'$,

$$V(\boldsymbol{\theta}) = (\boldsymbol{v}(\theta_1), \cdots, \boldsymbol{v}(\theta_p)), \ \boldsymbol{v}(\theta_k) = (1, e^{i\theta_k}, \cdots, e^{i(n-1)\theta_k})' \tag{1.5}$$

and $V^*(\boldsymbol{\theta})$ denotes the conjugate transpose of $V(\boldsymbol{\theta})$.

With $\hat{\boldsymbol{\omega}}$ obtained by maximizing (1.4), write $V(\boldsymbol{\omega}) = V$, $V(\hat{\boldsymbol{\omega}}) = \hat{V}$. The MLE's of $\boldsymbol{\alpha}$ and σ^2 are given by

$$\hat{\boldsymbol{\alpha}} = (\hat{V}^* \hat{V})^{-1} \hat{V}^* \boldsymbol{Y} \qquad \text{and}$$

$$\hat{\sigma}^2 = n^{-1} \boldsymbol{Y}^* (I - \hat{V}(\hat{V}^* V)^{-1} \hat{V}^*) \boldsymbol{Y}. \tag{1.6}$$

In this correspondence we develop asymptotic expansions of the estimates $\hat{\boldsymbol{\omega}}$, $\hat{\boldsymbol{\alpha}}$, $\hat{\sigma}^2$ and establish their strong consistency and asymptotic normality. From these results and the consideration of the Fisher information matrix derived in [2], it follows as a by-product that the MLE's of the parameters attain the Cramér-Rao (CR) lower bound, i.e., the best possible asymptotic covariances.

Some papers relevant to our work are by Bresler and Macovski [3], Bai *et al.* [1], Hurt [4], and Stoica and Nehorai [6], [7]. We note, however, that our aim is to establish asymptotic normality, as the problem does not come under the standard MLE theory, while the papers cited above concentrated on the CR bound.

We also refer to Whittle [11], Walker [10], Stoica and Nehorai, [8] and Stoica, Moses, Friedlander, and Söderström [9], where models similar to (1.1) are considered and asymptotic distributions of estimates are derived. There seems to be no reference where the model (1.1) with complex observations is considered and treated rigorously.

We prove the following main theorems:

Theorem: Suppose that $\alpha_j \neq 0$ and $\omega_j \neq \omega_k$ for $j \neq k$, $j, k = 1, \cdots, p$. Denote $A = \mathrm{diag}(\alpha_1, \cdots, \alpha_p)$, $A_1 = \mathrm{Im}\, A$, $A_2 = \mathrm{Re}\, A$, and $B = (A^*A)^{-1}$. If e_t satisfies (1.2), then

i) $\hat{\boldsymbol{\omega}}$, $\hat{\boldsymbol{\alpha}}$, and $\hat{\sigma}^2$ are strongly consistent estimates of $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$, σ^2.

ii) The limiting distribution of

$$(n^{3/2}(\hat{\boldsymbol{\omega}} - \boldsymbol{\omega}), n^{1/2}\, \mathrm{Re}\,(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}), n^{1/2}\, \mathrm{Im}\,(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}), \hat{\sigma}^2)$$