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Mixability is Bayes Risk Curvature Relative to Log Loss

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Abstract

Mixability of a loss governs the best possible performance when aggregating expert predictions with respect to that loss. The determination of the mixability constant for binary losses is straightforward but opaque. In the binary case we make this transparent and simpler by characterising mixability in terms of the second derivative of the Bayes risk of proper losses. We then extend this result to multiclass proper losses where there are few existing results. We show that mixability is governed by the maximum eigenvalue of the Hessian of the Bayes risk, relative to the Hessian of the Bayes risk for log loss. We conclude by comparing our result to other work that bounds prediction performance in terms of the geometry of the Bayes risk. Although all calculations are for proper losses, we also show how to carry the results across to improper losses.

1. Introduction

In *prediction with expert advice* (Vovk, 1990, 1995, 2001) a learner has to predict a sequence of outcomes, which might be chosen adversarially. The setting is online, meaning that learning proceeds in rounds; and the learner is aided by a finite number of experts. At the start of each round, all experts first announce their predictions for that round, then the learner has to make a prediction, and finally the real outcome is revealed. The discrepancy between a prediction and an outcome is measured by a *loss function*, and losses add up between rounds. Finally, the goal for the learner is to minimize the *regret*, which is the difference between their cumulative loss, and the cumulative loss of the best expert after T rounds.

The difficulty of the learning problem depends on the choice of loss function, which is considered a given. Kalnishkan and Vyugin (2008) characterise the losses for which the learner can guarantee a regret that grows no faster than $O(\sqrt{T})$, even if both the outcomes and the expert advice are chosen adversarially. But for some loss functions, which are *mixable*, the learner can do even better and guarantee a regret of $O(1)$, which does not grow with T (Vovk, 2001).

The mixability of several losses on binary outcomes and of the Brier score in the multiclass case (Vovk and Zhdanov, 2009) are known. However, a general characterisation of mixability in terms of other key properties of the loss has been missing. The present paper shows how mixability depends upon the curvature of the conditional Bayes risk when the loss is strictly proper, continuous and continuously differentiable (Theorems 10 and 13). Using these results a much shorter proof for mixability of the Brier score is possible (see Section 5). Although our results are stated for proper losses, in Section 6 we show how they carry across to losses that are not proper. There we also relate our results to recent work by Abernethy et al. (2009) in a related online learning setting.

2. Setting

Let $n \in \mathbb{N}$ and $\mathcal{Y} = \{1, \ldots, n\}$ be the outcome space. We will consider a prediction game where the loss of the learner making predictions $v_1, v_2, \ldots \in \mathcal{V}$ is measured by a loss function $\ell\colon \mathcal{Y} \times \mathcal{V} \to [0, \infty]$ cumulatively: for $T \in \mathbb{N}$, $\mathrm{Loss}(T) := \sum_{t=1}^{T} \ell(y_t, v_t)$, where $y_1, y_2, \ldots \in \mathcal{Y}$ are outcomes. The learner has access to predictions v_t^i, $t = 1, 2, \ldots$, $i \in \{1, \ldots, N\}$ generated by N experts $\mathcal{E}_1, \ldots, \mathcal{E}_N$ that attempt to predict the same sequence. The goal of the learner is to predict nearly as well as the best expert. A *merging strategy* $\mathcal{M}\colon \bigcup_{t=1}^{\infty} \left(\mathcal{Y}^{t-1} \times (\mathcal{V}^N)^t\right) \to \mathcal{V}$ takes the outcomes $y_1, \ldots, y_{t-1}$ and predictions v_s^i, $i = 1, \ldots, N$ for times $s = 1, \ldots, t$ and outputs an aggregated prediction $v_t^{\mathcal{M}}$, incurring loss $\ell(y_t, v_t^{\mathcal{M}})$ when y_t is revealed. After T rounds, the loss of $\mathcal{M}$ is $\mathrm{Loss}_{\mathcal{M}}(T) = \sum_{t=1}^{T} \ell(y_t, v_t^{\mathcal{M}})$. The loss of expert $\mathcal{E}_i$ is $\mathrm{Loss}_{\mathcal{E}_i}(T) = \sum_{t=1}^{T} \ell(y_t, v_t^i)$. If a loss ℓ is η-mixable for some $\eta > 0$ (see below for the definition), then using the *aggregating algorithm* (Vovk, 1995) as the merging strategy guarantees

$$\mathrm{Loss}_{\mathcal{M}}(t) \leq \mathrm{Loss}_{\mathcal{E}_i}(t) + \frac{\ln N}{\eta} \tag{1}$$

for all $t \in \mathbb{N}$, all $i \in \{1, \ldots, N\}$. Conversely, if there does not exist any $\eta > 0$ such that the loss function is η-mixable, then it is not possible to predict as well as the best expert up to an additive constant using any merging strategy.

Thus determining η_ℓ (the largest η such that ℓ is η-mixable) is equivalent to precisely bounding the prediction error of the aggregating algorithm. The mixability of several binary losses and the Brier score in the multiclass case (Vovk and Zhdanov, 2009) is known. However a general characterisation of η_ℓ in terms of other key properties of the loss has been missing. The present paper shows how η_ℓ depends upon the curvature of the conditional Bayes risk for ℓ when ℓ is a strictly proper continuously differentiable multiclass loss (see Theorem 13).

We use the following notation throughout. Let $[n] := \{1, \ldots, n\}$ and denote by $\mathbb{R}_+$ the non-negative reals. The transpose of a vector x is x'. If x is a n-vector, $A = \mathrm{diag}(x)$ is the $n \times n$ matrix with entries $A_{i,i} = x_i$, $i \in [n]$ and $A_{i,j} = 0$ for $i \neq j$. We also write $\mathrm{diag}(x_i)_{i=1}^n := \mathrm{diag}(x_1, \ldots, x_n) := \mathrm{diag}((x_1, \ldots, x_n)')$. The inner product of two n-vectors x and y is denoted by matrix product $x'y$. We sometimes write $A \cdot B$ for the matrix product AB for clarity when required. If $A - B$ is positive definite (resp. semi-definite), then we write $A \succ B$ (resp. $A \succcurlyeq B$). The n-simplex $\Delta^n := \{(x_1, \ldots, x_n)' \in \mathbb{R}^n \colon x_i \geq 0,\ i \in [n],\ \sum_{i=1}^n x_i = 1\}$. Other notation (the Kronecker product $\otimes$, the derivative D, and the Hessian H) is defined in Appendix A which also includes several matrix calculus results we use.

3. Proper Multiclass Losses

We consider multiclass losses for class probability estimation. A *loss function* $\ell : \Delta^n \to [0, \infty]^n$ assigns a loss vector $\ell(q) = (\ell_1(q), \ldots, \ell_n(q))'$ to each distribution $q \in \Delta^n$, where $\ell_i(q)$ $(= \ell(i, q)$ traditionally) is the penalty for predicting q when outcome $i \in [n]$ occurs. If the outcomes are distributed with probability $p \in \Delta^n$ then the *risk* for predicting q is just the expected loss

$$L(p, q) := p'\ell(q) = \sum_{i=1}^{n} p_i \ell_i(q).$$

The *Bayes risk* for p is the minimal achievable risk for that outcome distribution,

$$\underline{L}(p) := \inf_{q \in \Delta^n} L(p, q).$$

A loss is called *proper* whenever the minimal risk is always achieved by predicting the true outcome distribution, that is, $\underline{L}(p) = L(p, p)$ for all $p \in \Delta^n$. A proper loss is *strictly proper* if there exists no $q \neq p$ such that $L(p, q) = \underline{L}(p)$. For example, the *log loss* $\ell_{\log}(p) := (-\ln(p_1), \ldots, -\ln(p_n))'$ is strictly proper, and its corresponding Bayes risk is the entropy $\underline{L}_{\log}(p) = -\sum_{i=1}^{n} p_i \ln(p_i)$.

We call a proper loss ℓ *strongly invertible* if for all distributions $p, q \in \Delta^n$ there exists at least one outcome $i \in [n]$ such that $\ell_i(p) \neq \ell_i(q)$ and $p_i > 0$. Note that without the requirement that $p_i > 0$ this would be ordinary invertibility. One might also understand strong invertibility as saying that the loss should be invertible, and if we restrict the game to a face of the simplex (effectively removing one possible outcome), then the loss function for the resulting game should again be strongly invertible.

As it is central to our results, we will assume all losses are strictly proper for the remainder of the paper. Lemma 2 in the next section shows that strictness is not such a strong requirement.

Projecting Down to $n-1$ Dimensions Because probabilities sum up to one, any $p \in \Delta^n$ is fully determined by its first $n-1$ components $\tilde{p} = (p_1, \ldots, p_{n-1})$. It follows that any function of p can also be expressed as a function of $\tilde{p}$, which is often convenient in order to use the standard rules for derivatives. To go back and forth between the two, we define $p_n(\tilde{p}) := 1 - \sum_{i=1}^{n-1} \tilde{p}_i$ and the projection

$$\Pi_\Delta(p) := (p_1, \ldots, p_{n-1})',$$

$$\begin{tikzcd}
\Delta^n \arrow[r, "\ell"] \arrow[d, "\Pi_\Delta"] & \Lambda \arrow[r, "E_\eta"] \arrow[d, "\Pi_\Lambda"] & \Phi_\eta \arrow[d, "\Pi_\Phi"] \\
\tilde{\Delta}^n \arrow[r, "\tilde{\ell}"] \arrow[rr, bend right, "\tau_\eta"'] & \tilde{\Lambda} \arrow[r, "E_\eta"] & \tilde{\Phi}_\eta
\end{tikzcd}$$

Figure 1: Mappings and spaces.

which is a continuous and invertible function from Δ^n to $\tilde{\Delta}^n := \{(p_1, \ldots, p_{n-1})' : p \in \Delta^n\}$, with continuous inverse $\Pi_\Delta^{-1}(\tilde{p}) = (\tilde{p}_1, \ldots, \tilde{p}_{n-1}, p_n(\tilde{p}))$. For similar reasons, we sometimes project loss vectors $\ell(p)$ onto their first $n-1$ components $(\ell_1(p), \ldots, \ell_{n-1}(p))'$, using the projection

$$\Pi_\Lambda(\lambda) := (\lambda_1, \ldots, \lambda_{n-1})'.$$

We write $\Lambda := \ell(\Delta^n)$ for the domain of Π_Λ and $\tilde{\Lambda}$ for its range.

For loss functions $\ell(p)$, we will overload notation and abbreviate $\ell(\tilde{p}) := \ell(\Pi_\Delta^{-1}(\tilde{p}))$. In addition, we write

$$\tilde{\ell}(\tilde{p}) := \Pi_\Lambda(\ell(\tilde{p})) = (\ell_1(\tilde{p}), \ldots, \ell_{n-1}(\tilde{p}))'$$

for the first $n-1$ components of the loss (see Figure 1).

By contrast, for $\underline{L}(p)$ we will be more careful about its domain, and use the separate notation $\underline{\tilde{L}}(\tilde{p}) := \underline{L}(\Pi_\Delta^{-1}(\tilde{p}))$ when we consider it as a function of $\tilde{p}$.

3.1 First Properties

Our final result requires the following conditions on the loss:

Condition A *The loss $\ell(p)$ is strictly proper, continuous on Δ^n, and continuously differentiable on the relative interior* $\operatorname{rel\,int}(\Delta^n)$ *of its domain.*

As the projection Π_Δ is a linear function, differentiability of $\ell(p)$ is equivalent to differentiability of $\ell(\tilde{p})$, which will usually be easier to verify.

Lemma 1 *Let $\ell(p)$ be a strictly proper loss. Then the corresponding Bayes risk $\underline{L}(p)$ is strictly concave, and if $\ell(p)$ is differentiable on the relative interior* $\operatorname{rel\,int}(\Delta^n)$ *of Δ^n then the risk L satisfies the* stationarity condition

$$p'\mathsf{D}\ell(\tilde{p}) = 0_{n-1} \qquad \textit{for } p \in \operatorname{rel\,int}(\Delta^n). \tag{2}$$

If $\ell(p)$ is continuous on the whole simplex Δ^n, then Π_Λ, $\ell(p)$ and $\tilde{\ell}(\tilde{p})$ are all continuous and invertible, with continuous inverses.

Proof Let $p_0, p_1 \in \Delta^n$ and let $p_\lambda = (1-\lambda)p_0 + \lambda p_1$. Then for any $\lambda \in (0,1)$

$$\underline{L}(p_\lambda) = (1-\lambda)L(p, p_\lambda) + \lambda L(q, p_\lambda) > (1-\lambda)\underline{L}(p) + \lambda\underline{L}(q),$$

so $\underline{L}(p)$ is strictly concave. Properness guarantees that the function $L_p(\tilde{q}) := L(p, q(\tilde{q}))$ has a minimum at $\tilde{q} = \tilde{p}$. Hence $\mathsf{D}L_p(\tilde{q}) = p'\mathsf{D}\ell(\tilde{q}) = 0_{n-1}$ at $\tilde{q} = \tilde{p}$, giving the stationarity condition.

Now suppose ℓ is continuous on Δ^n, and observe that Π_Λ is also continuous. Then by tracing the relations in Figure 1, one sees that all remaining claims follow if we can establish invertibility of $\tilde{\ell}$ and continuity of its inverse. (Recall that Π_Δ is invertible with continuous inverse.)

To establish invertibility, suppose there exist $\tilde{p} \neq \tilde{q}$ in $\tilde{\Delta}^n$ such that $\tilde{\ell}(\tilde{p}) = \tilde{\ell}(\tilde{q})$ and assume without loss of generality that $\ell_n(p) \leq \ell_n(q)$ (otherwise, just swap them). Then $\underline{L}(q) = \tilde{q}'\tilde{\ell}(\tilde{q}) + q_n\ell_n(q) \geq \tilde{q}'\tilde{\ell}(\tilde{p}) + q_n\ell_n(p) = L(q,p)$, which contradicts strict properness. Hence $\tilde{\ell}$ must be invertible.

To establish continuity of $\tilde{\ell}^{-1}$, we need to show that $\tilde{\ell}(\tilde{p}_m) \to \tilde{\ell}(\tilde{p})$ implies $\tilde{p}_m \to \tilde{p}$ for any sequence $(\tilde{p}_m)_{m=1,2,\ldots}$ of elements from $\tilde{\Delta}^n$. To this end, let $\epsilon > 0$ be arbitrary. Then it is sufficient to show that there exist only a finite number of elements in $(\tilde{p}_m)$ such that $\|\tilde{p}_m - \tilde{p}\| > \epsilon$. Towards a contradiction, suppose that $(\tilde{q}_k)_{k=1,2,\ldots}$ is a subsequence of $(\tilde{p}_m)$ such that $\|\tilde{q}_k - \tilde{p}\| \geq \epsilon$ for all $\tilde{q}_k$. Then the fact that $\tilde{\Delta}^n$ is a compact subset of $\mathbb{R}^{n-1}$ implies (by the Bolzano-Weierstrass theorem) that $(\tilde{q}_k)$ contains a converging subsequence $\tilde{r}_v \to \tilde{r}$.

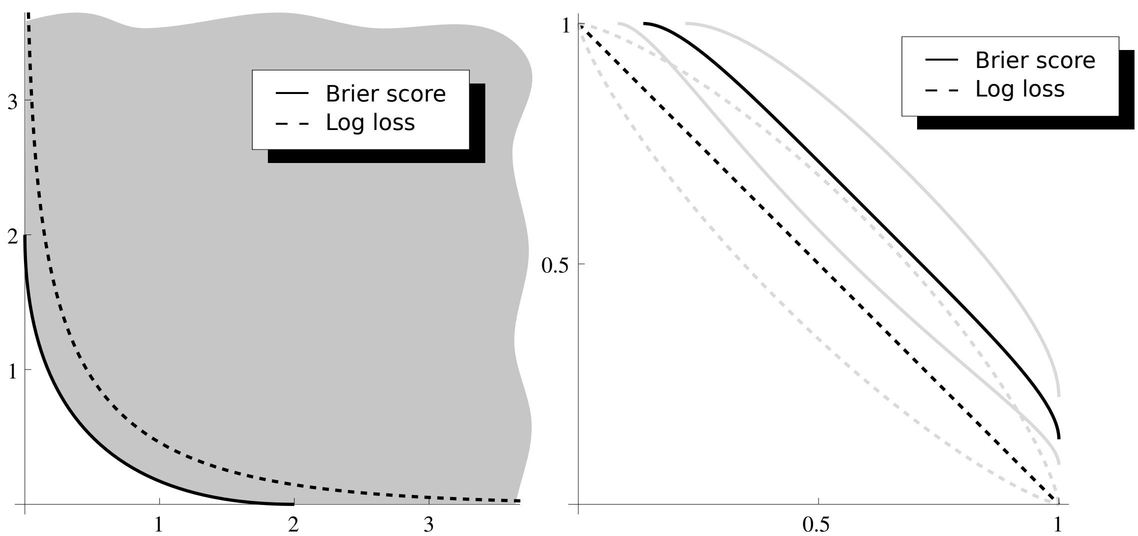

Figure 2: Left: the (boundary of the) super prediction set on two outcomes for the Brier score and the boundary of the super prediction set for log loss. Right: the same boundaries after applying the η-exponential operator for $\eta \in \{3/4, 1, 5/4\}$. The dark curve corresponds to $\eta = 1$.

Since continuity of ℓ and Π_Δ^{-1} imply continuity of $\tilde{\ell}$, we have $\tilde{\ell}(\tilde{r}_v) \to \tilde{\ell}(\tilde{r})$. But since $\tilde{r}_v$ is a subsequence of $(\tilde{p}_m)$, we also have that $\tilde{\ell}(\tilde{r}_v) \to \tilde{\ell}(\tilde{p})$ and hence $\tilde{\ell}(\tilde{r}) = \tilde{\ell}(\tilde{p})$. But then strict properness implies that $\tilde{r} = \tilde{p}$, which contradicts the assumption that $\|\tilde{r}_v - \tilde{p}\| \geq \epsilon$ for all v. ■

4. Mixability

We use the following characterisation of mixability (as discussed by Vovk and Zhdanov (2009)) and motivate our main result by looking at the binary case. To define mixability we need the notions of a superprediction set and a parametrised exponential operator. The *superprediction set* S_ℓ for a loss $\ell : \Delta^n \to [0,\infty]^n$ is the set of points in $[0,\infty]^n$ that point-wise dominate some point on the loss surface. That is,

$$S_\ell := \{\lambda \in [0,\infty]^n : \exists q \in \Delta^n,\ \forall i \in [n],\ \ell_i(q) \leq \lambda_i\}.$$

For any dimension m and $\eta \geq 0$, the *η-exponential operator* $E_\eta \colon [0,\infty]^m \to [0,1]^m$ is defined by

$$E_\eta(\lambda) := (e^{-\eta\lambda_1}, \ldots, e^{-\eta\lambda_m}).$$

For $\eta > 0$ it is clearly invertible, with inverse $E_\eta^{-1}(\phi) = -\eta^{-1}(\ln\phi_1, \ldots, \ln\phi_m)$. We will both apply it for $m = n$ and for $m = n - 1$. The dimension will always be clear from the context.

A loss ℓ is *η-mixable* when the set $E_\eta(S_\ell)$ is convex. The largest η such that a loss is η-mixable is of special interest, because it determines the best possible bound in (1). We call this the *mixability constant* and denote it by η_ℓ:

$$\eta_\ell := \max\{\eta \geq 0\colon\ \ell \text{ is } \eta\text{-mixable}\}.$$

A loss is always 0-mixable, so $\eta_\ell \geq 0$, but note that for $\eta_\ell = 0$ the bound in (1) is vacuous. A loss is therefore called *mixable* only if its mixability constant is positive, i.e. $\eta_\ell > 0$.

One may rewrite the definition of $E_\eta(S_\ell)$ as follows:

$$\begin{aligned} E_\eta(S_\ell) &= \{E_\eta(\lambda)\colon \lambda \in [0,\infty]^n,\ \exists q \in \Delta^n,\ \forall i \in [n],\ \ell_i(q) \leq \lambda_i\} \\ &= \{z \in [0,1]^n\colon \exists q \in \Delta^n,\ \forall i \in [n],\ e^{-\eta \ell_i(q)} \geq z_i\}, \end{aligned}$$

since $x \mapsto e^{-\eta x}$ is nonincreasing (in fact, decreasing for $\eta > 0$). Hence in order for $E_\eta(S_\ell)$ to be convex the

graph$(f_\eta) = \Phi_\eta := \{(e^{-\eta \ell_1(q)}, \ldots, e^{-\eta \ell_n(q)})\colon q \in \Delta^n\}$ needs to be *concave*. Observe that Φ_η is the (upper) boundary of $E_\eta(S_\ell)$; that is why concavity of f_η corresponds to *convexity* of $E_\eta(S_\ell)$.

Lemma 2 *If a proper, strongly invertible loss ℓ is mixable, then it is strictly proper.*

An example of a mixable proper loss that is not strictly proper, is when $\ell(p)$ does not depend on p. In this case the loss is not invertible.

Proof Suppose ℓ is not strictly proper. Then there exist $p \neq q$ such that $\underline{L}(p) = L(p,q)$. In addition, mixability implies that for any $\lambda \in (0,1)$ there exists a distribution r_λ such that for all $i \in [n]$

$$\ell_i(r_\lambda) \leq -\tfrac{1}{\eta_\ell} \log\Big((1-\lambda)e^{-\eta_\ell \ell_i(p)} + \lambda e^{-\eta_\ell \ell_i(q)}\Big) \leq (1-\lambda)\ell_i(p) + \lambda \ell_i(q),$$

where the second inequality follows from (strict) convexity of $x \mapsto e^{-x}$ and is strict when $\ell_i(p) \neq \ell_i(q)$. Since $\ell_i(p) \neq \ell_i(q)$ for at least one i with $p_i > 0$, it follows that

$$L(p, r_\lambda) = p'\ell(r_\lambda) < p'\big((1-\lambda)\ell(p) + \lambda \ell(q)\big) = \underline{L}(p),$$

which contradicts the definition of $\underline{L}(p)$. Thus mixability implies that ℓ must be strictly proper. ∎

4.1 The Binary Case

A loss is called binary of there are only two outcomes: $n = 2$. For twice differentiable binary losses ℓ it is known (Haussler et al., 1998) that

$$\eta_\ell = \min_{\tilde{p} \in [0,1]} \frac{\ell_1'(\tilde{p})\ell_2''(\tilde{p}) - \ell_1''(\tilde{p})\ell_2'(\tilde{p})}{\ell_1'(\tilde{p})\ell_2'(\tilde{p})(\ell_2'(\tilde{p}) - \ell_1'(\tilde{p}))}. \tag{3}$$

When a proper binary loss ℓ is differentiable, the stationarity condition (2) implies

$$\begin{aligned} & \tilde{p}\ell_1'(\tilde{p}) + (1-\tilde{p})\ell_2'(\tilde{p}) = 0 \\ \Rightarrow\quad & \tilde{p}\ell_1'(\tilde{p}) = (\tilde{p}-1)\ell_2'(\tilde{p}) \qquad (4) \\ \Rightarrow\quad & \frac{\ell_1'(\tilde{p})}{\tilde{p}-1} = \frac{\ell_2'(\tilde{p})}{\tilde{p}} =: w(\tilde{p}) =: w_\ell(\tilde{p}) \qquad (5) \end{aligned}$$

We have $\underline{\tilde{L}}(\tilde{p}) = \tilde{p}\ell_1(\tilde{p}) + (1-\tilde{p})\ell_2(\tilde{p})$. Thus by differentiating both sides of (4) and substituting into $\underline{\tilde{L}}''(\tilde{p})$ one obtains $\underline{\tilde{L}}''(\tilde{p}) = \frac{\ell_1'(\tilde{p})}{1-\tilde{p}} = -w(\tilde{p})$. (See Reid and Williamson (2011)). Equation 5 implies $\ell_1'(\tilde{p}) = (\tilde{p}-1)w(\tilde{p})$, $\ell_2'(\tilde{p}) = \tilde{p}w(\tilde{p})$ and hence $\ell_1''(\tilde{p}) = w(\tilde{p}) + (\tilde{p}-1)w'(\tilde{p})$ and $\ell_2''(\tilde{p}) = w(\tilde{p}) + \tilde{p}w'(\tilde{p})$. Substituting these expressions into (3) gives

$$\eta_\ell = \min_{\tilde{p}\in[0,1]} \frac{(\tilde{p}-1)w(\tilde{p})[w(\tilde{p}) + \tilde{p}w'(\tilde{p})] - [w(\tilde{p}) + (\tilde{p}-1)w'(\tilde{p})]\tilde{p}w(\tilde{p})}{(\tilde{p}-1)w(\tilde{p})\tilde{p}w(\tilde{p})[\tilde{p}w(\tilde{p}) - (\tilde{p}-1)w(\tilde{p})]} = \min_{\tilde{p}\in[0,1]} \frac{1}{\tilde{p}(1-\tilde{p})w(\tilde{p})}.$$

Observing that $\underline{L}_{\log}(p) = -p_1 \ln p_1 - p_2 \ln p_2$ we have $\underline{\tilde{L}}_{\log}(\tilde{p}) = -\tilde{p}\ln\tilde{p} - (1-\tilde{p})\ln(1-\tilde{p})$ and thus $\underline{\tilde{L}}''_{\log}(\tilde{p}) = \frac{-1}{\tilde{p}(1-\tilde{p})}$ and so $w_{\log}(\tilde{p}) = \frac{1}{\tilde{p}(1-\tilde{p})}$. Thus

$$\boxed{\eta_\ell = \min_{\tilde{p}\in[0,1]} \frac{w_{\log}(\tilde{p})}{w_\ell(\tilde{p})} = \min_{\tilde{p}\in[0,1]} \frac{\underline{\tilde{L}}''_{\log}(\tilde{p})}{\underline{\tilde{L}}''(\tilde{p})}.} \tag{6}$$

That is, the mixability constant of binary proper losses is the minimal ratio of the weight functions for log loss and the loss in question. The rest of this paper is devoted to the generalisation of (6) to the multiclass case. That there is a relationship between Bayes risk and mixability was also pointed out (in a less explicit form) by Kalnishkan et al. (2004).

By plugging $w_\ell(\tilde{p}) = \frac{\ell_1'(\tilde{p})}{\tilde{p}-1}$ and $w_{\log}(\tilde{p}) = \frac{1}{\tilde{p}(1-\tilde{p})}$ into (6), one obtains an expression to compute η_ℓ that is simpler than (3):

$$\boxed{\frac{-1}{\eta_\ell} = \min_{\tilde{p}\in[0,1]} \tilde{p}\ell_1'(\tilde{p}).} \tag{7}$$

This result also generalizes to the multiclass case.

4.2 Mixability and the Concavity of the Function f_η

Our aim is to relate mixability of a loss to the curvature of its Bayes risk surface. Since mixability is equivalent to concavity of the function f_η, which maps the first $n-1$ coordinates of Φ_η to the n-th coordinate, we will start by giving an explicit expression for f_η. We will assume throughout that the loss ℓ is strictly proper and continuous on Δ^n.

It is convenient to introduce an auxiliary function $\tau_\eta : \tilde{\Delta}^n \to [0,1]^{n-1}$ as

$$\tau_\eta(\tilde{p}) := E_\eta(\tilde{\ell}(\tilde{p})) = \left(e^{-\eta\ell_1(\tilde{p})}, \ldots, e^{-\eta\ell_{n-1}(\tilde{p})}\right), \tag{8}$$

which maps a distribution $\tilde{p}$ to the first $n-1$ coordinates of an element in Φ_η. The range of τ_η will be denoted $\tilde{\Phi}_\eta$ (see Figure 1). In addition, let the projection $\Pi_\Phi \colon \Phi_\eta \to \tilde{\Phi}_\eta$ map any element of $\phi \in \Phi_\eta$ to its first $n-1$ coordinates $(\phi_1, \ldots, \phi_{n-1})$. Then under our assumptions, all the maps we have defined are well-behaved:

Lemma 3 *Let ℓ be a continuous, strictly proper loss. Then for $\eta > 0$ all functions in Figure 1 are continuous and invertible with continuous inverse.*

Proof Lemma 1 already covers most of the functions. Given that E_η satisfies the required properties, they can be derived for the remaining functions by writing them as a composition of functions for which the properties are known. For example, $\tau_\eta = E_\eta \circ \tilde{\ell}$ is a composition of two continuous and invertible functions, which each have a continuous inverse. ■

It follows that, under the conditions of the lemma, the function $f_\eta \colon \tilde{\Phi}_\eta \to [0,1]$ may be defined as

$$f_\eta(\tilde{\phi}) = e^{-\eta \ell_n\left(\tau_\eta^{-1}(\tilde{\phi})\right)} \tag{9}$$

and is continuous. Moreover, as $\tilde{\Phi}_\eta$ (the domain of f_η) is the preimage under τ^{-1} of the closed set $\tilde{\Delta}^n$, continuity of τ^{-1} implies that $\tilde{\Phi}_\eta$ is closed as well. However, continuity implies that we may restrict attention to the interiors of $\tilde{\Phi}_\eta$ and of the probability simplex:

Lemma 4 *Let ℓ be a continuous, strictly proper loss. Then, for $\eta > 0$, f_η is concave if and only if it is concave on the interior $\operatorname{int}(\tilde{\Phi}_\eta)$ of its domain. Furthermore this set corresponds to a subset of the interior of the simplex: $\tau_\eta^{-1}(\operatorname{int}(\tilde{\Phi}_\eta)) \subseteq \operatorname{int}(\tilde{\Delta}^n) = \Pi_\Delta(\operatorname{rel\,int}(\Delta^n))$.*

Proof The restriction to $\operatorname{int}(\tilde{\Phi}_\eta)$ follows trivially from continuity of f_η. The set $\tau_\eta^{-1}(\operatorname{int}(\tilde{\Phi}_\eta))$ is the preimage under τ_η of the open set $\operatorname{int}(\tilde{\Phi}_\eta)$. Since τ_η is continuous, it follows that this set must also be open and hence be a subset of the interior of $\tilde{\Delta}^n$. ■

4.3 Relating Concavity of f_η to the Hessian of $\underline{L}$

The aim of this subsection is to express the Hessian of f_η in terms of the Bayes risk of the loss function defining f_η. We first note that a twice differentiable function $f : X \to \mathbb{R}$ defined on $X \subseteq \mathbb{R}^{n-1}$ is concave if and only if its Hessian at x, $\mathsf{H}f(x)$, is negative semi-definite for all $x \in X$ (Hiriart-Urruty and Lemaréchal, 1993). The argument that follows consists of repeated applications of the chain and inverse rules for Hessians to compute $\mathsf{H}f_\eta$.

We start the analysis by considering the η-exponential operator, used in the definition of τ (8):

Lemma 5 *Suppose $\eta > 0$. Then the derivatives of E_η and E_η^{-1} are*

$$\mathsf{D}E_\eta(\lambda) = -\eta \operatorname{diag}\left(E_\eta(\lambda)\right) \quad \textit{and} \quad \mathsf{D}E_\eta^{-1}(\phi) = -\eta^{-1}\left[\operatorname{diag}(\phi)\right]^{-1}.$$

And the Hessian of E_η^{-1} is

$$\mathsf{H}E_\eta^{-1}(\phi) = \frac{1}{\eta}\begin{bmatrix} \operatorname{diag}(\phi_1^{-2}, 0, \ldots, 0) \\ \vdots \\ \operatorname{diag}(0, \ldots, 0, \phi_n^{-2}) \end{bmatrix}. \tag{10}$$

If $\eta = 1$ and $\ell = \ell_{\log} = p \mapsto -(\ln p_1, \ldots, \ln p_n)'$ is the log loss, then the map τ_1 is the identity map (i.e., $\tilde{\phi} = \tau_1(\tilde{p}) = \tilde{p}$) and $E_1^{-1}(\tilde{p}) = \tilde{\ell}_{\log}(\tilde{p})$ is the (projected) log loss.

Proof The derivatives follow immediately from the definitions. By (28) the Hessian $\mathsf{H}E_\eta^{-1}(\phi) = \mathsf{D}\left(\mathsf{D}E_\eta^{-1}(\phi)\right)$ and so

$$\mathsf{H}E_\eta^{-1}(\phi) = \mathsf{D}\left(\left(-\frac{1}{\eta}\left[\operatorname{diag}(\phi)\right]^{-1}\right)'\right) = -\frac{1}{\eta}\mathsf{D}\operatorname{diag}(\phi_i^{-1})_{i=1}^n.$$

Let $h(\phi) = \mathrm{diag}(\phi_i^{-1})_{i=1}^n$. We have

$$\mathsf{D}h(\phi) = \mathsf{D}\,\mathrm{vec}\,h(\phi) = \begin{bmatrix} \mathrm{diag}(-\phi_1^{-2}, 0, \ldots, 0) \\ \vdots \\ \mathrm{diag}(0, \ldots, 0, -\phi_n^{-2}) \end{bmatrix}.$$

The result for $\eta = 1$ and $\ell_{\log}$ follows from $\tau_1(\tilde{p}) = E_1(\tilde{\ell}(\tilde{p})) = (e^{-1 \cdot -\ln \tilde{p}_1}, \ldots, e^{-1 \cdot -\ln \tilde{p}_{n-1}})'$. ■

Next we turn our attention to other components of f_η. Using the stationarity condition and invertibility of ℓ from Lemma 1, simple expressions can be derived for the Jacobian and Hessian of the projected Bayes risk $\underline{\tilde{L}}(\tilde{p}) := \underline{L}(\Pi_\Delta^{-1}(\tilde{p}))$:

Lemma 6 *Suppose the loss ℓ satisfied Condition A. Take $\tilde{p} \in \mathrm{int}(\tilde{\Delta}^n)$, and let $y(\tilde{p}) := -\tilde{p}/p_n(\tilde{p})$. Then*

$$Y(\tilde{p}) := -p_n(\tilde{p})\mathsf{D}y(\tilde{p}) = \left(I_{n-1} + \frac{1}{p_n}\tilde{p}\mathbb{1}'_{n-1}\right)$$

is invertible for all $\tilde{p}$, and

$$\mathsf{D}\ell_n(\tilde{p}) = y(\tilde{p})' \cdot \mathsf{D}\tilde{\ell}(\tilde{p}). \tag{11}$$

The projected Bayes risk function $\underline{\tilde{L}}(\tilde{p})$ satisfies

$$\mathsf{D}\underline{\tilde{L}}(\tilde{p}) = \tilde{\ell}(\tilde{p})' - \ell_n(\tilde{p})\mathbb{1}'_{n-1} \tag{12}$$

$$\mathsf{H}\underline{\tilde{L}}(\tilde{p}) = Y(\tilde{p})' \cdot \mathsf{D}\tilde{\ell}(\tilde{p}). \tag{13}$$

Furthermore, the matrix $\mathsf{H}\underline{\tilde{L}}(\tilde{p})$ is negative definite and invertible for all $\tilde{p}$, and when $\ell = \ell_{\log}$ is the log loss

$$\mathsf{H}\underline{\tilde{L}}_{\log}(\tilde{p}) = -Y(\tilde{p})' \cdot [\mathrm{diag}(\tilde{p})]^{-1}. \tag{14}$$

Proof The stationarity condition (Lemma 1) guarantees that $p'\mathsf{D}\ell(\tilde{p}) = 0_{n-1}$ for all $p \in \mathrm{rel\,int}(\Delta^n)$. This is equivalent to $\tilde{p}'\mathsf{D}\tilde{\ell}(\tilde{p}) + p_n(\tilde{p})\mathsf{D}\ell_n(\tilde{p}) = 0_{n-1}$, which can be rearranged to obtain (11).

By the product rule $\mathsf{D}a'b = (\mathsf{D}a')b + a'(\mathsf{D}b)$, we obtain

$$\begin{aligned}
\mathsf{D}y(\tilde{p}) &= -\tilde{p}\mathsf{D}[p_n(\tilde{p})^{-1}] - [p_n(\tilde{p})^{-1}]\mathsf{D}\tilde{p} \\
&= \tilde{p}[p_n(\tilde{p})^{-2}]\mathsf{D}p_n(\tilde{p}) - [p_n(\tilde{p})^{-1}]I_{n-1} \\
&= -\tilde{p}[p_n(\tilde{p})^{-2}]\mathbb{1}'_{n-1} - [p_n(\tilde{p})^{-1}]I_{n-1} \\
&= -\frac{1}{p_n(\tilde{p})}\left[I_{n-1} + \frac{1}{p_n(\tilde{p})}\tilde{p}\mathbb{1}'_{n-1}\right],
\end{aligned}$$

since $p_n(\tilde{p}) = 1 - \sum_{i\in[n-1]} \tilde{p}_i$ implies $\mathsf{D}p_n(\tilde{p}) = -\mathbb{1}'_{n-1}$. This establishes that $Y(\tilde{p}) = I_{n-1} + \frac{1}{p_n(\tilde{p})}\tilde{p}\mathbb{1}'_{n-1}$. That this matrix is invertible can be easily checked since

$$(I_{n-1} - \tilde{p}\mathbb{1}'_{n-1})(I_{n-1} + \frac{1}{p_n(\tilde{p})}\tilde{p}\mathbb{1}'_{n-1}) = I_{n-1}$$

by expanding and noting $\tilde{p}\mathbb{1}_{n-1}'\tilde{p}\mathbb{1}_{n-1}' = (1-p_n)\tilde{p}\mathbb{1}_{n-1}'$.

The Bayes risk is $\underline{\tilde{L}}(\tilde{p}) = \tilde{p}'\tilde{\ell}(\tilde{p}) + p_n(\tilde{p})\ell_n(\tilde{p})$. Taking the derivative and using the product rule gives

$$\begin{aligned}
\mathsf{D}\underline{\tilde{L}}(\tilde{p}) &= \mathsf{D}\left[\tilde{p}'\tilde{\ell}(\tilde{p})\right] + \mathsf{D}\left[p_n(\tilde{p})\ell_n(\tilde{p})\right] \\
&= \tilde{\ell}(\tilde{p}) + \tilde{p}'\mathsf{D}\tilde{\ell}(\tilde{p}) + \left[\mathsf{D}p_n(\tilde{p})\right]\ell_n(\tilde{p}) + p_n(\tilde{p})\mathsf{D}\ell_n(\tilde{p}) \\
&= \tilde{\ell}(\tilde{p}) - p_n(\tilde{p})\mathsf{D}\ell_n(\tilde{p}) - \ell_n(\tilde{p})\mathbb{1}_{n-1}' + p_n(\tilde{p})\mathsf{D}\ell_n(\tilde{p})
\end{aligned}$$

by (11). Thus, $\mathsf{D}\underline{\tilde{L}}(\tilde{p}) = \tilde{\ell}(\tilde{p})' - \ell_n(\tilde{p})\mathbb{1}_{n-1}'$, establishing (12).

Equation 13 is obtained by taking derivatives once more and using (11) again, yielding

$$\mathsf{H}\underline{\tilde{L}}(\tilde{p}) = \mathsf{D}\left(\left(\mathsf{D}\underline{\tilde{L}}(\tilde{p})\right)'\right) = \mathsf{D}\tilde{\ell}(\tilde{p}) - \mathbb{1}_{n-1}\cdot\mathsf{D}\ell_n(\tilde{p}) = \left(I_{n-1} + \frac{1}{p_n}\mathbb{1}_{n-1}\tilde{p}'\right)\mathsf{D}\tilde{\ell}(\tilde{p})$$

as required. Now $\underline{\tilde{L}}(\tilde{p}) = \underline{L}(p_1,\ldots,p_{n-1},p_n(\tilde{p})) = \underline{L}(p_1,\ldots,p_{n-1},1-\sum_{i=1}^{n-1}p_i) = \underline{L}(C(\tilde{p}))$ where C is affine. Since $p \mapsto \underline{L}(p)$ is strictly concave (Lemma 1) it follows (Hiriart-Urruty and Lemaréchal, 1993) that $\tilde{p} \mapsto \underline{\tilde{L}}(\tilde{p})$ is also strictly concave and thus $\mathsf{H}\underline{\tilde{L}}(\tilde{p})$ is negative definite. It is invertible since we have shown $Y(\tilde{p})$ is invertible and $\mathsf{D}\tilde{\ell}$ is invertible by the inverse function theorem and the invertibility of $\tilde{\ell}$ (Lemma 1).

Finally, Equation 14 holds since Lemma 5 gives us $E_1^{-1} = \tilde{\ell}_{\log}$ so (13) specialises to $\mathsf{H}\underline{\tilde{L}}_{\log}(\tilde{p}) = Y(\tilde{p})'\cdot\mathsf{D}\tilde{\ell}_{\log}(\tilde{p}) = Y(\tilde{p})'\cdot\mathsf{D}E_1^{-1}(\tilde{p}) = -Y(\tilde{p})'\cdot[\mathrm{diag}(\tilde{p})]^{-1}$, also by Lemma 5. ■

4.4 Completion of the Argument

Recall that our aim is to compute the Hessian of the function describing the boundary of the η-exponentiated superprediction set and determine when it is negative semi-definite. The boundary is described by the function f_η which can be written as the composition $h_\eta \circ g_\eta$ where $h_\eta(z) := e^{-\eta z}$ and $g_\eta(\tilde{\phi}) := \ell_n\left(\tau_\eta^{-1}(\tilde{\phi})\right)$. The Hessian of f_η can be expanded in terms of g_η using the chain rule for the Hessian (Theorem 18) as follows.

Lemma 7 *Suppose the loss ℓ satisfies Condition A and $\eta > 0$. Then for all $\tilde{\phi} \in \mathrm{int}(\tilde{\Phi})$, the Hessian of f_η at $\tilde{\phi}$ is*

$$\mathsf{H}f_\eta(\tilde{\phi}) = \eta e^{-\eta g_\eta(\tilde{\phi})}\Gamma_\eta(\tilde{\phi}), \tag{15}$$

where $\Gamma_\eta(\tilde{\phi}) := \eta\mathsf{D}g_\eta(\tilde{\phi})'\cdot\mathsf{D}g_\eta(\tilde{\phi}) - \mathsf{H}g_\eta(\tilde{\phi})$. Furthermore, for $\eta > 0$ the negative semi-definiteness of $\mathsf{H}f_\eta(\tilde{\phi})$ (and thus the concavity of f_η) is equivalent to the negative semi-definiteness of $\Gamma_\eta(\tilde{\phi})$.

Proof Using $f := f_\eta$ and $g := g_\eta$ temporarily and letting $z = g(\tilde{\phi})$, the chain rule for H gives

$$\begin{aligned}
\mathsf{H}f(\tilde{\phi}) &= \left(I_1 \otimes \mathsf{D}g(\tilde{\phi})'\right)\cdot(\mathsf{H}h_\eta(z))\cdot\mathsf{D}g(\tilde{\phi}) + (\mathsf{D}h_\eta(z)\otimes I_{n-1})\cdot\mathsf{H}g(\tilde{\phi}) \\
&= \eta^2 e^{-\eta z}\mathsf{D}g(\tilde{\phi})'\cdot\mathsf{D}g(\tilde{\phi}) - \eta e^{-\eta z}\mathsf{H}g(\tilde{\phi}) \\
&= \eta e^{-\eta g(\tilde{\phi})}\left[\eta\mathsf{D}g(\tilde{\phi})'\cdot\mathsf{D}g(\tilde{\phi}) - \mathsf{H}g(\tilde{\phi})\right],
\end{aligned}$$

since $\alpha \otimes A = \alpha A$ for scalar α and matrix A and $\mathsf{D}h_\eta(z) = \mathsf{D}[\exp(-\eta z)] = -\eta e^{-\eta z}$ so $\mathsf{H}h(z) = \eta^2 e^{-\eta z}$. Whether $\mathsf{H}f \preccurlyeq 0$ depends only on Γ_η since $\eta e^{-\eta g(\tilde{\phi})}$ is positive for all $\eta > 0$ and $\tilde{\phi}$. ■

We proceed to compute the derivative and Hessian of g_η:

Lemma 8 *Suppose ℓ satisfies Condition A. For $\eta > 0$ and $\tilde{\phi} \in \operatorname{int}(\tilde{\Phi}_\eta)$, let $\lambda := E_\eta^{-1}(\tilde{\phi})$ and $\tilde{p} := \tilde{\ell}^{-1}(\lambda)$. Then*

$$\mathsf{D}g_\eta(\tilde{\phi}) = y(\tilde{p})' A_\eta(\tilde{\phi}) \tag{16}$$

$$\mathsf{H}g_\eta(\tilde{\phi}) = -\frac{1}{p_n(\tilde{p})} A_\eta(\tilde{\phi})' \cdot \left[\eta \operatorname{diag}(\tilde{p}) + Y(\tilde{p}) \cdot \left[\mathsf{H}\underline{\tilde{L}}(\tilde{p})\right]^{-1} \cdot Y(\tilde{p})'\right] \cdot A_\eta(\tilde{\phi}), \tag{17}$$

where $A_\eta(\tilde{\phi}) := \mathsf{D}E_\eta^{-1}(\tilde{\phi})$.

Proof By definition, $g_\eta(\tilde{\phi}) := \ell_n(\tau_\eta^{-1}(\tilde{\phi}))$. Since $\tau_\eta^{-1} = \tilde{\ell}^{-1} \circ E_\eta^{-1}$ we have $g_\eta = \ell_n \circ \tilde{\ell}^{-1} \circ E_\eta^{-1}$. Thus, by the chain rule, Equation 11 from Lemma 6, and the inverse function theorem, we obtain

$$\mathsf{D}g_\eta(\tilde{\phi}) = \mathsf{D}\ell_n(\tilde{p}) \cdot \mathsf{D}\tilde{\ell}^{-1}(\lambda) \cdot \mathsf{D}E_\eta^{-1}(\tilde{\phi}) = y(\tilde{p})'\mathsf{D}\tilde{\ell}(\tilde{p}) \cdot \left[\mathsf{D}\tilde{\ell}(\tilde{p})\right]^{-1} \cdot \left[\mathsf{D}E_\eta^{-1}(\tilde{\phi})\right] = y(\tilde{p})' A_\eta(\tilde{\phi})$$

yielding (16). Since $\tilde{p} = \tau_\eta^{-1}(\tilde{\phi})$ and $\mathsf{H}g_\eta = \mathsf{D}((\mathsf{D}g_\eta)')$ (see (28)), the chain and product rules give

$$\begin{aligned}
\mathsf{H}g_\eta(\tilde{\phi}) &= \mathsf{D}\left[\left(\mathsf{D}E_\eta^{-1}(\tilde{\phi})\right)' \cdot y\left(\tau_\eta^{-1}(\tilde{\phi})\right)\right] \\
&= \left(y(\tau_\eta^{-1}(\tilde{\phi}))' \otimes I_{n-1}\right) \cdot \mathsf{D}\left(\mathsf{D}E_\eta^{-1}(\tilde{\phi})'\right) + \left(I_1 \otimes (\mathsf{D}E_\eta^{-1}(\tilde{\phi}))'\right) \cdot \mathsf{D}\left(y\left(\tau_\eta^{-1}(\tilde{\phi})\right)\right) \\
&= \left(y(\tilde{p})' \otimes I_{n-1}\right) \cdot \mathsf{H}E_\eta^{-1}(\tilde{\phi}) + \left(\mathsf{D}E_\eta^{-1}(\tilde{\phi})\right)' \cdot \mathsf{D}y(\tilde{p}) \cdot \mathsf{D}\tau_\eta^{-1}(\tilde{\phi}) \\
&= -\frac{\eta}{p_n(\tilde{p})} A_\eta(\tilde{\phi}) \cdot \operatorname{diag}(\tilde{p}) \cdot A_\eta(\tilde{\phi}) + A_\eta(\tilde{\phi})' \cdot \mathsf{D}y(\tilde{p}) \cdot \mathsf{D}\tau_\eta^{-1}(\tilde{\phi}).
\end{aligned} \tag{18}$$

The first summand in (18) is due to (10) and the fact that

$$\begin{aligned}
(y \otimes I_{n-1}) \cdot \mathsf{H}E_\eta^{-1}(\tilde{\phi}) &= \frac{1}{\eta}[y_1 I_{n-1}, \ldots, y_{n-1} I_{n-1}] \cdot \begin{bmatrix} \operatorname{diag}(\phi_1^{-2}, 0, \ldots, 0) \\ \vdots \\ \operatorname{diag}(0, \ldots, 0, \phi_{n-1}^{-2}) \end{bmatrix} \\
&= \frac{1}{\eta} \sum_{i=1}^{n-1} y_i \cdot I_{n-1} \cdot \operatorname{diag}(0, \ldots, 0, \phi_i^{-2}, 0, \ldots, 0) \\
&= \frac{1}{\eta} \operatorname{diag}(y_i \phi_i^{-2})_{i=1}^{n-1} \\
&= \frac{-\eta}{p_n(\tilde{p})} A_\eta(\tilde{\phi})' \cdot \operatorname{diag}(\tilde{p}) \cdot A_\eta(\tilde{\phi}).
\end{aligned}$$

The last equality holds because $A_\eta(\tilde{\phi})' \cdot A_\eta(\tilde{\phi}) = \eta^{-2} \operatorname{diag}(\tilde{\phi}_i^{-2})_{i=1}^{n-1}$ by Lemma 5, the definition of $y(\tilde{p}) = -[p_n(\tilde{p})]^{-1}\tilde{p}$, and because all the matrices are diagonal and thus commute.

The second summand in (18) reduces by $\mathsf{D}y(\tilde{p}) = -\frac{1}{p_n(\tilde{p})}Y(\tilde{p})$ from Lemma 6 and $\tau_\eta = E_\eta \circ \tilde{\ell}$:

$$\begin{aligned}\mathsf{D}\tau_\eta^{-1}(\tilde{\phi}) &= \left[\mathsf{D}E_\eta(\lambda)\cdot \mathsf{D}\tilde{\ell}(\tilde{p})\right]^{-1} \\ &= \left[\mathsf{D}E_\eta(\lambda)\cdot (Y(\tilde{p})')^{-1}\cdot \mathsf{H}\underline{\tilde{L}}(\tilde{p})\right]^{-1} \\ &= \left[\mathsf{H}\underline{\tilde{L}}(\tilde{p})\right]^{-1}\cdot Y(\tilde{p})'\cdot \mathsf{D}E_\eta^{-1}(\lambda).\end{aligned}$$

Substituting these into (18) gives

$$\mathsf{H}g_\eta(\tilde{\phi}) = -\frac{\eta}{p_n(\tilde{p})}A_\eta(\tilde{\phi})\cdot \operatorname{diag}(\tilde{p})\cdot A_\eta(\tilde{\phi}) - \frac{1}{p_n(\tilde{p})}A_\eta(\tilde{\phi})'\cdot Y(\tilde{p})\cdot\left[\mathsf{H}\underline{\tilde{L}}(\tilde{p})\right]^{-1}\cdot Y(\tilde{p})'\cdot A_\eta(\tilde{\phi}),$$

which can be factored into the required result. ■

We can now use the last two lemmata to express the function Γ_η in terms of the Hessian of the Bayes risk functions for the specified loss ℓ and the log loss.

Lemma 9 *Suppose a loss ℓ satisfies Condition A. Then for $\eta > 0$ the matrix-valued function Γ_η satisfies, for all $\tilde{\phi} \in \operatorname{int}(\tilde{\Phi}_\eta)$ and $\tilde{p} = \tau_\eta^{-1}(\tilde{\phi})$,*

$$\Gamma_\eta(\tilde{\phi}) = \frac{1}{p_n}A_\eta(\tilde{\phi})'\cdot Y(\tilde{p})\cdot\left[\left[\mathsf{H}\underline{\tilde{L}}(\tilde{p})\right]^{-1} - \eta\left[\mathsf{H}\underline{\tilde{L}}_{\log}(\tilde{p})\right]^{-1}\right]\cdot Y(\tilde{p})'\cdot A_\eta(\tilde{\phi}), \tag{19}$$

and is negative semi-definite if and only if $R(\eta,\ell,\tilde{p}) := \left[\mathsf{H}\underline{\tilde{L}}(\tilde{p})\right]^{-1} - \eta\left[\mathsf{H}\underline{\tilde{L}}_{\log}(\tilde{p})\right]^{-1}$ is negative semi-definite.

Proof Substituting the values of $\mathsf{D}g_\eta$ and $\mathsf{H}g_\eta$ from Lemma 8 into the definition of Γ_η from Lemma 7 and then using Lemma 5 and the definition of $y(\tilde{p})$, we obtain

$$\begin{aligned}\Gamma_\eta(\tilde{\phi}) &= \eta A_\eta(\tilde{\phi})'\cdot y(\tilde{p})\cdot y(\tilde{p})'\cdot A_\eta(\tilde{\phi}) \qquad (20)\\ &\quad + \frac{1}{p_n(\tilde{p})}A_\eta(\tilde{\phi})'\cdot\left[\eta\operatorname{diag}(\tilde{p}) + Y(\tilde{p})\cdot\left[\mathsf{H}\underline{\tilde{L}}(\tilde{p})\right]^{-1}\cdot Y(\tilde{p})'\right]\cdot A_\eta(\tilde{\phi}) \\ &= \frac{1}{p_n}A_\eta(\tilde{\phi})'\cdot\left[\eta\frac{1}{p_n}\tilde{p}\cdot\tilde{p}' + \eta\operatorname{diag}(\tilde{p}) + Y(\tilde{p})\cdot\left[\mathsf{H}\underline{\tilde{L}}(\tilde{p})\right]^{-1}\cdot Y(\tilde{p})'\right]\cdot A_\eta(\tilde{\phi}). \qquad (21)\end{aligned}$$

Using Lemma 6 we then see that

$$\begin{aligned}-Y(\tilde{p})\cdot\left[\mathsf{H}\underline{\tilde{L}}_{\log}(\tilde{p})\right]^{-1}\cdot Y(\tilde{p})' &= -Y(\tilde{p})\cdot\left[-Y(\tilde{p})'\operatorname{diag}(\tilde{p})^{-1}\right]^{-1}\cdot Y(\tilde{p})' \\ &= Y(\tilde{p})\cdot\operatorname{diag}(\tilde{p})\cdot(Y(\tilde{p})')^{-1}\cdot Y(\tilde{p})' \\ &= (I_{n-1} + \frac{1}{p_n}\mathbb{1}_{n-1}\tilde{p}')\cdot\operatorname{diag}(\tilde{p}) \\ &= \operatorname{diag}(\tilde{p}) + \frac{1}{p_n}\tilde{p}\cdot\tilde{p}'.\end{aligned}$$

Substituting this for the appropriate terms in (21) gives

$$\Gamma_\eta(\tilde{\phi}) = \frac{1}{p_n} A_\eta(\tilde{\phi})' \cdot \left[Y(\tilde{p}) \cdot \left[\mathsf{H}\underline{\tilde{L}}(\tilde{p}) \right]^{-1} \cdot Y(\tilde{p})' - \eta Y(\tilde{p}) \cdot \left[\mathsf{H}\underline{\tilde{L}}_{\log}(\tilde{p}) \right]^{-1} \cdot Y(\tilde{p})' \right] \cdot A_\eta(\tilde{\phi})$$

which equals (19).

Since $\Gamma_\eta = [p_n]^{-1} BRB'$ where $B = A_\eta(\tilde{\phi})' Y(\tilde{p})$ and $R = R(\eta, \ell, \tilde{p})$ the definition of negative semi-definiteness and the positivity of p_n means we need to show that $\forall x : x'\Gamma_\eta x \leq 0 \iff \forall y : y'Ry \leq 0$. It suffices to show that B is invertible, since we can let $y = Bx$ to establish the equivalence. The matrix $A_\eta(\tilde{\phi})$ is invertible since, by definition, $A_\eta(\tilde{\phi}) = \mathsf{D}E_\eta^{-1}(\tilde{\phi}) = -\eta^{-1}[\operatorname{diag}(\tilde{\phi})]^{-1}$ by Lemma 5 and so has matrix inverse $-\eta \operatorname{diag}(\tilde{\phi})$. The matrix $Y(\tilde{p})$ is invertible by Lemma 8. Thus, B is invertible because it is the product of two invertible matrices. ■

The above arguments result in a characterisation of the concavity of the function f_η (via its Hessian)—and hence the convexity of the η-exponentiated superprediction set—in terms of the Hessian of the Bayes risk function of the loss ℓ and the log loss $\ell_{\log}$. As in the binary case (cf. (6)), this means we are now able to specify the mixability constant η_ℓ in terms of the curvature $\mathsf{H}\underline{\tilde{L}}$ of the Bayes risk for ℓ relative to the curvature $\mathsf{H}\underline{\tilde{L}}_{\log}$ of the Bayes risk for log loss.

Theorem 10 *Suppose a loss ℓ satisfies Condition A. Let $\underline{\tilde{L}}(\tilde{p})$ be the Bayes risk for ℓ and $\underline{\tilde{L}}_{\log}(\tilde{p})$ be the Bayes risk for the log loss. Then the following statements are equivalent:*

(i.) ℓ is η-mixable;

(ii.) $\eta \mathsf{H}\underline{\tilde{L}}(\tilde{p}) \succcurlyeq \mathsf{H}\underline{\tilde{L}}_{\log}(\tilde{p})$ for all $\tilde{p} \in \operatorname{int}(\tilde{\Delta}^n)$;

(iii.) $\eta \underline{L}(p) - \underline{L}_{\log}(p)$ is convex on $\operatorname{rel\,int}(\Delta^n)$;

(iv.) $\eta \underline{\tilde{L}}(\tilde{p}) - \underline{\tilde{L}}_{\log}(\tilde{p})$ is convex on $\operatorname{int}(\tilde{\Delta}^n)$.

Note that the largest η that satisfies any one of (i)–(iv) is the mixability constant for the loss. For example,

$$\eta_\ell = \max\{\eta \geq 0 \colon \forall \tilde{p} \in \operatorname{int}(\tilde{\Delta}^n)\ ,\ \eta \underline{\tilde{L}}(\tilde{p}) \succcurlyeq \mathsf{H}\underline{\tilde{L}}_{\log}(\tilde{p})\}. \tag{22}$$

Proof The case $\eta = 0$ is trivial, so suppose $\eta > 0$. Then by Lemmas 7 and 9 we know $\mathsf{H}f_\eta(\tilde{p}) \preccurlyeq 0 \iff R(\eta, \ell, \tilde{p}) \preccurlyeq 0$. By Lemma 6, $\mathsf{H}\underline{\tilde{L}}(\tilde{p}) \prec 0$ and $\mathsf{H}\underline{\tilde{L}}_{\log}(\tilde{p}) \prec 0$ for all $\tilde{p}$ and so we can use the fact that for positive definite matrices A and B we have $A \succcurlyeq B \iff B^{-1} \succcurlyeq A^{-1}$ (Horn and Johnson, 1985, Corollary 7.7.4). This means $R(\eta, \ell, \tilde{p}) \preccurlyeq 0 \iff \mathsf{H}\underline{\tilde{L}}(\tilde{p})^{-1} \preccurlyeq \eta \mathsf{H}\underline{\tilde{L}}_{\log}(\tilde{p})^{-1} \iff \eta^{-1}\mathsf{H}\underline{\tilde{L}}_{\log}(\tilde{p}) \preccurlyeq \mathsf{H}\underline{\tilde{L}}(\tilde{p}) \iff \eta\mathsf{H}\underline{\tilde{L}}(\tilde{p}) \succcurlyeq \mathsf{H}\underline{\tilde{L}}_{\log}(\tilde{p})$. Therefore f_η is concave at $\tilde{p}$ if and only if $\eta \mathsf{H}\underline{\tilde{L}}(\tilde{p}) \succcurlyeq \mathsf{H}\underline{\tilde{L}}_{\log}(\tilde{p})$. And as concavity of f_η was equivalent to η-mixability, this establishes equivalence of (i) and (ii).

Since $\eta \mathsf{H}\underline{\tilde{L}}(\tilde{p}) \succcurlyeq \mathsf{H}\underline{\tilde{L}}_{\log}(\tilde{p}) \iff \mathsf{H}\big(\eta\underline{\tilde{L}}(\tilde{p}) - \underline{\tilde{L}}_{\log}(\tilde{p})\big) \succcurlyeq 0$, Equivalence of (ii) and (iv) follows from the fact that positive semi-definiteness of the Hessian of a function on an open set is equivalent to convexity of the function (Hiriart-Urruty and Lemaréchal, 1993). Finally, equivalence of (iv) and (iii) follows by linearity of the map $p_n(\tilde{p}) = 1 - \sum_{i=1}^{n-1} \tilde{p}_i$. ■

The lemma allows one to derive η-mixability of an average of two η-mixable proper losses that satisfy its conditions:

Corollary 11 *Suppose ℓ_A and ℓ_B are two η-mixable losses that satisfy Condition A. Then, for any $\lambda \in (0,1)$, the loss $\ell = (1-\lambda)\ell_A + \lambda\ell_B$ is also η-mixable.*

Proof Clearly ℓ is continuous and continuously differentiable. And because properness of ℓ_A and ℓ_B implies that $\underline{L}_\ell(p) = (1-\lambda)\underline{L}_{\ell_A}(p) + \lambda\underline{L}_{\ell_B}(p)$, it is also strictly proper. Thus Theorem 10 applies to ℓ, and we just need to verify that $\eta\underline{L}_\ell(p) - \underline{L}_{\log}(p)$ is convex. Noting that

$$\eta\underline{L}_\ell(p) - \underline{L}_{\log}(p) = (1-\lambda)\Big(\eta\underline{L}_{\ell_A}(p) - \underline{L}_{\log}(p)\Big) + \lambda\Big(\eta\underline{L}_{\ell_B}(p) - \underline{L}_{\log}(p)\Big)$$

is a convex combination of two convex functions, the result follows. ∎

One may wonder which loss is the most mixable. In the following we derive a straightforward result that shows the (perhaps unsurprising) answer is log loss. Let $e_i \in \Delta^n$ denote the point-mass on the i-th outcome. Then we call a proper loss *fair* if $L(e_i, e_i) = \underline{L}(e_i) = 0$ for all i (Reid and Williamson, 2011). That is, if one is certain that outcome i will occur and this is correct, then it is only fair if one incurs no loss. Any loss can be made fair by subtracting an affine function from its Bayes risk. Note that Theorem 10 implies that all such losses have the same mixability constant (provided the conditions of the theorem are satisfied). We will call a proper loss *normalised* if it is fair and $\max_{p\in\Delta^n} \underline{L}(p) = 1$. If a fair proper loss is not normalised, one may normalise it by dividing the loss on all outcomes by $\max_{p\in\Delta^n} \underline{L}(p)$. This scales up the mixability constant by $\max_{p\in\Delta^n} \underline{L}(p)$. For example, log loss is fair, but in order to normalise it, one needs to divide by $\max_{p\in\Delta^n} \underline{L}_{\log}(p) = \log(n)$, and the mixability constant η_ℓ for the resulting loss is $\log(n)$.

Corollary 12 *Suppose a loss ℓ satisfies Condition A. Then, if ℓ is normalised and $\underline{L}(p)$ is continuous, it can only be η-mixable for $\eta \leq \log(n)$. This bound is achieved if ℓ is the normalised log loss.*

Proof As $\underline{L}(p)$ is continuous and has a compact domain, there exists a $p^* = \arg\max_{p\in\Delta^n} \underline{L}(p)$ that achieves its maximum, which is 1 by assumption. Now by Theorem 10, η-mixability implies convexity of $\eta\underline{L}(p) - \underline{L}_{\log}(p)$ on $\operatorname{int}(\Delta^n)$, which extends to convexity on Δ^n by continuity of $\underline{L}(p)$ and $\underline{L}_{\log}(p)$, and hence

$$\begin{aligned} 0 = \mathbb{E}_{i\sim p^*}\Big[\eta\underline{L}(e_i) - \underline{L}_{\log}(e_i)\Big] \geq \eta\underline{L}(p^*) - \underline{L}_{\log}(p^*) = \eta - \underline{L}_{\log}(p^*) \\ \eta \leq \underline{L}_{\log}(p^*) \leq \underline{L}_{\log}\big(\tfrac{1}{n},\ldots,\tfrac{1}{n}\big) = \log(n), \end{aligned}$$

where the first equality follows from fairness of ℓ and log loss. ∎

The mixability constant can also be expressed in terms of the maximal eigenvalue of the "ratio" of the Hessian matrices for the Bayes risk for log loss and the loss in question. In the following, $\lambda_i(A)$ will denote the ith largest (possibly repeated) eigenvalue of the $n \times n$ symmetric matrix A. That is, $\lambda_{\min}(A) := \lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n =: \lambda_{\max}(A)$ where each $\lambda_i(A)$ satisfies $|A - \lambda_i(A)I| = 0$.

Theorem 13 *Suppose a loss ℓ satisfies Condition A. Then its mixability constant is*

$$\boxed{\eta_\ell = \inf_{\tilde{p}\in\mathrm{int}(\tilde{\Delta}^n)} \lambda_{\max}\left((\mathsf{H}\underline{\tilde{L}}(\tilde{p}))^{-1}\cdot\mathsf{H}\underline{\tilde{L}}_{\log}(\tilde{p})\right).} \tag{23}$$

Equation 23 reduces to (6) when $n=2$ since the maximum eigenvalue of a 1×1 matrix is simply its single entry.

Proof For $\tilde{p}\in\mathrm{int}(\tilde{\Delta}^n)$, we define $C_\eta(\tilde{p}) := \eta\mathsf{H}\underline{\tilde{L}}(\tilde{p})-\mathsf{H}\underline{\tilde{L}}_{\log}(\tilde{p})$ and $\rho(\tilde{p}) := \mathsf{H}\underline{\tilde{L}}(\tilde{p})^{-1}\cdot\mathsf{H}\underline{\tilde{L}}_{\log}(\tilde{p})$ and first show that zero is an eigenvalue of $C_\eta(\tilde{p})$ if and only if η is an eigenvalue of $\rho(\tilde{p})$. This can be seen since $\mathsf{H}\underline{\tilde{L}}(\tilde{p})$ is invertible (Lemma 6) so

$$\begin{aligned}&|C_\eta(\tilde{p})-0I|=0 \iff |\eta\mathsf{H}\underline{\tilde{L}}(\tilde{p})-\mathsf{H}\underline{\tilde{L}}_{\log}(\tilde{p})|=0 \iff |\mathsf{H}\underline{\tilde{L}}(\tilde{p})^{-1}||\eta\mathsf{H}\underline{\tilde{L}}(\tilde{p})-\mathsf{H}\underline{\tilde{L}}_{\log}(\tilde{p})|=0\\ &\iff \left|\mathsf{H}\underline{\tilde{L}}(\tilde{p})^{-1}\cdot\left[\eta\mathsf{H}\underline{\tilde{L}}(\tilde{p})-\mathsf{H}\underline{\tilde{L}}_{\log}(\tilde{p})\right]\right|=0 \iff |\eta I-\mathsf{H}\underline{\tilde{L}}(\tilde{p})^{-1}\cdot\mathsf{H}\underline{\tilde{L}}_{\log}(\tilde{p})|=0.\end{aligned}$$

Since a symmetric matrix is positive semidefinite if and only if all its eigenvalues are non-negative it must be the case that if $\lambda_{\min}(C_\eta(\tilde{p}))\ge 0$ then $C_\eta(\tilde{p})\succcurlyeq 0$ since every other eigenvalue is bigger than the minimum one. Conversely, if $C_\eta(\tilde{p})\not\succcurlyeq 0$ then at least one eigenvalue must be negative, thus the smallest eigenvalue must be negative. Thus, $\lambda_{\min}(C_\eta(\tilde{p}))\ge 0 \iff C_\eta(\tilde{p})\succcurlyeq 0$. Now define $\eta(\tilde{p}) := \sup\{\eta>0 : C_\eta(\tilde{p})\succcurlyeq 0\} = \sup\{\eta > 0 : \lambda_{\min}(C_\eta(\tilde{p}))\ge 0\}$. We show that for each $\tilde{p}$ the function $\eta\mapsto\lambda_{\min}(C_\eta(\tilde{p}))$ is continuous and only has a single root. First, continuity is because the entries of $C_\eta(\tilde{p})$ are continuous in η for each $\tilde{p}$ and eigenvalues are continuous functions of their matrix's entries (Horn and Johnson, 1985, Appendix D). Second, as a function of its matrix arguments, the minimum eigenvalue $\lambda_{\min}$ is known to be concave (Magnus and Neudecker, 1999, §11.6). Thus, for any fixed $\tilde{p}$, its restriction to the convex set of matrices $\{C_\eta(\tilde{p}) : \eta>0\}$ is also concave in its entries and so in η. Since $C_0(\tilde{p}) = -\mathsf{H}\underline{\tilde{L}}_{\log}(\tilde{p})$ is positive definite for every $\tilde{p}$ (Lemma 6) we have $\lambda_{\min}(C_0(\tilde{p}))>0$ and so, by the concavity of the map $\eta\mapsto\lambda_{\min}(C_\eta(\tilde{p}))$, there can be only one $\eta>0$ for which $\lambda_{\min}(C_\eta(\tilde{p}))=0$ and by continuity it must be largest non-negative one, that is, $\eta(\tilde{p})$.

Thus $\eta(\tilde{p}) = \sup\{\eta>0 : \lambda_{\min}(C_\eta(\tilde{p}))=0\} = \sup\{\eta>0 : \eta \text{ is an eigenvalue of } \rho(\tilde{p})\} = \lambda_{\max}(\rho(\tilde{p}))$. Now let $\eta^* := \inf_{\tilde{p}\in\mathrm{int}(\tilde{\Delta}^n)}\eta(\tilde{p}) = \inf_{\tilde{p}\in\mathrm{int}(\tilde{\Delta}^n)}\lambda_{\max}(\rho(\tilde{p}))$. We now claim that $C_{\eta^*}(\tilde{p})\succcurlyeq 0$ for all $\tilde{p}$ since if there was some $\tilde{q}\in\tilde{\Delta}^n$ such that $C_{\eta^*}(\tilde{q})\not\succcurlyeq 0$ we would have $\eta(\tilde{q})<\eta^*$ since $\eta\mapsto\lambda_{\min}(C_\eta(\tilde{q}))$ only has a single root—a contradiction. Thus, since we have shown η^* is the largest η such that $C_{\eta^*}(\tilde{p})\succcurlyeq 0$ it must be η_ℓ, by Theorem 10, as required. ∎

The following Corollary gives an expression for η_ℓ that is simpler than (23), generalising (7) from the binary case.

Corollary 14 *Suppose ℓ satisfies Condition A. Then its mixability constant satisfies*

$$\boxed{\frac{-1}{\eta_\ell} = \inf_{\tilde{p}\in\mathrm{int}(\tilde{\Delta}^n)}\lambda_{\max}\left(\mathrm{diag}(\tilde{p})\cdot\mathsf{D}\tilde{\ell}(\tilde{p})\right).} \tag{24}$$

Proof Theorem 13 combined with Lemma 6 allows us to write

$$\begin{aligned}
\eta_\ell &= \inf_{\tilde{p}\in \operatorname{int}\tilde{\Delta}^n} \lambda_{\max}\left(\left(Y(\tilde{p})'\cdot \mathsf{D}\tilde{\ell}(\tilde{p})\right)^{-1}\cdot\left(Y(\tilde{p})'\cdot \mathsf{D}\tilde{\ell}_{\log}(\tilde{p})\right)\right)\\
&= \inf_{\tilde{p}\in \operatorname{int}\tilde{\Delta}^n} \lambda_{\max}\left((\mathsf{D}\tilde{\ell}(\tilde{p}))^{-1}\cdot \mathsf{D}\tilde{\ell}_{\log}(\tilde{p})\right)\\
&= \inf_{\tilde{p}\in \operatorname{int}\tilde{\Delta}^n} \lambda_{\max}\left((\mathsf{D}\tilde{\ell}(\tilde{p}))^{-1}\cdot \operatorname{diag}(-1/p_i)_{i=1}^{n-1}\right).\\
&= -\sup_{\tilde{p}\in \operatorname{int}\tilde{\Delta}^n} \lambda_{\min}\left((\mathsf{D}\tilde{\ell}(\tilde{p}))^{-1}\cdot \operatorname{diag}(1/p_i)_{i=1}^{n-1}\right)
\end{aligned}$$

and thus (24) follows since $\lambda_{\max}(A) = 1/\lambda_{\min}(A^{-1})$. ■

5. Mixability of the Brier Score

We will now apply the results from the previous section to show that the multiclass Brier score is mixable with mixability constant 1, as first proved by Vovk and Zhdanov (2009). The n-class Brier score is[1]

$$\ell_{\text{Brier}}(y,\hat{p}) = \sum_{i=1}^{n}([\![y_i = 1]\!] - \hat{p}_i)^2,$$

where $y\in\{0,1\}^n$ and $\hat{p}\in\Delta^n$. Thus

$$L_{\text{Brier}}(p,\hat{p}) = \sum_{i=1}^{n}\mathbb{E}_{\mathsf{Y}\sim p}([\![\mathsf{Y}_i = 1]\!] - \hat{p}_i)^2 = \sum_{i=1}^{n}(p_i - 2p_i\hat{p}_i + \hat{p}_i^2).$$

Hence $\underline{L}_{\text{Brier}}(p) = L_{\text{Brier}}(p,p) = \sum_{i=1}^{n}(p_i - 2p_ip_i + p_i^2) = 1 - \sum_{i=1}^{n} p_i^2$ since $\sum_{i=1}^{n} p_i = 1$, and $\tilde{\underline{L}}_{\text{Brier}}(\tilde{p}) = 1 - \sum_{i=1}^{n-1} p_i^2 - \left(1 - \sum_{i=1}^{n-1} p_i\right)^2$.

Theorem 15 *The Brier score is mixable, with mixability constant $\eta_{\text{Brier}} = 1$.*

Proof It can be verified by basic calculus that ℓ_{Brier} is continuous and continuously differentiable on $\operatorname{int}(\tilde{\Delta}^n)$. To see that it is strictly proper, note that for $\hat{p}\neq p$ the inequality $L_{\text{Brier}}(p,\hat{p}) > \underline{L}_{\text{Brier}}(p)$ is equivalent to

$$\sum_{i=1}^{n}(p_i^2 - 2p_i\hat{p}_i + \hat{p}_i^2) > 0 \qquad \text{or} \qquad \sum_{i=1}^{n}(p_i - \hat{p}_i)^2 > 0,$$

and the latter inequality is true because $p_i \neq \hat{p}_i$ for at least one i by assumption. Hence the conditions of Theorem 10 are satisfied.

1. This is the definition used by Vovk and Zhdanov (2009). Cesa-Bianchi and Lugosi (2006) use a different definition (for the binary case) which differs by a constant. Their definition results in $\tilde{\underline{L}}(\tilde{p}) = \tilde{p}(1-\tilde{p})$ and thus $\tilde{\underline{L}}''(\tilde{p}) = -2$. If $n=2$, then $\tilde{\underline{L}}_{\text{Brier}}$ as defined above leads to $\tilde{\underline{L}}''_{\text{Brier}}(\tilde{p}) = \mathsf{H}\tilde{\underline{L}}_{\text{Brier}}(\tilde{p}) = -2(1+1) = -4$.

We will first prove that $\eta_{\text{Brier}} \leq 1$ by showing that convexity of $\eta\underline{\tilde{L}}_{\text{Brier}}(\tilde{p}) - \underline{\tilde{L}}_{\log}(\tilde{p})$ on $\text{int}(\tilde{\Delta}^n)$ implies $\eta \leq 1$. If $\eta\underline{\tilde{L}}_{\text{Brier}}(\tilde{p}) - \underline{\tilde{L}}_{\log}(\tilde{p})$ is convex, then it is convex as a function of p_1 when all other elements of $\tilde{p}$ are kept fixed. Consequently, the second derivative with respect to p_1 must be nonnegative:

$$0 \leq \frac{\partial^2}{\partial p_1^2}\Big(\eta\underline{\tilde{L}}_{\text{Brier}}(\tilde{p}) - \underline{\tilde{L}}_{\log}(\tilde{p})\Big) = \frac{1}{p_1} + \frac{1}{p_n} - 4\eta.$$

By letting p_1 and p_n both tend to $1/2$, it follows that $\eta \leq 1$.

It remains to show that $\eta_{\text{Brier}} \geq 1$. By Theorem 10 it is sufficient to show that, for $\eta \leq 1$, $\eta\underline{L}_{\text{Brier}}(p) - \underline{L}_{\log}(p)$ is convex on $\operatorname{rel\,int}(\Delta^n)$. We proceed by induction. For $n = 1$, the required convexity holds trivially. Suppose the lemma holds for $n-1$, and let $f_n(p_1,\ldots,p_n) = \eta\underline{L}_{\text{Brier}}(p) - \underline{L}_{\log}(p)$ for all n. Then for $n \geq 2$

$$f_n(p_1,\ldots,p_n) = f_{n-1}(p_1+p_2,p_3,\ldots,p_n) + g(p_1,p_2),$$

where $g(p_1,p_2) = -\eta p_1^2 - \eta p_2^2 + \eta(p_1+p_2)^2 + p_1\ln p_1 + p_2 \ln p_2 - (p_1+p_2)\ln(p_1+p_2)$. Since f_{n-1} is convex by inductive assumption and the sum of two convex functions is convex, it is therefore sufficient to show that $g(p_1,p_2)$ is convex or, equivalently, that its Hessian is positive semi-definite. Abbreviating $q = p_1 + p_2$, we have that

$$\mathsf{H}g(p_1,p_2) = \begin{pmatrix} 1/p_1 - 1/q & 2\eta - 1/q \\ 2\eta - 1/q & 1/p_2 - 1/q \end{pmatrix}.$$

A 2×2 matrix is positive semi-definite if its trace and determinant are both non-negative, which is easily verified in the present case: $\mathsf{Tr}(\mathsf{H}g(p_1,p_2)) = 1/p_1 + 1/p_2 - 2/q \geq 0$ and $|\mathsf{H}g(p_1,p_2)| = (1/p_1 - 1/q)(1/p_2 - 1/q) - (2\eta - 1/q)^2$, which is non-negative if

$$\begin{aligned} \frac{1}{p_1p_2} - \frac{1}{p_1 q} - \frac{1}{p_2 q} &\geq 4\eta^2 - \frac{4\eta}{q} \\ 0 &\geq 4\eta^2 q - 4\eta \\ \eta q &\leq 1. \end{aligned}$$

Since $q = p_1 + p_2 \leq 1$, this inequality holds for $\eta \leq 1$, which shows that $g(p_1,p_2)$ is convex and thereby completes the proof. ■

6. Discussion

In combination with the existing results on mixability, our result bounds the performance of certain predictors in terms of the Hessian of the Bayes risk $\mathsf{H}\underline{L}(p)$ which depends on the choice of loss function.

The main result is stated for proper losses. However it turns out that this is not really a limitation[2]. Suppose $\ell_{\text{imp}} \colon \mathcal{V} \to [0,+\infty]^n$ is an *improper* loss (i.e. not proper). Let $L_{\text{imp}} \colon \Delta^n \times \mathcal{V} \to [0,+\infty]$ and $\underline{L}_{\text{imp}} \colon \Delta^n \to [0,+\infty]$ denote the corresponding conditional

2. We thank a COLT2011 referee for pointing this out by referring us to Chernov et al. (2010).

risk and conditional Bayes risk respectively. Let $\psi_{\mathrm{imp}}\colon \Delta^n \to \mathcal{V}$ be a *reference link* (cf. Reid and Williamson (2010))—that is, a (possibly non-unique) function satisfying

$$L_{\mathrm{imp}}(p, \psi_{\mathrm{imp}}(p)) = \underline{L}_{\mathrm{imp}}(p).$$

This function can be seen as one which "calibrates" ℓ_{imp} by returning $\psi_{\mathrm{imp}}(p)$, the best possible prediction under labels distributed by p. Let

$$\ell(y, q) := \ell_{\mathrm{imp}}(y, \psi_{\mathrm{imp}}(q)), \quad y \in [n],\ q \in \Delta^n \tag{25}$$

and thus

$$L(p, q) = L_{\mathrm{imp}}(p, \psi_{\mathrm{imp}}(q)), \quad p, q \in \Delta^n.$$

We claim that ℓ is proper. It suffices to show that $p \in \arg\min_{q\in\Delta^n} L(p, q)$ which we demonstrate by contradiction. Suppose that for arbitrary $p \in \Delta^n$ there exists $p^* \neq p$ such that

$$\begin{aligned} & L(p, p^*) && < L(p, p) \\ \Leftrightarrow \quad & L_{\mathrm{imp}}(p, \psi_{\mathrm{imp}}(p^*)) && < L_{\mathrm{imp}}(p, \psi_{\mathrm{imp}}(p)) = \underline{L}_{\mathrm{imp}}(p) = \min_{v\in\mathcal{V}} L_{\mathrm{imp}}(p, v), \end{aligned}$$

which is indeed a contradiction. Thus ℓ defined by (25) is proper. Observe too that $\underline{L}_{\mathrm{imp}}(p) = L_{\mathrm{imp}}(p, \psi_{\mathrm{imp}}(p)) = L(p, p) = \underline{L}(p)$. Thus the method of identifying the conditional Bayes risk of an improper loss with that of a proper loss (confer Grünwald and Dawid (2004, §3.4) and Chernov et al. (2010)) is equivalent to the above use of the reference link.

We now briefly relate our result to recent work by Abernethy et al. (2009). They formulate the problem slightly differently. They do not restrict themselves to proper losses and so the predictions are not restricted to the simplex. This means it is not necessary to go to a submanifold in order for derivatives to be well defined. (It may well be that one can avoid the explicit projection down to $\tilde{\Delta}^n$ using the intrinsic methods of differential geometry (Thorpe, 1979); we have been unable to prove our result using that machinery.)

Abernethy et al. (2009) have developed their own bounds on cumulative loss in terms of the α-flatness (defined below) of $\underline{L}(p)$. They show that α-flatness is implied by strong convexity of the loss ℓ. The duality between the loss surface and Bayes risk that they established through the use of support functions can also be seen in Lemma 6 in the relationship between the Hessian of $\tilde{\underline{L}}$ and the derivative of $\tilde{\ell}$. Although it is obscured somewhat due to our use of functions of $\tilde{p}$, this relationship is due to the properness of ℓ guaranteeing that ℓ^{-1} is the (homogeneously extended) Gauss map for the surface $\tilde{\underline{L}}$. Below we point out the relationship between α-flatness and the positive definiteness of $\mathsf{H}\underline{L}(p)$ (we stress that in our work we used $\mathsf{H}\tilde{\underline{L}}(\tilde{p})$). Whilst the two results are not precisely comparable, the comparison below seems to suggest that the condition of Abernethy et al. (2009) is stronger than necessary.

Suppose $\mathcal{X}$ is a Banach space with norm $\|\cdot\|$. Given a real number $\alpha > 0$ and a function $\sigma : \mathbb{R}_+ \to [0, \infty]$ such that $\sigma(0) = 0$, a convex function $f\colon \mathcal{X} \to \mathbb{R}$ is said to be $(\alpha, \sigma, \|\cdot\|)$-*flat* (or $(\alpha, \sigma, \|\cdot\|)$-smooth)[3] if for all $x, x_0 \in \mathcal{X}$,

$$f(x) - f(x_0) \le \mathsf{D}f(x_0)\cdot(x - x_0) + \alpha\sigma(\|x - x_0\|). \tag{26}$$

3. This definition is redundantly parametrised — $(\alpha, \sigma, \|\cdot\|)$-flatness is equivalent to $(1, \alpha\sigma, \|\cdot\|)$-flatness. We have defined the notion as above in order to relate to existing definitions and because in fact one

A concave function g is flat if the convex function $-g$ is flat. When $\|\cdot\| = \|\cdot\|_2$, and $\sigma(x) = x^2$, it is known (Hiriart-Urruty and Lemaréchal, 1993) that for $\alpha > 0$, f is $(\alpha, x \mapsto x^2, \|\cdot\|_2)$-flat if and only if $f - \alpha\|\cdot\|^2$ is concave. Thus f is α-flat if and only if $\mathsf{H}(f - \alpha\|\cdot\|^2)$ is negative semi-definite, which is equivalent to $\mathsf{H}f - 2\alpha I \preccurlyeq 0 \iff \mathsf{H}f \preccurlyeq 2\alpha I$.

Abernethy et al. (2009) show that if $\underline{L}$ is $(\alpha, x \mapsto x^2, \|\cdot\|_1)$-flat, then the minimax regret for a prediction game with T rounds is bounded above by $4\alpha \log T$. It is thus of interest to relate their assumption on $\underline{L}$ to the mixability condition (which guarantees constant regret, in the prediction with experts setting).

In contrast to the above quoted result for $\|\cdot\|_2$, we only get a one-way implication for $\|\cdot\|_1$.

Lemma 16 *If $f - \alpha\|\cdot\|_1^2$ is concave on $\mathbb{R}_+^n$ then f is $(\alpha, x \mapsto x^2, \|\cdot\|_1)$-flat.*

Proof It is known (Hiriart-Urruty and Lemaréchal, 1993, page 183) that a function h is concave if and only if $h(x) \le h(x_0) + \mathsf{D}h(x_0) \cdot (x - x_0)$ for all x, x_0. Hence $f - \alpha\|\cdot\|_1^2$ is concave on $\mathbb{R}_+^n$ if and only if for all $x, x_0 \in \mathbb{R}_+^n$,

$$
\begin{aligned}
& f(x) - \alpha\|x\|_1^2 \;\le\; f(x_0) - \alpha\|x_0\|_1^2 + \mathsf{D}(f(x_0) - \alpha\|x_0\|_1^2) \cdot (x - x_0) \\
\Leftrightarrow\;\; & f(x) - f(x_0) \;\le\; \alpha\|x\|_1^2 - \alpha\|x_0\|_1^2 + \mathsf{D}f(x_0) \cdot (x - x_0) - \alpha\mathsf{D}(\|x_0\|_1^2) \cdot (x - x_0).
\end{aligned}
$$

Since $\mathsf{D}(\|x_0\|_1^2) = 2\|x_0\|_1 \mathbb{1}$ and $2\|x_0\|_1(\mathbb{1} \cdot (x - x_0)) = 2\|x_0\|_1(\|x\|_1 - \|x_0\|_1) = 2\|x_0\|_1\|x\|_1 - 2\|x_0\|_1^2$,

$$
\begin{aligned}
\Leftrightarrow\;\; & f(x) - f(x_0) \;\le\; \alpha\left(\|x\|_1^2 + \|x_0\|^2 - 2\|x_0\|_1\|x\|_1\right) + \mathsf{D}f(x_0) \cdot (x - x_0) \\
\Leftrightarrow\;\; & f(x) - f(x_0) \;\le\; Df(x_0) \cdot (x - x_0) + \alpha\left(\|x\|_1 - \|x_0\|_1\right)^2
\end{aligned}
$$

By the reverse triangle inequality $\|x - x_0\|_1 \ge |\|x\|_1 - \|x_0\|_1| \ge \|x\|_1 - \|x_0\|_1$ and thus $\|x - x_0\|_1^2 \ge (\|x\|_1 - \|x_0\|_1)^2$, which gives

$$
\Rightarrow\;\; f(x) - f(x_0) \;\le\; \mathsf{D}f(x_0) \cdot (x - x_0) + \alpha\|x - x_0\|_1^2.
$$

■

Now $f - \alpha\|\cdot\|_1^2$ is concave if and only if $\mathsf{H}(f - \alpha\|\cdot\|_1^2) \preccurlyeq 0$. We have (again for $x \in \mathbb{R}_+^n$) $\mathsf{H}(f - \alpha\|\cdot\|_1^2) = \mathsf{H}f - \alpha\mathsf{H}(\|\cdot\|_1^2)$. Let $\phi(x) = \|x\|_1^2$. Then $\mathsf{D}\phi(x) = 2\|x\|_1\mathsf{D}(\|x\|_1) = 2\|x\|_1\mathbb{1}$. Hence $\mathsf{H}\phi(x) = \mathsf{D}\left(\mathsf{D}\phi(x)\right)' = \mathsf{D}(2\|x\|_1\mathbb{1}') = 2\mathbb{1} \cdot \mathbb{1}'$. Thus $(\alpha, x \mapsto x^2, \|\cdot\|_1)$-flatness of $\underline{L}$ is implied by negative semi-definiteness of the Hessian of $\underline{L}$ relative to $2\alpha\mathbb{1} \cdot \mathbb{1}'$, instead of (see Theorem 10, part ii) $\underline{L}_{\log}$. The comparison with log loss is not that surprising in light of the observations regarding mixability by Grünwald (2007, §17.9).

The above analysis is not entirely satisfactory for three reasons: 1) Lemma 16 does not characterise the flatness condition (it is only a sufficient condition); 2) we have glossed over

sometimes fixes σ and then is interested in the effect of varying α. When $\sigma(x) = x^2$, Abernethy et al. (2009) and Kakade et al. (2010) call this α-flat with respect to $\|\cdot\|$. Azé and Penot (1995) and Zălinescu (1983) would say f is σ-flat with respect to an implicitly given norm if f is (in our definition) $(\alpha, \sigma, \|\cdot\|)$-flat for some $\alpha > 0$ (which in their setup is effectively bundled into σ). These differences do not matter (unless one wishes to utilise results from the earlier literature, which we do not).

the fact that in order to compute derivatives one needs to work in $\tilde{\Delta}^n$; and 3) the learning protocols for the two situations are not identical. These last two points can be potentially addressed in future work. However the first seems impossible since there can not exist a characterisation of $(\alpha, x \mapsto x^2, \|\cdot\|_1)$-flatness in terms of concavity of some function. To see this, consider the one dimensional case and suppose there was some function g such that f was flat if g was concave. Then we would require $\mathsf{D}g(x) \cdot (x - x_0) = \alpha \|x - x_0\|_1^2 \Rightarrow \mathsf{D}g(x)(x - x_0) = \alpha |x - x_0|^2 = \alpha(x - x_0)^2 \Rightarrow \mathsf{D}g(x) = \alpha(x - x_0)$ which is impossible because the left hand side $\mathsf{D}g(x)$ does not depend upon x_0. On the other hand, perhaps it is not worth further investigation since the result due to Abernethy et al. (2009) is only a *sufficient* condition for logarithmic regret.

7. Conclusion

We have characterised the mixability constant for strictly proper multiclass losses (and shown how the result also applies to improper losses). The result shows in a precise and intuitive way the effect of the choice of loss function on the performance of an aggregating forecaster and the special role played by log loss in such settings.

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Appendix A. Matrix Calculus

We adopt notation from Magnus and Neudecker (1999): I_n is the $n \times n$ identity matrix, A' is the transpose of A, the n-vector $\mathbb{1}_n := (1, \ldots, 1)'$, and $0_{n\times m}$ denotes the zero matrix with n rows and m columns. The unit n-vector $e_i^n := (0, \ldots, 0, 1, 0, \ldots, 0)'$ has a 1 in the ith coordinate and zeroes elsewhere. If $A = [a_{ij}]$ is an $n \times m$ matrix, $\operatorname{vec} A$ is the vector of columns of A stacked on top of each other. The *Kronecker product* of an $m \times n$ matrix A with a $p \times q$ matrix B is the $mp \times nq$ matrix

$$A \otimes B := \begin{pmatrix} A_{1,1}B & \cdots & A_{1,n}B \\ \vdots & \ddots & \vdots \\ A_{m,1}B & \cdots & A_{m,n}B \end{pmatrix}.$$

We use the following properties of Kronecker products (see Chapter 2 of Magnus and Neudecker (1999)): $(A \otimes B)(C \otimes D) = (AC \otimes BD)$ for all appropriately sized A, B, C, D and $(A \otimes B)^{-1} = (A^{-1} \otimes B^{-1})$ for invertible A and B.

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at c then the *partial derivative* of f_i with respect to the jth coordinate at c is denoted $\mathsf{D}_j f_i(c)$ and is often[4] also written as $[\partial f_i/\partial x_j]_{x=c}$. The $m \times n$ matrix of partial derivatives of f is the *Jacobian* of f and denoted

$$(\mathsf{D}f(c))_{i,j} := \mathsf{D}_j f_i(c) \quad \text{for } i \in [m], j \in [n].$$

The *inverse function theorem* relates the Jacobians of a function and its inverse (cf. Fleming (1977, §4.5)):

Theorem 17 *Let $S \subset \mathbb{R}^n$ be an open set and $g : S \to \mathbb{R}^n$ be a C^q function with $q \geq 1$ (i.e., continuous with at least one continuous derivative). If $\mathsf{D}g(s) \neq 0$ then: there exists an open set S_0 such that $s \in S_0$ and the restriction of g to S_0 is invertible; $g(S_0)$ is open; f, the inverse of the restriction of g to S_0, is C^q; and $\mathsf{D}f(t) = [\mathsf{D}g(s)]^{-1}$ for $t = g(s)$ and $s \in S_0$.*

If F is a matrix valued function $\mathsf{D}F(X) := \mathsf{D}f(\operatorname{vec} X)$ where $f(X) = \operatorname{vec} F(X)$.

We will require the product rule for matrix valued functions (Fackler, 2005): Suppose $f \colon \mathbb{R}^n \to \mathbb{R}^{m\times p}$, $g \colon \mathbb{R}^n \to \mathbb{R}^{p\times q}$ so that $(f \times g) \colon \mathbb{R}^n \to \mathbb{R}^{m\times q}$. Then

$$\mathsf{D}(f \times g)(x) = (g(x)' \otimes I_m) \cdot \mathsf{D}f(x) + (I_q \otimes f(x)) \cdot \mathsf{D}g(x). \tag{27}$$

The *Hessian* at $x \in X \subseteq \mathbb{R}^n$ of a real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is the $n \times n$ real, symmetric matrix of second derivatives at x

$$(\mathsf{H}f(x))_{j,k} := \mathsf{D}_{k,j} f(x) = \frac{\partial^2 f}{\partial x_k \partial x_j}.$$

Note that the derivative $D_{k,j}$ is in row j, column k. It is easy to establish that the Jacobian of the transpose of the Jacobian of f is the Hessian of f. That is,

$$\mathsf{H}f(x) = \mathsf{D}\left((\mathsf{D}f(x))'\right) \tag{28}$$

4. See Chapter 9 of Magnus and Neudecker (1999) for why the $\partial/\partial x$ notation is a poor one for multivariate differential calculus despite its popularity.

(Magnus and Neudecker, 1999, Chapter 10). If $f : X \to \mathbb{R}^m$ for $X \subseteq \mathbb{R}^n$ is a vector valued function then the Hessian of f at $x \in X$ is the $mn \times n$ matrix that consists of the Hessians of the functions f_i stacked vertically:

$$\mathsf{H}f(x) := \begin{pmatrix} \mathsf{H}f_1(x) \\ \vdots \\ \mathsf{H}f_m(x) \end{pmatrix}.$$

The following theorem regarding the chain rule for Hessian matrices can be found in (Magnus and Neudecker, 1999, pg. 110).

Theorem 18 *Let S be a subset of $\mathbb{R}^n$, and $f : S \to \mathbb{R}^m$ be twice differentiable at a point c in the interior of S. Let T be a subset of $\mathbb{R}^m$ containing $f(S)$, and $g : T \to \mathbb{R}^p$ be twice differentiable at the interior point $b = f(c)$. Then the function $h(x) := g(f(x))$ is twice differentiable at c and*

$$\mathsf{H}h(c) = (I_p \otimes \mathsf{D}f(c))' \cdot (\mathsf{H}g(b)) \cdot \mathsf{D}f(c) + (\mathsf{D}g(b) \otimes I_n) \cdot \mathsf{H}f(c).$$

Applying the chain rule to functions that are inverses of each other gives the following corollary.

Corollary 19 *Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is invertible with inverse $g := f^{-1}$. If $b = f(c)$ then*

$$\mathsf{H}f^{-1}(b) = -\left(G \otimes G'\right) \mathsf{H}f(c) G$$

where $G := [\mathsf{D}f(c)]^{-1} = \mathsf{D}g(b)$.

Proof Since $f \circ g = \mathrm{id}$ and $\mathsf{H}[\mathrm{id}] = 0_{n^2 \times n}$ Theorem 18 implies that for c in the interior of the domain of f and $b = f(c)$

$$\mathsf{H}(g \circ f)(c) = (I_n \otimes \mathsf{D}f(c))' \cdot \mathsf{H}g(b) \cdot \mathsf{D}f(c) + (\mathsf{D}g(b) \otimes I_n) \cdot \mathsf{H}f(c) = 0_{n^2 \times n}.$$

Solving this for $\mathsf{H}g(b)$ gives

$$\mathsf{H}g(b) = -\left[(I_n \otimes \mathsf{D}f(c))'\right]^{-1} \cdot (\mathsf{D}g(b)) \otimes I_n) \cdot \mathsf{H}f(c) \cdot [\mathsf{D}f(c)]^{-1}.$$

Since $(A \otimes B)^{-1} = (A^{-1} \otimes B^{-1})$ and $(A')^{-1} = (A^{-1})'$ we have $[(I \otimes B)']^{-1} = [(I \otimes B)^{-1}]' = (I^{-1} \otimes B^{-1})' = (I \otimes B^{-1})'$ so the first term in the above product simplifies to $-\left[(I_n \otimes \mathsf{D}f(c)^{-1})\right]'$. The inverse function theorem implies $\mathsf{D}g(b) = [\mathsf{D}f(c)]^{-1} =: G$ and so

$$\begin{aligned} \mathsf{H}g(b) &= -(I_n \otimes G)' \cdot (G \otimes I_n) \cdot \mathsf{H}f(c) \cdot G \\ &= -(G \otimes G') \cdot \mathsf{H}f(c) \cdot G \end{aligned}$$

as required, since $(A \otimes B)(C \otimes D) = (AC \otimes BD)$. ■