THE CIRCULAR NATURE OF DISCRETE-TIME FREQUENCY ESTIMATES

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ABSTRACT

Although it is well-known that sampling in the time-domain leads to periodicity in the frequency domain, this periodic or circular nature of discrete-time frequency estimates is often neglected in subsequent calculations yielding misleading results. Circular data, such as discrete-time frequency or phase estimates, may be visualized by points on a circle whereas linear data may be visualized by points on a line. The different algebraic structures of the circle and the line produce statistics and operations with quite different behaviour. Yet, few researchers dealing with circular data have realized that the fundamental difficulty stems from the inappropriate application of mathematical operations which are designed to be used with linear data. An elegant solution to this problem is to define mathematical operations for circular data which exhibit analogous behaviour to familiar operations for linear data. These operations can be immensely useful and should be more widely known. To highlight the practical utility of such techniques, we use concepts from the analysis of circular data to reformulate a recently proposed frequency estimator to remove its bias and elevated threshold. We also show that maximum likelihood frequency estimation is equivalent to circular least squares regression on the phase.

1 INTRODUCTION

There are many well-known cases where discrete-time signal processing does not correspond to continuous-time processing. Take, for example, the product of two Fourier transforms. In the continuous-time domain, this operation is equivalent to linear convolution of the two time functions, whereas in the discrete-time domain, it corresponds to circular convolution [1]. With careful zero-padding, the circular convolution operation can be used to perform linear convolution, but it is still, nonetheless, a distinctly different operation. The signal samples processed by the discrete Fourier transform (DFT) may be regarded as samples on a *circular* domain rather than a *linear* domain, and this is the reason for the discrepency. Since the DFT makes no distinction between the frequencies f_0 and f_0+kf_s, $k \in \mathcal{Z}$, the discrete-time frequency estimates produced by the DFT should also be visualized on a circular domain of circumference f_s, where f_s is the sampling frequency.

Although the odd behaviour of the DFT is well-documented, the circular nature of some common types of data causes other effects which are rarely mentioned. In particular, the statistical analysis of circular data, such as discrete-time frequency estimates, raises some interesting problems. Fortunately, similar problems have already been encountered in the study of directional data such as declination data in geology, seasonal fluctuation data in medicine, and wind direction data in meteorology [2]. The temptation to use conventional linear analysis techniques can lead to paradoxes; for example, the arithmetic mean of the angles 1° and 359° is 180°, whereas by geometrical intuition the mean should be 0°.

In this paper, we give a short introduction to the analysis of circular data and then show how these concepts can be applied to a problem of discrete-time frequency estimation.

2 ANALYSIS OF CIRCULAR DATA

We have already seen that the arithmetic mean is unsuitable for circular data since the answer obtained is very dependent on the arbitrary choice of origin for the circular domain. In the case of a set of N angular estimates, a sensible way to obtain the mean is to represent the estimates by N unit phasors with arguments equal to the corresponding angular estimates. Then the mean angle is given by the argument of the phasor sum and this value is *independent* of the choice of origin. The magnitude S of the phasor sum gives a measure of the concentration of the estimates about the mean direction. If $S = N$ then all of the estimates must be identically equal to the mean; if $S = 0$ then the estimates are uniformly distributed on the circle and the mean direction is undefined. This phasor sum construction is used as the basis for the following general definitions of circular mean and variance.

Definition 1 *Circular Sample Mean and Sample Variance: Let* $\{\hat{\alpha}(k)\}$, $\hat{\alpha} : \mathcal{Z} \mapsto \mathcal{R}$, *be a set of* N *observations of a random variable in the circular domain* $[0, P)$. *Then the circular sample mean* $\hat{\mu}_P$ *and circular sample variance* $\hat{V}p$ *are defined by*

$$\hat{\mu}_P = \frac{P}{2\pi}\left(\left(\arg\left[\sum_{k=0}^{N-1} e^{j2\pi\hat{\alpha}(k)/P}\right]\right)\right)_{2\pi} \tag{1}$$

and

$$\hat{V}_P = \frac{P^2}{4\pi^2}\left[1 - \frac{1}{N}\left|\sum_{k=0}^{N-1} e^{j2\pi\hat{\alpha}(k)/P}\right|\right] \tag{2}$$

where $((\quad))_{2\pi}$ *denotes reduction modulo* 2π *onto* $[0, 2\pi)$.

Since the circular variance takes values in $[0, P^2/4\pi^2]$ unlike the arithmetic (linear) variance which has a domain of $[0, \infty)$, the circular and linear variance estimators cannot be directly compared. However we can transform $\hat{V}_P$ to the range $[0, \infty)$ by using a transformation based on the relationship between the normal distribution on the circle (wrapped normal, [2, page 74]) and the normal distribution on the line. We can also add a term to account for estimator bias to obtain the *linearized* circular mean square error (CMSE) estimator [3].

Definition 2 *Circular Mean Square Error (CMSE): Let* $\{\hat{\alpha}(k)\}$, $\hat{\alpha} : \mathcal{Z} \mapsto \mathcal{R}$, *be a set of* N *noisy observations of a random variable with known value* α *defined on the circular domain* $[0, P)$. *Then the sample estimator of the (linearized) circular mean square error is defined by*

$$e_P^2 = -\frac{P^2}{2\pi^2}\ln\left[\frac{1}{N}\left|\sum_{k=0}^{N-1} e^{j2\pi\hat{\alpha}(k)/P}\right|\right.$$

$$+\left[\frac{P}{2\pi}\left\{\left(\left(2\pi\frac{(\hat{\mu}_P-\alpha)}{P}+\pi\right)\right)_{2\pi}-\pi\right\}\right]^2 \quad (3)$$

where $\hat{\mu}_P$ is the circular sample mean obtained from (1).

The first term in (3) is the linearized circular sample variance and the second term is due to bias. The CMSE can be used to evaluate the performance of discrete-time frequency estimators in noise and it gives sensible results for all values of frequency. This contrasts with the performance of the conventional linear estimator of sample variance which gives absurd results if the mean frequency is close to the end-points of the circular domain (0 Hz and $f_s/2$ Hz in this paper) as observed by Rife and Boorstyn [4].

The phasor sum construction also leads to the following definition for the periodic first moment of a discrete circular distribution function such as a periodogram.

Definition 3 *Periodic First Moment: Let $\theta(k)$ be a discrete function which is periodic in k with period M. Then the first periodic moment i_1^P is defined by*

$$i_1^P=\frac{M}{2\pi}\left(\left(\arg\left[\sum_{k=0}^{M-1}\theta(k)e^{j2\pi k/M}\right]\right)\right)_{2\pi}. \quad (4)$$

If a circular distribution function has all of its energy confined to a small region of the circular domain, then we can unwrap the domain about an appropriate point and use the conventional linear first moment to obtain the same result as the periodic moment. However, in general, we should use the periodic first moment as a measure of location for circular data since it is the more general, mathematically-rigorous concept.

If we wish to filter a sequence of circular random variables such as discrete-time frequency estimates, we cannot simply apply a standard FIR filter using *linear* convolution. Instead, we should use the *modulo-λ* convolution operation from [5,6].

Definition 4 *Modulo-λ Convolution: Let the sequence $\tilde{f}$ be of the form $\tilde{f}(n)=((\ f(n)\ ))_\lambda$, $f:\mathcal{Z}\mapsto\mathcal{R}$ and $\lambda\in\mathcal{R}$. If we convolve $\tilde{f}$ with a smoothing function h of odd length $P=2Q+1, h:\mathcal{Z}\mapsto\mathcal{R}$, then we must use the modulo-λ convolution operation defined by*

$$\tilde{f}(n)\ ((*))_\lambda\ h(n)=\frac{\lambda}{2\pi}\left(\left(\arg\left[\sum_{p=-Q}^{Q}h(n-p)e^{j2\pi\tilde{f}(p)/\lambda}\right]\right)\right)_{2\pi} \quad (5)$$

This operation is effectively a running, weighted periodic mean of the sequence $\tilde{f}$. Modulo-λ convolution appears to be a new operation which is the natural form of convolution for circular variables.

3 APPLICATION TO FREQUENCY ESTIMATION

We wish to examine discrete-time frequency estimators based on the discrete-time analytic signal defined as follows:

Definition 5 *Discrete-Time Analytic Signal: The discrete-time analytic signal z associated with the real discrete-time signal x is defined by*

$$\begin{aligned} z &= A[x] \\ &= x+jH[x] \end{aligned} \quad (6)$$

where $A[\ ]$ is the linear operator which forms the analytic signal and $H[\ ]$ is the discrete-time Hilbert transform defined by

$$H[x](n)=\sum_{\substack{-\infty \\ m \text{ odd}}}^{+\infty}\frac{2x(n-m)}{m\pi}. \quad (7)$$

One of the simplest estimators is based on finite differencing of the phase of the analytic signal as follows. It can be shown [3,7,8] that one need only consider estimators which use either the odd or the even samples of the analytic signal, since the odd samples may be reconstructed from the even, or the even from the odd.

Definition 6 *Central Finite Difference (CFD) Frequency Estimator: Let $z=A[x]$ where x is a real discrete-time signal. Then the frequency of x at sample n is estimated by*

$$\hat{f}_i^c(n)=\frac{f_s}{4\pi}((\arg[z(n+1)]-\arg[z(n-1)]))_{2\pi}. \quad (8)$$

As the CFD estimate only uses two samples of the analytic signal, it has a relatively high variance compared to estimators which employ more statistical averaging. Yet it is easy to show that the CFD estimator is efficient since it asymptotically meets the Cramer-Rao bound on estimator variance [6]. Since the CFD estimates have a circular nature, the question arises: "How do we average a set of CFD estimates to obtain a low variance estimate?" Before we tackle this question, we will say a few words about optimal frequency estimation.

3.1 Optimal Frequency Estimation

Rife and Boorstyn [4] have determined the Cramer-Rao lower bound for the variance of any unbiased frequency estimator in the case of a complex sinusoid with unknown phase and amplitude in white, Gaussian noise. We have adapted their result so it can be applied to real signals.

Theorem 1 *Cramer-Rao (CR) Lower Bound for a Real Signal: Let $\hat{x}(n)=x(n)+\epsilon(n)$ where x is a real sinusoid of the form $x(n)=a_c\cos[2\pi f_0 n]$ and ϵ is a zero-mean white Gaussian noise sequence with variance σ_ϵ^2. Then the Cramer-Rao lower bound on the variance of an unbiased estimator of the frequency f_0 is given by*

$$var[\hat{f}_o]\geq\frac{f_s^2}{(4\pi)^2}\frac{6}{sN_i(N_i^2-1)} \quad (9)$$

where $N_i=(M+1)/2$ and M is the number of samples in the data window, presumed odd. The signal-to-noise ratio is given by $s=a_c^2/2\sigma_\epsilon^2$.

The maximum likelihood (ML) estimator of a complex sinusoid in complex white Gaussian noise is given by the location of the peak of the periodogram [4] and a coarse estimate can be made directly from the peak of the DFT magnitude of the signal as was done by Palmer [9]. The ML estimator is asymptotically efficient since its variance meets the CR bound at sufficiently high signal-to-noise ratios. There is a signal-to-noise ratio, called the *threshold*, below which the dispersion of the frequency estimator rises very rapidly as the signal-to-noise ratio decreases. This threshold decreases with increasing data window length in the case of the ML estimator [4]. Although several other frequency estimators are asymptotically efficient at high signal-to-noise ratios, they all exhibit higher thresholds than the ML estimator for a given data set. Unfortunately, in many instances, the high computational load of the ML estimator may be prohibitive.

3.2 The Kay Estimator

Kay [7] recently proposed a discrete-time frequency estimator based on FIR filtering of CFD estimates. This new estimator involves far less computation than the ML estimator and yet it was claimed to be asymptotically efficient at moderate signal-to-noise ratios. Kay formulated his estimator by using linear regression on the phase as proposed by Tretter [10].

Kay's estimator is given by

$$\begin{aligned}\hat{f}_i^k(n) &= \sum_{p=-Q}^{Q} h_p(p)\hat{f}_i^c(n-p) \\ &= h_p(n) * \hat{f}_i^c(n) \qquad (10)\end{aligned}$$

where

$$h_p(p) = \begin{cases} \frac{3N_i}{2(N_i^2-1)}\left(1-\left[\frac{p}{N_i}\right]^2\right) & \text{for even } |p| \leq Q \\ 0 & \text{otherwise} \end{cases} \qquad (11)$$

is an FIR filter unit sample response with length $P = 2Q + 1$.

Kay uses a *parabolic* window function h_p which minimizes the variance of the estimator according to his analysis. The parabolic shape arises because of the dependency between successive CFD estimates.

In [6] we examined Kay's claims and found that the estimator is, in fact, biased and that it exhibits a elevated threshold when the frequency approaches the end-points of the circular domain, 0 and $f_s/2$ Hz. This effect occurs at low frequencies, for example, because the influence of noise causes some of the $\hat{f}_i^c$ in the summation in (10) to *wrap around* the circular domain — estimates that should be near 0 Hz sometimes appear near $f_s/2$ Hz and this effect greatly increases the variance of the sum. The poor performance of the Kay estimator is a direct result of using the *linear* convolution operation on circular data.

3.3 The PSCFD Estimator

These problems may be overcome by replacing the linear convolution operation in (10) with the modulo-λ convolution operation from definition 4 to obtain the parabolic smoothed central finite difference (PSCFD) estimator given by

$$\hat{f}_i^o(n) = h_p(p)\ ((*))_{f_s/2}\ \hat{f}_i^c(p). \qquad (12)$$

Figure 1 compares the CMSE of the original Kay estimator with the PSCFD estimator and the ML estimator for a tone of normalised frequency $0.05 f_s$ Hz. Note the elevated threshold (15 dB) of the Kay estimator which is far larger than the value quoted in the original paper (6 dB). The threshold of the PSCFD estimator is about 8 dB and is independent of frequency. Unlike the ML estimator of Figure 2, the threshold of the PSCFD estimator is largely independent of the FIR filter length as shown in Figure 3. This is in agreement with the general analysis presented in [3,11].

4 LEAST SQUARES REGRESSION ON PHASE ESTIMATES

Tretter [10] and Kay [7] attempted to determine frequency estimators by performing linear regression on phase estimates. We will show that the correct formulation of the regression problem using circular statistics leads, quite naturally, to the ML estimator.

Consider the problem of estimating the frequency of a complex sinusoid in white, zero-mean noise by regression on the phase estimates . Let $\hat{z} = z + \epsilon$ denote the sequence of N samples of the noise-corrupted signal where z is the complex sinusoidal sequence and ϵ is the noise sequence. From $\hat{z}$ we obtain the following phase estimates

$$\hat{\phi}(0) = \arg[\hat{z}(0)], \hat{\phi}(1) = \arg[\hat{z}(1)], \ldots, \hat{\phi}(N-1) = \arg[\hat{z}(N-1)].$$

Since z is a complex sinusoid, we need to regress these phase estimates onto a straight line reduced modulo 2π of the form

$$\phi(n) = ((\ 2\pi f n + \phi(0)\ ))_{2\pi}.$$

If the phase estimates were linear random variables, we would simply vary the frequency f and initial phase $\phi(0)$ to minimize the mean square error e_l^2 given by

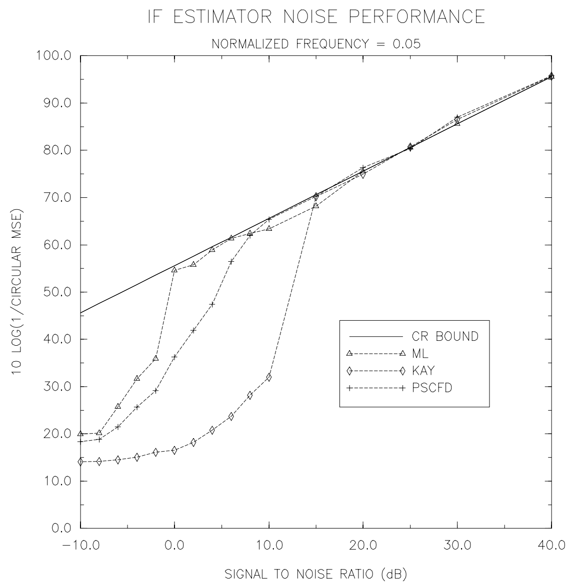

Figure 1: Comparison of the dispersion of the ML, PSCFD and Kay estimators against signal-to-noise ratio for a normalized frequency of 0.05. The window length $M = 47$ for the ML estimator and the smoothing window length $P = 45$ for the PSCFD and Kay estimators.

$$e_l^2 = \sum_{n=0}^{N-1}[\hat{\phi}(n) - \phi(n)]^2.$$

However, since they are really circular random varibles, we must minimize the CMSE of definition 2. This operation may be performed in two parts; minimization of the bias term of the CMSE yields the initial phase estimate $\hat{\phi}(0)$, and minimization of the linearized variance term yields the frequency estimate $\hat{f}$. This latter operation is equivalent to minimizing the circular sample variance $\hat{V}_p$ given by

$$\hat{V}_p = 1 - \frac{1}{N}\left|\sum_{n=0}^{N-1} e^{j[\hat{\phi}(n)-\phi(n)]}\right|. \qquad (13)$$

Minimizing $\hat{V}_p$ is equivalent to maximizing the function P given by

$$\begin{aligned}P &= \left|\sum_{n=0}^{N-1} e^{j[\hat{\phi}(n)-\phi(n)]}\right| \\ &= \left|\sum_{n=0}^{N-1} e^{j[\arg[\hat{z}(n)]-2\pi f n-\phi(0)]}\right| \\ &= \left|\sum_{n=0}^{N-1} e^{j\arg[\hat{z}(n)]}e^{-j2\pi f n}\right| \\ &= \left|\sum_{n=0}^{N-1} \frac{1}{|\hat{z}(n)|}\hat{z}(n)e^{-j2\pi f n}\right| \qquad (14)\end{aligned}$$

Now if the signal-to-noise ratio is high, we can say that $|\hat{z}| \approx |z|$

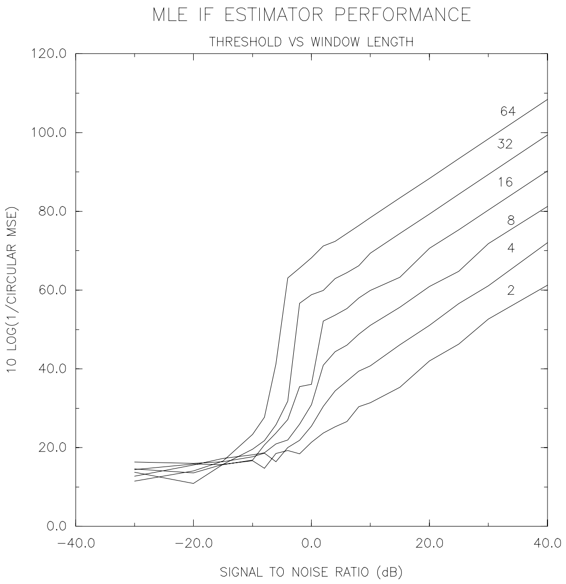

Figure 2: Dispersion of the ML estimator against signal-to-noise ratio for various window lengths.

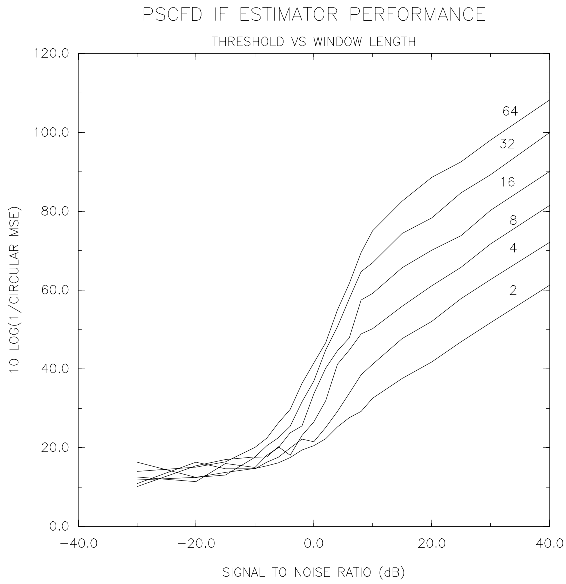

Figure 3: Dispersion of the PSCFD estimator against signal-to-noise ratio for various window lengths.

and so (14) becomes

$$P \approx \frac{1}{|z|} \left| \sum_{n=0}^{N-1} \hat{z}(n) e^{-j2\pi fn} \right|. \quad (15)$$

Thus, for high signal-to-noise ratios, maximizing P corresponds to finding the peak of the magnitude of the discrete-time Fourier transform of the noisy signal $\hat{z}$, which is just the ML estimator of frequency. Simulations have shown that the performance of the least squares estimator obtained from the peak of (14) is virtually identical to the ML estimator. However there is a risk of the least squares estimator becoming undefined if $|\hat{z}(n)| = 0$, and so the ML estimator is preferable.

5 CONCLUSIONS

We have shown that the conventional (linear) definitions of mean and variance can yield absurd results when applied to data defined on a circular domain. Appropriate definitions for the mean, variance and moments of circular data have been given and these concepts have been applied to reformulate a recently proposed frequency estimator to avoid bias and elevated threshold effects. The maximum likelihood frequency estimator has been shown to be equivalent to least squares regression on phase estimates. These results demonstrate the importance of appreciating the true circular nature of many quantities commonly encountered in digital signal processing.

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