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Finite Model Theory

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Overview of Topics

- **Part 1**: – **Introduction**
  1. What is finite model theory?
  2. Connections to some areas in CS
     - Database theory
     - Complexity theory
  3. Basic definitions and terminology
  4. Inexpressibility proofs
  5. Classical results over finite structures
     - Failures
     - Successes
     - Open questions

Finite Model Theory

- Classical model theory concentrates on **all structures** - the origin is in mathematics.
  - Boolean algebras, random graphs, algebraically closed fields, various models of arithmetic, etc.
- Finite model theory studies logics over **finite structures** - the origin is in computer science.
  - Finite relations
  - Finite graphs
  - Finite strings
  - Finite classes of arithmetic structures
  - ...

Connections to Database Theory

- Finite model theory plays a central role in the development of database theory.
  - A database can be naturally viewed as a finite structure, e.g.,

Database	Structure
Relational databases	finite relations
XML databases	finite trees
Graph databases	finite graphs

  - A query language is often measured in terms of logic, e.g.,

Query language	Logic
Relational Calculus	FO
Datalog$\neg$	$\exists$LFP
Basic SQL + aggregation	FO + counting extension
Core XPath	MSO

- Database theory supplies finite model theory with key motivations and problems.

Connections to Database Theory

- **Example**: Reachability queries
  - **Q1.** Find pairs of cities (s, d) such that one can fly from s to d with at most one stop.
  - **Q2.** Find pairs of cities (s, d) such that one can fly from s to d with at most two stops.
  - **Q3.** Find pairs of cities (s, d) such that one can fly from s to d.

FLIGHTS	
Source	**Destination**
Berlin	Beijing
Beijing	Auckland
Auckland	Sydney

Connections to Database Theory

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FLIGHTS	
Source	**Destination**
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ANSWER	
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Beijing	Auckland
Auckland	Sydney
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Connections to Database Theory

- **Example**: Reachability queries
  - **Q2.** Find pairs of cities (s, d) such that one can fly from s to d with at most two stops.

$$\{(x_s, x_d) \mid \text{FLIGHTS}(x_s, x_d) \vee \exists x_1.(\text{FLIGHTS}(x_s, x_1) \wedge \text{FLIGHTS}(x_1, x_d)) \vee \exists x_1, x_2.(\text{FLIGHTS}(x_s, x_1) \wedge \text{FLIGHTS}(x_1, x_2) \wedge \text{FLIGHTS}(x_2, x_d))\}$$

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Connections to Database Theory

- **Example**: Reachability queries
  - **Q3.** Find pairs of cities (s, d) such that one can fly from s to d.

Connections to Database Theory

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Connections to Database Theory

- **Example**: **Reachability queries**
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Cannot be expressed in Relational Calculus (FO)

- *How can we tell whether a query CAN or CANNOT be expressed in a query language?*

Connections to Complexity Theory

- Computational complexity measures the amount of resources (e.g., time and space) that are needed to solve a problem.
  - Many models of computation have been invented, e.g.,

Models of Computation	
Lambda calculus	Church
Recursive functions	Gödel
Turing machines	Turing

  - Two key questions:
    - What can be automatically computed (i.e., computability)?
      $\hookrightarrow$ Turing/Church thesis
    - How difficult it is to solve a problem (i.e., complexity)?
      $\hookrightarrow$ Complexity classes (P, NP, Pspace,...)

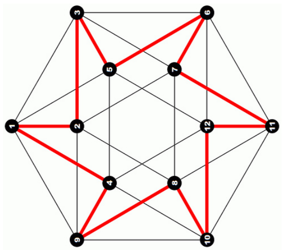
Connections to Complexity Theory

- The development of descriptive complexity is one of the most striking results in finite model theory.
- **How are different logics and complexity classes related?**
  1. What logic can be used to express a query?
  2. What is the complexity of evaluating a query?
- Some known results:

Query	Logic	Complexity class
Transitive closure	IFP + $<$	P
Connectivity	IFP + $<$	P
Evenness	IFP + $<$	P
Hamiltonicity	$\exists$SO	NP
3-colorability	$\exists$SO	NP
Clique	$\exists$SO	NP
Quantified SAT	PFP + $<$	Pspace

Connections to Complexity Theory

- **Example**: Hamiltonicity
  - A **Hamiltonian cycle** is a cycle that visits each vertex exactly once. A graph that contains a Hamiltonian cycle is called a **Hamiltonian graph**.



Connections to Complexity Theory

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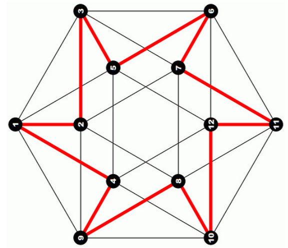
$\exists L \exists S$ linear order$(L) \wedge$
S is the successor relation of $L \wedge$
$\forall x \exists y (L(x,y) \vee L(y,x)) \wedge$
$\forall x \forall y (S(x,y) \Rightarrow E(x,y))$

L is a linear ordering relation.

S is a circular successor relation:
$\forall x \forall y S(x,y) \Leftrightarrow$
$((L(x,y) \wedge \neg \exists z (L(x,z) \wedge L(z,y))) \wedge$
$(\neg \exists z L(x,z) \wedge \neg \exists z L(z,y)))$



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  - Testing Hamiltonicity is an NP-complete problem.
  - So, *is there any connection between* $\exists$*SO and NP?*

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  - So, *is there any connection between* $\exists$*SO and NP?*

$\exists$SO	Existential SO quantifiers	+	FO formula
NP	Guess stage	+	Verify stage

  - **Descriptive complexity** aims to characterize complexity classes by means of logics.

Finite Model Theory

- Apparently, finite structures are a subclass of "all structures". Is finite model theory just a subfield of classical model theory?
- Some history:
  - Before 1970s, some problems about FO over finite structures were studied.
  - In 1970s,
    - Fagin published several papers relating to finite model theory.
    - Hájek proposed to "develop logic (classical and generalized) modified by allowing only finite models".
    - ...
  - Since 1980s, finite model theory becomes an active line of research.

Finite Model Theory – Central Themes

- **Logic and expressiveness**
  - Ehrenfeucht-Fraïssé games
  - Locality
- **Logic and complexity**
  - Descriptive complexity
  - Data complexity and expression complexity
- **Logic and combinatorics**
  - Zero-one laws
  - Asymptotic probabilities
- **Various logics**
  - Finite variable logics
  - Fixed point logics
  - Logics with counting
  - ...

Structures

- A **vocabulary** σ is a *finite* set of relation symbols, each with a fixed arity.

  Note that, we restrict vocabularies to be relational, and there are *no function symbols* in σ.
- A **structure** (also called a **model**) of vocabulary σ is $\mathfrak{A} = \langle A, (R^{\mathfrak{A}})_{R \in \sigma} \rangle$, where
  - A, called the **universe** of $\mathfrak{A}$, is a nonempty set, and
  - each $R^{\mathfrak{A}} \subseteq A^n$ is an interpretation of a n-ary relation symbol from σ.
- A structure $\mathfrak{A}$ is called **finite** if its universe A is a finite set.
- *Conventions:*
  - We denote universes by using Roman letters corresponding to their structures, e.g., the universe of $\mathfrak{A}$ is A, the universe of $\mathfrak{B}$ is B, etc.
  - We use the same symbol R for both a relation symbol in σ, and its interpretation $R^{\mathfrak{A}}$.

First-order Logic

- Recall terms and formulas of first-order logic (FO) (we assume a countably infinite set of variables).
- **Syntax**:
  - **Terms** of FO are defined by:
    - Each variable x and constant c is a term.
    - $f(t_1, \ldots, t_n)$ is a term, where f is a relation symbol, and $t_1, \ldots, t_n$ are terms.
  - **Formulas** of FO can be inductively defined by:
    - atomic formulas: $R(t_1, \ldots, t_n)$, $t_1 = t_2$;
    - Boolean operations: $\varphi_1 \wedge \varphi_2$, $\varphi_1 \vee \varphi_2$, $\neg\varphi$;
    - first-order quantifiers: $\exists\varphi$, $\forall\varphi$.
- **Semantics**: we skip the details as it has been covered in the last week

First-order Logic

- A **sentence** is a formula without free variables.
- A sentence φ is **satisfiable** if it has a model, and is **valid** if it is true in every structure.
  - φ is not valid iff $\neg\varphi$ is satisfiable.
- A sentence φ is **finitely satisfiable** if it has a finite model, and is **finitely valid** if it is true in every finite structure.
  - φ is not finitely valid iff $\neg\varphi$ is finitely satisfiable.
- We use $\mathfrak{A} \models \varphi(a_1, \ldots, a_n)$ to denote that $\varphi(a_1, \ldots, a_n)$ is true in $\mathfrak{A}$.

Queries

- A formula $\varphi(x_1, \ldots, x_n)$ with free variables $x_1, \ldots, x_n$ defines **a mapping** Q that associates to every structure $\mathfrak{A}$ a (n-ary) relation on $\mathfrak{A}$:
$$Q(\mathfrak{A}) = \{(a_1, \ldots, a_n) | \mathfrak{A} \models \varphi(a_1, \ldots, a_n)\}.$$
- A **n-ary query** Q is such a mapping **closed under isomorphism**, i.e., if $h : A \to B$ is an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$, it is also an isomorphism between $Q(\mathfrak{A})$ and $Q(\mathfrak{B})$.
- If $n = 0$, a 0-ary query is a mapping from structures to $\{true, false\}$, which is called a **Boolean query**.
- A query Q is **definable** in a logic L if there is a formula $\varphi(x_1, \ldots, x_n)$ of L that defines Q. A Boolean query Q is **definable** in a logic L if there is a formula φ of L such that $Q(\mathfrak{A}) = true$ iff $\mathfrak{A} \models \varphi$ for all $\mathfrak{A}$.

Queries Definable in FO

- Consider $\sigma = \{E\}$ and $G = \langle V, E\rangle$.
  1. Graphs whose edges are antireflexive and symmetric:
  $$\forall x \neg E(x, x) \wedge \forall x \forall y (E(x, y) \Rightarrow E(y, x)).$$
  2. Graphs that contain at least one triangle:
  $$\exists x \exists y \exists z (x \neq y \wedge y \neq z \wedge x \neq z \wedge E(x, y) \wedge E(x, z) \wedge E(y, z)).$$
  3. Graphs that contain at least n vertices:
  $$\exists x_1 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j.$$

Limitations of FO

- The expressive power of FO on finite structures is limited:
  - Cannot express counting properties, e.g.,
    1. **Evenness**: Given a graph G, is the number of vertices in G even?
    $$even(V) = \begin{cases} 1 & \text{if } |V| \text{ is even} \\ 0 & \text{otherwise .} \end{cases}$$
  - Cannot express properties that require iterative algorithms, e.g.,
    1. **Connectivity**: Given a graph G, is it connected?
    I.e., there exists a path between any two nodes a and b in G.
  - Cannot express properties that require to quantify relations, e.g.,
    1. **3-Colorability**
    2. **Clique**
    3. ...

Inexpressibility Proofs

- *How can one prove that a property is inexpressible in a logic, e.g., FO?*
- Some techniques are available for inexpressibility proofs in FO.
  - Compactness theorem
  - Ehrenfeucht-Fraïssé games
  - ...

Inexpressibility Proofs

- **Compactness theorem**

  Let Φ be a set of first-order sentences. If every finite subset of Φ is satisfiable, then Φ is satisfiable.
- Main ideas of using the compactness theorem to prove inexpressibility of a property P:
  - Assume P is expressible by a FO-sentence φ.
  - Construct a set of sentences Ψ so that each model of Ψ does not satisfy P, but each finite subset of Ψ has a model satisfying P.
  - By compactness we would know that $\Psi \cup \{\varphi\}$ has a model.
  - Contradiction with the assumption.

Inexpressibility Proofs

- **Connectivity** of *arbitrary graphs* is not FO-definable.

  Proof:
  - Assume that connectivity is definable by ϕ.
  - Expand the vocabulary with two constants c_1 and c_2, and let $T = \{\psi_n | n > 0\} \cup \{\phi\}$, where
  $$\psi_n = \neg(\exists x_1 ... \exists x_n x_1 = c_1 \wedge x_n = c_2 \wedge \bigwedge_{1 \leq i \leq n-1} E(x_i, x_{i+1}))$$
  i.e., there is no path of length n+1 from c_1 to c_2.
  - Every finite subset $T' \subseteq T$ is satisfiable, because there exists N, s.t. for all $\psi_n \in T'$, $n < N$, and T' has a model with a path of length N+1. By compactness, T is satisfiable.
  - However, T has no model. *Contradiction.*

Inexpressibility Proofs

- *Does the previous proof tell us that FO cannot express connectivity over finite graphs*?

  The previous proof:
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  - However, T has no model. *Contradiction.*

Failure of Compactness Theorem

- *Does the previous proof tell us that FO cannot express connectivity over finite graphs*?
- To modify the previous proof for finite models, one would be to use compactness over finite models.
- But compactness fails over finite models.

  Proposition: There is a set T of FO-sentences s.t.

  1. T has no finite models, and
  2. every finite subset of T has a finite model.

Failure of Compactness Theorem

- **Proposition**: There is a set T of FO-sentences s.t.

  1. T has no finite models, and
  2. every finite subset of T has a finite model.

Proof:

- Assume that $\sigma = \emptyset$, and define

$$\psi_n = \exists x_1 ... \exists x_n \bigwedge_{i \neq j} \neg(x_i = x_j).$$

  i.e., the universe has at least n distinct elements.
- Let $T = \{\psi_n | n > 0\}$.
- T has no finite model. But for each finite subset of T, a set whose cardinality exceeds the maximal n is a model.

Inexpressibility Proofs

- For the techniques available for inexpressibility proofs in FO:
  - **Compactness theorem**
    $\hookrightarrow$ fails over finite structures.
  - **Ehrenfeucht-Fraïssé games**
    $\hookrightarrow$ used as a central tool on classes of finite structures.

Classical Model Theory

- Main topics:
  - Logical definability of structural properties
  - Classification of models of theories
  - Algebraic properties in axiomatic theories
- Some key results:
  1. Completeness theorem (and Compactness theorem)
  2. Löwenheim-Skolem theorem
  3. Beth's definability theorem
  4. Craig's interpolation theorem
  5. Los-Tarski preservation theorem and Lyndon's positivity theorem

Classical Model Theory

- Some key results:
  1. Completeness theorem (and Compactness theorem)
  2. Löwenheim-Skolem theorem
  3. Beth's definability theorem
  4. Craig's interpolation theorem
  5. Los-Tarski preservation theorem and Lyndon's positivity theorem
- *Can these results of classical model theory hold over finite structures?*
- Unfortunately, **all the above results fail when we restrict to finite structures.**

Failure of Completeness Theorem

- **Trakhtenbrot's Theorem**[1]

  For every relational vocabulary σ with at least one binary relation symbol, it is undecidable whether a sentence Φ of σ is finitely satisfiable.

  Basic idea:
  - For any Turing machine M, there is a FO sentence φ_M such that
    M halts iff φ_M has a finite model.
- **Corollary**

  For every relational vocabulary σ with at least one binary relation symbol, **the set of finitely valid FO sentences is not recursively enumerable**.

  Basic idea:
  - Because the set of non-halting Turing machines is not recursively enumerable, $\{\varphi | \neg\varphi$ has no finite models$\}$ is also not recursively enumerable.

[1] B. Trakhtenbrot, The Impossibility of an Algorithm for the Decidability Problem on Finite Classes. Proceedings of the USSR Academy of Sciences (in Russian) 70 (4): 569572, 1950.

Failure of Completeness Theorem

- Some consequences of Trakhtenbrot's Theorem:
  - Unsatisfiability of FO is semi-decidable, but finite unsatisfiability is not semi-decidable.
  - Satisfiability of FO is not semi-decidable, but finite satisfiability is semi-decidable.

Failure of Löwenheim-Skolem Theorem

- **Corollary**

  There is no recursive function f s.t. if a FO sentence φ has a finite model, then it has a model of size at most $f(\varphi)$.
- Recall the Löwenheim-Skolem theorem in classical model theory:

  If a countable first-order theory has an infinite model, then for every infinite cardinal number k it has a model of size k.

Trakhtenbrot Theorem

- **Theorem** (Trakhtenbrot, 1950)

  For every relational vocabulary σ with at least one binary relation symbol, it is undecidable whether a sentence Φ of σ is finitely satisfiable.

- **Proof**:

  For every Turing machine $M = (S, \Sigma, \Delta, \delta, q_0, S_a, S_r)$, construct a sentence φ_M of σ s.t. φ_M **is finitely satisfiable iff** M **halts on the empty input**.

  - Let $\sigma = \{<, min, T_0, T_1, (H_q)_{q \in S}\}$, and φ_M states that each relation symbol in σ is interpreted below, and M eventually halts.
    - $<$ is a linear order, and *min* is the minimal element w.r.t. $<$;
    - $T_0(p, t)$ and $T_1(p, t)$ are *tape predicates*: indicate that position p at time t contains 0 or 1;
    - $H_q(p, t)$'s are *head predicates*: indicate that at time t, M is in state q and its head is in position p.
  - If φ_M has a finite model, then such a model represents a computation of M that halts on an empty input, and vice versa.

Classical Result – Revisited for Finite Models

- *Are there any results of classical model theory that survive on finite models?*
- Rosen wrote:[2]

  > *"there seems to be no example of a theorem [of classical model theory] that remains true when relativized to finite structures but for which there are entirely different proofs for the two cases. It would be interesting to find a theorem proved using the compactness theorem that can be established using a new method over finite structures."*

[2]Rosen, E. Some aspects of model theory and finite structures. Bull. Symbolic Logic 8, 380-403, 2002.

Classical Result – Revisited for Finite Models

- A FO sentence $\varphi(\vec{x})$ is **preserved under homomorphisms** if $\mathfrak{A} \models \varphi(\vec{a})$ implies $\mathfrak{B} \models \varphi(h(\vec{a}))$ whenever $h : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism.

- Recall the homomorphism preservation theorem in classical model theory:

  The following statements are equivalent for any FO sentence φ:

  1. φ is preserved under homomorphisms on all structures;
  2. $\varphi \equiv \varphi^*$ on all structures for some existential positive FO-sentence φ^*.

- **Theorem** (Rossman, 2005)

  If a FO sentence φ is preserved under homomorphisms on all finite structures, then there is an existential positive FO-sentence φ^* that is equivalent to φ on all finite structures.

Classical Result – Revisited for Finite Models

- First-order logic has a special place in classical model theory.

- **Theorem** (Lindström, 1969)

  First-order logic is a maximal logic possessing both the Compactness Theorem and the Löwenheim-Skolem Theorem, i.e., no logic that is compact and satisfies the Löwenheim-Skolem property can properly extend FO.

- **Question**
  - Is there a similar characterization of first-order logic/least fixed-point logic on finite structures?

Useful References

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