# An internal model principle for observers

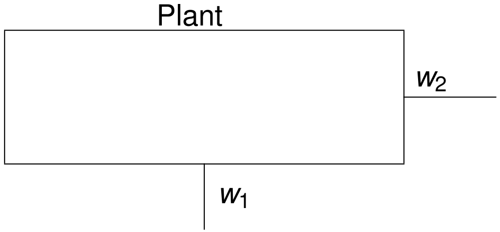
J. Trumpf J.C. Willems

July 2007

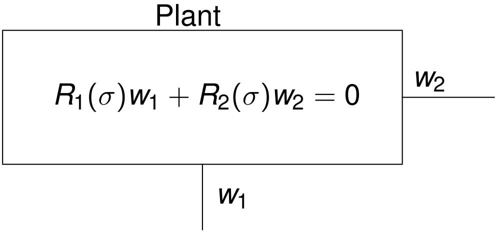
# Outline

# Outline

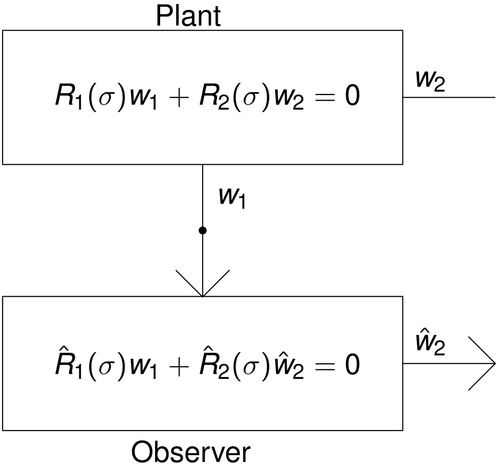
# What is an observer?



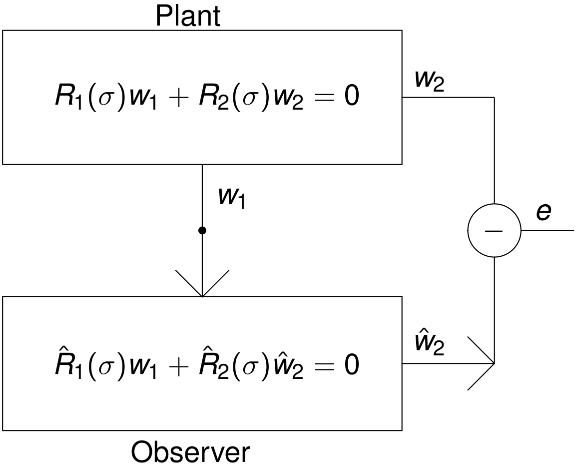
# What is an observer?



# What is an observer?



# What is an observer?



# Outline

# The error system

Given a plant

$$\mathcal{B} = \{(w_1, w_2) \mid R_1(\sigma)w_1 + R_2(\sigma)w_2 = 0\}$$

and an observer

$$\hat{\mathcal{B}} = \{(w_1, \hat{w}_2) \mid \hat{R}_1(\sigma)w_1 + \hat{R}_2(\sigma)\hat{w}_2 = 0\}$$

for $\mathcal{B}$, the *error system* is defined as

$$\mathcal{B}_e = \{e \mid \exists (w_1, w_2) \in \mathcal{B}, (w_1, \hat{w}_2) \in \hat{\mathcal{B}} : \; e = \hat{w}_2 - w_2\}$$

The elimination theorem says that $\mathcal{B}_e$ is an LTID system.

# What is a good observer?

We *postulate* that the fundamental property any *reasonable* observer should have is

$$\boxed{B_e \text{ is autonomous.}}$$

The classical cases are:

- $B_e$ stable $\Rightarrow$ asymptotic observer
- (discrete time) $B_e$ nilpotent $\Rightarrow$ dead-beat observer
- $B_e = 0 \Rightarrow$ exact observer

# What is a good observer?

We say that $\hat{B}$ contains an *internal model* of $B$ if

$$\boxed{B \subseteq \hat{B}}$$

This is equivalent to the existence of $S$ such that

$$\begin{pmatrix} \hat{R}_1 & \hat{R}_2 \end{pmatrix} = S \begin{pmatrix} R_1 & R_2 \end{pmatrix}$$

Then the error system is

$$B_e = \{ e \,|\, S R_2 e = 0 \}$$

which is autonomous ($SR_2 = \hat{R}_2$ is nonsingular square since $w_1$ is full input to $\hat{B}$).

# What is a good observer?

$$\boxed{B \subseteq \hat{B} \;\Rightarrow\; \exists S:\; B_e = \text{Ker}\, SR_2(\sigma)}$$

Recall: $w_2$ is *observable* from $w_1$ $\Leftrightarrow$
$(w_1, w_2), (w_1, \tilde{w}_2) \in B$ implies $w_2 = \tilde{w}_2$ $\Leftrightarrow$
$R_2(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$

Under this condition we get full *pole placement*: for any $\pi$ there exists $S$ such that det $SR_2 = \pi$, can even choose $SR_2 = I$.

Note: same story for $w_2$ detectable or reconstructible from $w_1$.

# Outline

# Main result

## Theorem

*$B_e$ autonomous implies $B_{\text{contr.}} \subseteq \hat{B}$.*

## Corollary

*Any asymptotic (dead-beat, exact) observer for a controllable system contains an internal model.*

# Proof sketch

$$\mathcal{B}_{\text{contr.}} = \operatorname{Im}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}, \quad \hat{\mathcal{B}}_{\text{contr.}} = \operatorname{Im}\begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix}$$

where $\Gamma_1$ is square and has full rank ($w_1$ is full input to $\hat{\mathcal{B}}$).

Then, $(\mathcal{B}_{\text{contr.}})_e$ is given by

$$\begin{pmatrix} 0 \\ e \end{pmatrix} = \begin{pmatrix} M_1 & -\Gamma_1 \\ M_2 & -\Gamma_2 \end{pmatrix} \begin{pmatrix} l \\ l' \end{pmatrix}$$

and is autonomous iff

$$\operatorname{rk}\begin{pmatrix} M_1 & -\Gamma_1 \\ M_2 & -\Gamma_2 \end{pmatrix} = \operatorname{rk}\begin{pmatrix} M_1 & -\Gamma_1 \end{pmatrix} = \operatorname{rk}\Gamma_1 = \operatorname{rk}\begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix}$$

# Proof sketch

Since $\Gamma_1$ has full column rank this implies the existence of a rational $T$ such that

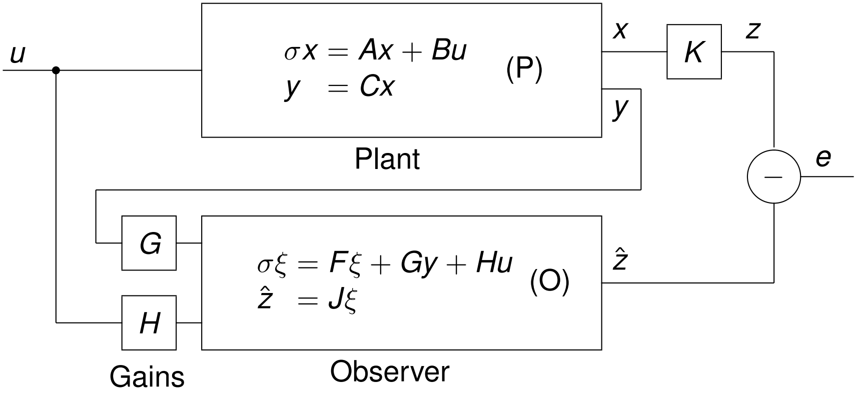
$$\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} T$$

i.e.

$$B_{\text{contr.}} \subseteq \hat{B}_{\text{contr.}}$$

# Outline

# What is an observer?



# Luenberger's equations

The existence of $Z$ such that

$$ZA - FZ = GC$$
$$H = ZB$$
$$K = JZ$$

implies ($d := \xi - Zx$)

$$\sigma d = Fd$$
$$e = Jd$$

$$\sigma x = Ax + Bu$$
$$y = Cx$$
$$z = Kx$$

$$\sigma \xi = F\xi + Gy + Hu$$
$$\hat{z} = J\xi$$

Z then maps $x$-trajectories to corresponding $\xi$-trajectories.

# Fuhrmann's interpretation

$$\gamma: \ B_f \longrightarrow \hat{B}_f, \quad \begin{pmatrix} x \\ u \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} Zx \\ u \\ y \\ z \end{pmatrix} =: \begin{pmatrix} \xi \\ u \\ y \\ \hat{z} \end{pmatrix}$$

is an injective behavior homomorphism, i.e. the observer contains an internal model!

We know from the above theorem that this is true for every asymptotic observer if the system is controllable. Fuhrmann and Helmke proved this statement in the state space case in 2002.

# Outline

# Summary

- Everything is easy once we have an internal model.
- This is the case for reasonable observers.

- Outlook
  - properness
  - algorithms
  - nD case