A manifold structure on the set of functional observers

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joint work with U. Helmke
Definition. Let $(C, A) \in \mathbb{R}^{p \times n} \times \mathbb{R}^{n \times n}$. A linear subspace $\mathcal{V} \subset \mathbb{R}^n$ is called $(C, A)$-invariant if there exists an output injection matrix $J$ such that

$$(A - JC)\mathcal{V} \subset \mathcal{V}$$

holds. Such a $J$ is called a friend of $\mathcal{V}$. 
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**Problem:** How much do perturbations in \(J\) affect \(\mathcal{V}\)?
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cf. related work on stable subspaces by

- L. Rodman (various articles) or
- F. Velasco (LAA 301, pp. 15–49, 1999)
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Let $P \in \mathbb{R}^{n \times n}$ be the orthogonal projector on $\mathcal{V}$. Then

$$(A - JC)\mathcal{V} \subset \mathcal{V} \iff f(P, J) := (I_n - P)(A - JC)P = 0$$
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Let $(P_0, J_0)$ be such that $f(P_0, J_0) = 0$. Consider

$$\frac{\partial f}{\partial P}|_{(P_0, J_0)}(\dot{P}) = -\dot{P}A_0P_0 + (I_n - P_0)A_0\dot{P}, \quad A_0 := A - J_0C$$
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in the basis where

$$P_0 = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix} \quad \text{and} \quad \dot{P} = [P_0, \Omega] = \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}$$

$(\Omega$ is skew-symmetric, here.)
Motivating problem

We get

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\frac{\partial f}{\partial P}|_{(P_0,J_0)}(\dot{P}) = \begin{pmatrix}
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Result. Let \(f(P_0, J_0) = 0\) and

\[
\sigma(A_0|_{\text{Im} P_0}) \cap \sigma(A_0|_{\mathbb{R}^n/\text{Im} P_0}) = \emptyset
\]

Then locally around \(J_0\) there exists a Lipschitz continuous function \(J \mapsto P(J)\) such that

\[
f(J, P(J)) = 0
\]
Tracking observers

Consider the linear, time-invariant, finite-dimensional control system in state space form

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\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}
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**(sys)**

**Definition.** A *tracking observer* for \( Vx \) is a dynamical system

\[
\dot{v} = Kv + Ly + Mu
\]

**(obs)**

which is driven by \( u \) and by \( y \) and has the *tracking property*:

\[
v(0) := Vx(0) \Rightarrow v(t) = Vx(t) \quad \text{for all } t \in \mathbb{R}
\]

where \( x(0) \) and \( u(.) \) are arbitrary.
Theorem. (Luenberger, 1964) System (obs) is a tracking observer for $V x$ if and only if

$$VA - KV = LC$$

$$M = VB$$

In this case the tracking error $e(t) = v(t) - Vx(t)$ is governed by the differential equation $\dot{e} = Ke$. 
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**Theorem.** (Willems et al., ≈ 1980) Let \(V\) be of full row rank. For every tracking observer for \(Vx\) there exists a friend \(J\) of \(\text{Ker } V\) such that \((A - JC)|_{\mathbb{R}^n/\text{Ker } V}\) is similar to \(K\). Conversely, for every friend \(J\) of \(\text{Ker } V\) there exists a unique tracking observer for \(Vx\) such that \(K\) is similar to \((A - JC)|_{\mathbb{R}^n/\text{Ker } V}\). Especially, there exists a tracking observer for \(Vx\) if and only if \(\text{Ker } V\) is \((C, A)\)-invariant.
The manifold of tracking observers

**Theorem.** (T., 2002) Let \((C, A)\) be observable and let \(k\) and \(p\) be the numbers of rows of \(V\) and \(C\), respectively. Then the set

\[
\text{Obs}_{k,k} := \{(K, L, M, V) \mid VA - KV = LC, M = VB, \text{rk} V = k\}
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of tracking observer parameters is a smooth (sub)manifold of dimension \(k^2 + kp\).
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**Proof.** The value \((0, 0)\) is a regular value of the map

\[
f : (K, L, M, V) \mapsto (VA - KV - LC, M - VB)
\]

The requirement \(\text{rk} V = k\) yields an open subset.
The manifold of tracking observers

**Theorem.** (T., 2002) Consider the similarity action

$$\sigma : \text{GL}(k) \times \text{Obs}_{k,k} \longrightarrow \text{Obs}_{k,k},$$

$$(S, (K, L, M, V)) \mapsto (SKS^{-1}, SL, SM, SV)$$

The $\sigma$-orbit space $\text{Obs}_{k,k}^\sigma$ of similarity classes

$$[K, L, M, V]_\sigma = \{(SKS^{-1}, SL, SM, SV) \mid S \in \text{GL}(k)\}.$$ 

is a smooth manifold of dimension $kp$. 

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is a smooth manifold of dimension \( kp \).

**Proof.** The equations \( VA - KV = LC \) and \( M = VB \) are invariant under \( \sigma \). The similarity action is free and has a closed graph mapping. Furthermore, \( \dim \text{GL}(k) = k^2 \).
**Theorem.** (Helmke/T., 2002) Let $(C, A)$ be observable, let $p \times n$ be the format of $C$ and let $0 \leq k < n$. Then the set

$$\text{Inv}_k = \{(V, J) \mid (A - JC)V \subset V, \text{codim } V = k\}$$

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is a smooth manifold of dimension \(np\). Furthermore, the map

\[ f : \text{Inv}_k \longrightarrow \text{Obs}_{k,k}^\sigma, \]

\[ (V, J) \mapsto [K, L, M, V]_\sigma, \]

defined by \(\text{Ker } V = V, M = VB, L = VJ\) and \(KV = VA - LC = V(A - JC)\) is a smooth vector bundle.
**Theorem.** (Helmke/T., 2002) Let \((C, A)\) be observable, let \(p \times n\) be the format of \(C\) and let \(0 \leq k < n\). Then the set

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defined by \(\text{Ker} V = \mathcal{V}, M = VB, L = VJ\) and \(KV = VA - LC = V(A - JC)\) is a smooth vector bundle.

**Proof.** http://statistik.mathematik.uni-wuerzburg.de/~jochen
Application: OAF-compensators

One way of stabilizing system (sys) is to dynamically feed back the state $v$ of an appropriately designed tracking observer (obs) via

$$u = Fv + r$$

Here the observer matrix $K$ as well as $(A + BF)$ have to be stable. $r$ denotes an external reference signal.
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\( \rightarrow \) Minimize the \( L^2 \)-sensitivity of the closed loop transfer function from \( r \) to \( y \) over the previously defined observer manifold to get the OAF-compensator best suited to fixed point arithmetics as used in hardware signal processors.
Outlook

- extend the previous results to compensator couples and MA-compensators.
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Thank you.