Error models for nonlinear observers

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Abstract— **We revisit the concept of observation error for nonlinear observers of nonlinear systems. In order to obtain a coordinate free notion of such an error we define it using fiber bundles over the system manifold. The new notion ties in nicely with Brockett's and Willems' classical description of nonlinear systems as bundles as well as with the related description of observers as bundles due to van der Schaft. It identifies the nonlinear observer design problem as the problem of choosing a nonlinear connection form (horizontal distribution plus affine offset) in this bundle with certain desirable properties. We demonstrate how a particular solution to this connection design problem is given by the invariant observers of Bonnabel, Martin and Rouchon. In the case of systems on Lie groups we recover the recent results by Lageman and two of the authors.**

I. INTRODUCTION

The need for abstract observation error models arises from two principle limitations of the usual approach of taking the difference between the current estimate produced by an observer and the value of the observed system variable in nonlinear observer theory. Clearly, this approach only makes sense where the observer estimate as well as the system variable have been expressed in the same (vector space) coordinates, and hence the value of the observation error depends on that choice of coordinates. The first limitation stems from the well known fact that some nonlinear systems (in particular many systems on compact manifolds) can not be expressed in globally valid coordinates of minimal dimension [3] and hence a switch of coordinates needs to be performed during the operation of the observer if singular points can not be avoided (think for example of a spherical pendulum that goes through full turns). The second limitation stems from the less obvious fact that it is not always clear how to choose "good" coordinates for a given observation problem. Indeed, this last point motivated our study of error models. For a thorough discussion of the pros and cons of various error coordinates for the attitude estimation problem (an observer design problem on the Special Orthogonal Group $\mathrm{SO}(3)$) see the recent work by Lageman and two of the authors [5]. There is also related work by Bonnabel et al. on invariant observers [1], [2].

Our proposed solution to these issues is inspired by the physical interpretation of different coordinate choices in the attitude estimation problem. They can be interpreted as choosing an inertial reference frame with respect to which measurements of physical quantities are made. Physics also suggests that the "proper" way to model this situation are coordinate bundles. Hence, we *define* an observation error with the help of a fiber bundle over the system manifold. It turns out that this approach ties in nicely with the fiber bundle based descriptions of general nonlinear systems due to Brockett [3] and Willems [11], as well as with the related descriptions of observers due to van der Schaft [10]. In particular, the observer design problem reduces to the problem of choosing a nonlinear connection form (a horizontal distribution plus an affine offset term) in the error bundle whose properties will then translate directly to the respective observer properties (such as the tracking property or asymptotic error convergence).

The main contribution of this paper is a general geometric framework for the design and analysis of nonlinear state observers for nonlinear input-state-output systems in fiber bundle form. This includes the classical nonlinear setup with state space $\mathbb{R}^n$ as a special case. We also show how some recent contributions to nonlinear observer theory fit in this general framework.

II. SYSTEM MODEL AND PROBLEM STATEMENT

We consider smooth nonlinear systems given in form of a bundle map $f\colon S \longrightarrow TX$ where $\pi_S\colon S \longrightarrow X$ is a

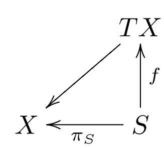

Fig. 1. Basic system model

smooth fiber bundle with the state space manifold X as base manifold and the fibers $U(x) := \pi_S^{-1}(x)$, $x \in X$ encoding the state depending input spaces. TX denotes the tangent bundle of X. This generalization of smooth nonlinear systems in $\mathbb{R}^n$ has been proposed by Brockett [3] more than 30 years ago to allow for the global modelling of systems whose input spaces can not be made state independent by a global change of coordinates. A simple example is a spherical pendulum where the input is a force applied to the tip of the pendulum. This force is tangential to the sphere at the current location of the pendulum and it is well known that the tangent bundle of the sphere (the union of all such tangential input spaces) is not globally trivial, i.e. it can not be globally written as a direct product of the sphere with an (input) vector space. In local coordinates x in X and (x, u) in S we have $\pi_S(x, u) = u$ and the system can locally be

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written in the more familiar form

$$\dot{x} = f(x, u).$$

In general, several such local descriptions in overlapping coordinate charts are needed to fully describe the global system.

We follow a suggestion by Willems [11] and add a second smooth map $h\colon S \longrightarrow W$ to our system description where W is a smooth manifold encoding the external variables in the system (for example inputs and/or outputs, or any other variables of external interest). We refer the reader to van der Schaft [9] for a detailed discussion of additional conditions on h and the structure of W that allow to distinguish inputs from outputs in this formalism and hence to rewrite the system locally or globally in the classical form

$$\begin{aligned} \dot{x} &= f(x, u), \\ y &= \tilde{h}(x, u). \end{aligned}$$

For the purpose of this paper we do not need to distinguish between different roles for the external variables in W and will hence not assume any of these additional conditions *a priori*. In local coordinates x and u as above and w for W the system then locally takes the form

$$\begin{aligned} \dot{x} &= f(x, u), \\ w &= h(x, u). \end{aligned}$$

We require the following addition to our system model that appears not to have been considered in prior literature. We wish to be able to globally express what it means to *not* control the system, i.e. what it means *not* to apply an input. The proper mathematical object to encode this information is a global section in the bundle $\pi_S\colon S \longrightarrow X$, i.e. a smooth map $z_u\colon X \longrightarrow S$ that respects base points, meaning $\pi_S(z_u(x)) = x$ for all $x \in X$. We will call such a section z_u the *zero input section* of the system. Geometrically, the zero input section identifies a submanifold of the total space S that can be thought of as an "uncontrolled" copy of the state space X. In a vector bundle S where the state dependent input spaces $U(x)$ are vector spaces (such as in the example of the spherical pendulum), there is of course a natural choice for such a zero input section that corresponds to a value of zero for u. In general bundles, a global section need not exist (for example in nontrivial principal bundles), so this requirement places a restriction on the class of systems we consider in this paper. We emphasize, though, that all the practical examples of models for control systems we have come across so far do allow such a global section. From a philosophical point of view one could argue that speaking of inputs is logically flawed if such a global section does not exist since then these "inputs" can not be globally turned off. A more appropriate notion in these cases might be "local inputs".

The zero input section z_u induces a tangent map $Dz_u\colon TX \longrightarrow TS$ and hence imbeds the (local) vector fields $(x, u) \mapsto f(x, u)$ in TX in a globally consistent manner into the larger tangent bundle TS via $\bar{f}(s) := Dz_u \circ f(s) \in T_sS$ for all $s \in S$. This observation turns out to be a key first step in our observer construction detailed below. The complete system diagram then takes the form shown in Figure 2. Note that this model also allows for a difference between

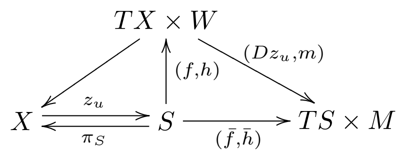

Fig. 2. Full system model

the maps h and $\bar{h}\colon S \longrightarrow M$ given by a map $m\colon W \longrightarrow M$ such that $\bar{h} = m \circ h$. The map m can be used to encode a measurement process for (all or some of) the external variables if a distinction between the "real" external map h and its measured variant $\bar{h}$ is desired. To simplify the discussion we will choose $m = \mathrm{id}$, the identity map on W in the examples in this paper.

The problem treated in this paper is how to systematically design a geometric observer for the state of the system shown in Figure 2. The observer system itself should be modelled as a bundle map to be consistent with the overall setup. In the next section, we will start the design process by examining possible objectives for such an observer design. Central to this will be the notion of *observation error* that is used to quantify how well the observer is achieving its objectives.

We finish this section with two examples of system models that fit this framework.

Example 1: **(Spherical pendulum)** Here the state space $X = TS^2$, the tangent bundle of the unit sphere in $\mathbb{R}^3$, encoding position and velocity of the pendulum on the sphere. The total space $S = T^2S^2$, the bundle of pairs of tangent vectors on the unit sphere, whence the state dependent input spaces are the tangent spaces $U(x) = T_xS^2$ at points $x \in S^2$ encoding tangential forces applied to the pendulum. As the zero input section we choose the zero section in the vector bundle $S \longrightarrow X$ and as external variable space we choose $W = TS^2$ to measure the position of the pendulum and the applied tangential force. Local coordinate expressions for the map f can be found on page 9 of Brockett's paper [3] and $h(x, \dot{x}, u) = (x, u)$ in local coordinates. Note that $TX = TTS^2$, the double tangent bundle of the sphere, here. The map Dz_u just adds a zero vector in the double tangent "force" direction in $TS = TT^2S^2$. We choose $m = \mathrm{id}$ and hence $Y = W = TS^2$ and $\bar{h} = h$.

Example 2: **(Invariant systems on Lie groups)** Here the state space $X = G$, a finite dimensional connected Lie group. The total space $S = TG \simeq G \times \mathfrak{g}$, the trivial tangent bundle of G. Here $\mathfrak{g}$ denotes the Lie algebra associated with G. The input space can be globally identified with the Lie algebra $\mathfrak{g}$ and the map f will be associated with a left invariant vector field $\dot{x} = xu$, where $x \in G$, $u \in \mathfrak{g}$ and xu is shorthand notation for T_1L_xu, the tangent map of the left multiplication with $x \in G$ at the identity element $1 \in G$ applied to u. An important example of such a system is given by the orientation kinematics of a rigid body in 3D space, cf. [5], [4]. Here, the zero input section is again the zero section globally given by $z_u(x) = (x, 0)$ and hence $\bar{f}(x,u) = (xu, 0)$ with $TS \simeq TG \times \mathfrak{g}$. We choose $W = Y \times \mathfrak{g}$ where Y is a homogeneous space of G under a right group action denoted by '$\cdot$' and $h(x, u) = (x \cdot y_0, u)$, where $y_0 \in Y$ is a reference measurement. For convenience we choose $m = \mathrm{id}$ and hence $M = W$ and $\bar{h} = h$.

III. ERROR MODELS

The *observation error* will be encoded using a second fiber bundle structure $\pi_e\colon O \longrightarrow S$ over the system manifold S that we will call the *error bundle*. The fibers $E(s) = \pi_e^{-1}(s)$, $s \in S$ of π_e should be thought of as (potentially) system state and input dependent error spaces. Similar to the situation with the system bundle, we will additionally require the existence of a global *zero error section* $z_e\colon S \longrightarrow O$ encoding globally what the meaning of an exact observation is. The reason for calling the total space in the error bundle O (for *O*bserver) will become clear in the next section.

The error bundle will be the space in which the combined dynamics of the system and the observer evolve, so the usual objective of achieving asymptotically zero observation error translates to a requirement to render the image of the zero error section asymptotically stable in O.

Another fundamental objective in observer design is the ability for the observer to not make the observation error *worse* than the initial, instantaneous observation error. This is in fact close to Luenberger's original way of thinking about observers, see Theorem 1 in [6]. The mental concept we are using to describe this objective is that of *synchrony*, i.e. the observation error being constant along corresponding pairs of trajectories of the system and the observer, cf. [5]. Where the error spaces $E(s)$ are state and/or input dependent, this concept does not make immediate sense. Geometrically, what we want is for the combined dynamics to evolve "parallel" to the zero error section in O. The mathematical structure that allows to capture this geometric picture is that of an *Ehresmann connection* [7]. This consists of a complement $H(s)$ to each *vertical* subspace $V(s) = \operatorname{Ker} D\pi_e(s)$ of the tangent space T_sO at $s \in O$, where this complement $H(s)$ varies smoothly with $s \in O$. The vertical subspace at $s \in O$ can be thought of as the tangent space to the fiber $\pi_e^{-1}(s)$. The collection of the complementary subspaces $H(s)$ is called a *horizontal distribution*. In more sophisticated language, this process splits the tangent bundle TO into a Whitney sum of vertical and horizontal subbundles, $TO = V \oplus H$. A combined system and observer trajectory in O then corresponds to "constant error" if its tangent at every point $s \in O$ lies in the horizontal subspace $H(s)$. Obviously, we want to choose our Ehresmann connection such that the image of the zero error section is horizontal in that sense.

Another restriction for the selection of such an Ehresmann connection results from the desire to express the error dynamics in "sensible" local coordinates. Ideally, we want to choose local trivializations $\phi_e\colon (\hat{x}, x, u) \mapsto (e(\hat{x}, x, u), x, u)$ for $\pi_e\colon O \longrightarrow S$, where (x, u) are local coordinates for S as before and e stands for *e*rror, such that $\operatorname{Ker} De = H(s)$ and hence a "constant error" trajectory actually corresponds to error dynamics $\dot{e} = 0$ in these local coordinates in the error fiber. Hence the Ehresmann connection must be integrable (or, equivalently, the horizontal distribution $H(s)$ must be involutive) and hence must have trivial curvature.

In the following we will assume that our error bundle is endowed with an integrable Ehresmann connection and a horizontal global zero error section. We will briefly discuss how the error bundle could look like in Example 2 from the previous section before we discuss observer design in the next section.

Example 3: **(Example 2 continued)** Recall that the system manifold $S \simeq G \times \mathfrak{g}$ in the case of left invariant kinematics on a Lie group G. We wish to observe the Lie group state of the system so an appropriate error bundle is given by the trivial bundle $O = G \times S \simeq G \times G \times \mathfrak{g}$ with zero error section $z_e\colon (x, u) \mapsto (x, x, u)$ in coordinates. The tangent space $T_{(\hat{x},x,u)}O$ can be indentified with $T_{\hat{x}}G \times T_xG \times \mathfrak{g}$ using the vector space structure of the Lie algebra $\mathfrak{g}$. The vertical subspace $V(\hat{x}, x, u) \subset T_{\hat{x}}G \times T_xG \times \mathfrak{g}$ is then simply $V(\hat{x}, x, u) = T_{\hat{x}}G \times \{0\} \times \{0\}$ and an Ehresmann connection can be specified by a vertical vector valued 1-form, a so called connection form $c(\hat{x}, x, u)\colon T_{\hat{x}}G \times T_xG \times \mathfrak{g} \longrightarrow V(\hat{x}, x, u)$ such that the horizontal subspace $H(\hat{x}, x, u) = \operatorname{Ker} c(\hat{x}, x, u)$. Since the second two components of elements of $V(\hat{x}, x, u)$ are zero we can think of $c(\hat{x}, x, u)$ as being $T_{\hat{x}}G$-valued in this example. We choose $c(\hat{x}, x, u)(\dot{\hat{x}}, \dot{x}, \dot{u}) = \dot{\hat{x}} - T_1L_{\hat{x}}T_xL_{x^{-1}}\dot{x} = \dot{\hat{x}} - \hat{x}x^{-1}\dot{x} \in T_{\hat{x}}G$ where we have used the previous shorthand notation for the tangent maps of the left multiplication and hence $H(\hat{x}, x, u) = \{(\hat{x}x^{-1}\dot{x}, \dot{x}, \dot{u}) \,|\, (\dot{x}, \dot{u}) \in T_xG \times \mathfrak{g}\}$. Now obviously $\operatorname{Im} Dz_e(x, u) = \{(\dot{x}, \dot{x}, u) \,|\, (\dot{x}, \dot{u}) \in T_xG \times \mathfrak{g}\} = H(x, x, u) = H(z_e(x, u))$ and hence the zero error section is horizontal. A "sensible" set of error coordinates is given by the global trivialization $\phi_e\colon (\hat{x}, x, u) \mapsto (\hat{x}x^{-1}, x, u)$ where $e(\hat{x}, x, u) = \hat{x}x^{-1}$ is the right invariant error considered in [5]. Indeed, $De(\hat{x}, x, u)(\dot{\hat{x}}, \dot{x}, \dot{u}) = \dot{\hat{x}}x^{-1} - \hat{x}x^{-1}\dot{x}x^{-1}$ and hence $\operatorname{Ker} De(\hat{x}, x, u) = H(\hat{x}, x, u)$. Constant error in this error bundle hence corresponds to right synchrony [5]. This motivates the initially somewhat curios choice of Ehresmann connection in this example.

Before we move on to the next section we stress that this approach to modelling the observation error inherently allows for different choices of error coordinates in the form of different (local) trivializations of the error bundle. In fact, from a mathematical point of view the specification of a fiber bundle is incomplete without the specification of a family of local trivializations and the associated transformation group [8]. It is the intrinsic capability of the fiber bundle formalism to account for and clearly express the choice of different *frames of reference* in the modelling of physical systems that motivated us to this study in the first place. The reader is invited to work this out for the case $G = \mathrm{SO}(3)$ in Example 2, i.e. for the Special Orthogonal group carrying the orientation kinematics of a rigid body. With respect to what frame of reference is the above right invariant error measured? More on this topic can be found in [5] and the references therein.

IV. OBSERVER DESIGN

In the previous section we already hinted at the idea that the total space O of the error bundle will carry the combined system and observer dynamics and the observer dynamics themselves are hence a projection of these dynamics. Luenberger's original approach to observer design [6] was to study the class of systems whose trajectories "follow" the observed system's trajectories and then select one that has the additional capability to asymptotically reduce any initial, instantaneous observation error. In the classical observer construction for linear systems this is achieved by embedding a copy of the system dynamics within the observer dynamics that is then modified by a stabilizing innovation term.[1]

In analogy to this two step design procedure we will specify the total dynamics as a sum of two terms, one generating "constant error", i.e. horizontal trajectories and the other "reducing" the error, i.e. rendering the zero error section asymptotically stable. The horizontal part of the combined dynamics is given by the unique *horizontal lift* of $\bar{f}\colon S \longrightarrow TS$ (recall Figure 2) with respect to the Ehresmann connection in the error bundle. This results in a vector field $\hat{\bar{f}}\colon O \longrightarrow TO$ whose integral curves will be horizontal by design, i.e. the vector field can be viewed as a map $\hat{\bar{f}}\colon O \longrightarrow H$. To this we add a vertical *innovation term*, i.e. a vector field $\alpha\colon O \longrightarrow V$ whose value can in principle be any nonlinear function $\alpha(\hat{x}, x, u)$ in coordinates. The extended diagram describing the combined dynamics then looks as shown in Figure 3.

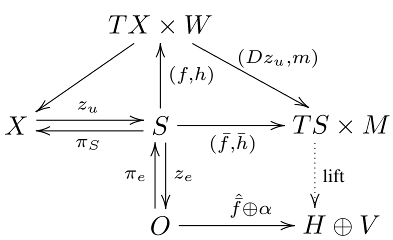

Fig. 3. Combined dynamics

All that is left to do is to project the combined dynamics onto state space dynamics on a smooth manifold $\hat{X}$ with the result then constituting an *observer*. This amounts to completing the diagram in Figure 3 as shown in Figure 4. The value of the observer state in $\hat{X}$ can then be interpreted as an estimate of $\Phi(x)$ where $x \in X$ is the system state and $\Phi\colon X \longrightarrow \hat{X}$ is defined by the diagram in Figure 4. This map Φ plays the same role in our theory as in van der Schaft's original work on nonlinear observers given by bundle maps [10]. Our construction is however considerably more general. Note that our construction doesn't require *a priori* that the observer state manifold X be diffeomorphic to $\hat{X}$ and hence also allows for the construction of partial state observers. There are however implicit restrictions on

[1]Note that this is a modern interpretation of how a Luenberger observer works and was developed subsequent to Luenberger's original work.

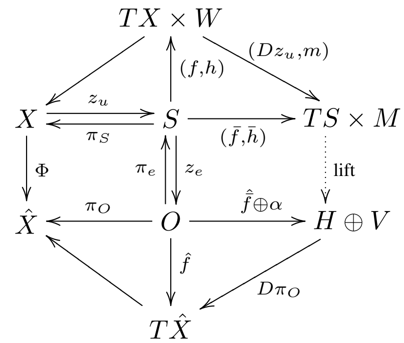

Fig. 4. The complete system and observer diagram

the choice of $\hat{X}$ in the sense that it must allow a diagram of the form shown in Figure 4. In the special case treated in van der Schaft's work [10] this restriction is shown to be locally equivalent to the existence of an involutive conditioned invariant distribution on X.

Another, independent question raised by this observer construction method is that of *implementability* of the resulting observer. In particular, we must be able to choose local coordinates such that the resulting local observer dynamics do not explicitly depend on the system state x. We don't pursue this question in full generality here, but we will show how this works out in the example given above. In particular, implementability requires that the innovation term can be locally written as a function of the observer state $\hat{x}$ and the measured system variables m only.

Example 4: **(Example 2 continued)** Recall that $O = G \times G \times \mathfrak{g}$ with coordinates $(\hat{x}, x, u)$ in this example. The horizontal lift of the vector field $\bar{f}(x, u) = (xu, 0)$ is given by $\hat{\bar{f}}(\hat{x}, x, u) = \hat{x}u$ in this example and the analogy to the linear situation (copy of the system dynamics) is evident. In [4] it was shown that if Y in the measurement space $M = Y \times \mathfrak{g}$ is a reductive homogeneous space of G then a vertical innovation term can be constructed in form of a lifted gradient of an invariant cost function on Y ("lifted" in this context refers to the group action on Y and not to our Ehresmann connection in the error bundle). This construction yields gradient error dynamics for the right invariant error e with strong asymptotic stability properties for suitable (Morse-Bott) cost functions. For the details see [4].

V. CONCLUSIONS

We have introduced a geometric framework for the design and analysis of nonlinear input-state-output systems in fiber bundle form. We have shown that this construction generalizes van der Schaft's fiber bundle observers and also contains recent work by Lageman and two of the authors on observers for invariant systems on Lie groups as a special case. Similarly, it can be shown that the invariant observers of Bonnabel, Martin and Rouchon fit in this framework as does the nonlinear equivalent of the classical Luenberger observer construction.

Similar to other general frameworks for observer design, our new framework does not present any direct, constructive method for choosing a stabilizing innovation term. However, by clarifying the geometry of the observer dynamics as being composed of a horizontal part and a vertical innovation term it provides additional structure that aids in the subsequent design and analysis process. The framework also clarifies the role of error coordinates as local trivializations of the fiber bundle supporting the combined system and observer dynamics.

In future work we aim to clarify some of the issues around projectability of the combined dynamics as well as implementability of the resulting observers. We will also look at an extension of the formalism that allows for external variables in the observer.

VI. ACKNOWLEDGMENTS

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