A Converse Liapunov Theorem for Uniformly Locally Exponentially Stable Systems Admitting Carathéodory Solutions ⋆

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Abstract: This paper provides a converse Liapunov theorem for uniformly locally exponentially stable, locally Lipschitz, non-linear, time-varying, possibly non-smooth systems that admit Carathéodory solutions. The main result proves that a critical point of such a system is uniformly locally exponentially stable if and only if the system admits a local (possibly non-smooth, time-varying) Liapunov function.

Keywords: converse Liapunov theorem, Carathéodory solution, uniform local exponential stability

1. INTRODUCTION

Converse Liapunov results are an important tool in the analysis of stability of classical "smooth" systems and have been well studied, see for example Hahn (1967); Khalil (1996); Corless and Glielmo (1998) and citations therein. Non-smooth systems are important in a wide range of applications; for example, switched and hybrid systems, piecewise linear or discontinuous systems, sliding mode systems, non-smooth control designs for non-holonomic systems, networked control systems, and robust stability analysis where the plant variation is non-smooth or discontinuous. In a recent survey article Cortés (2008) provides an excellent overview of the existence and uniqueness of different solution classes, principally Fillipov or Carathéodory solutions, for non-smooth systems and discusses some questions of stability. There is a considerable body of work available for the stability analysis of solutions in the sense of Filippov (1988) of non-smooth systems. In particular, we mention the work of Clarke et al. (1998b), Aubin and Celina (1994) and Bacciotti and Rosier (2005), as well as Cortés (2008) recent review article and the references contained in these works. Analysis of Carathéodory solutions appears to be somewhat less developed than for Fillipov solutions. The existence of Carathéodory solutions in switched non-linear control systems has been considered by a number of authors Polycarpou and Ioannou (1993); Kim and Ha (2004). Ancona and Bressan have studied the stabilization problem for Carathéodory solutions of vector fields patched together from smooth vector fields Ancona and Bressan (1999, 2002, 2004). Bacciotti and Ceragioli (2006) provide sufficient conditions for stability and asymptotic stability of Carathéodory solutions. A recent article by Grzanek et al. (2008) uses a general form of the integral theorem of calculus to guarantee (Liapunov) stability of solutions, often the most difficult step in all the above results, and go on to prove asymptotic stability for general Carathéodory solutions of system with a Liapunov function. Aeyels and Peuteman (1998) have studied stability of systems using an integral formulation of the Liapunov decent condition. Although their work is focused on applications in time-varying and averaged systems, it has direct application to analysis of non-smooth systems with Carathéodory solutions. Aeyels et al. also consider the question of exponential stability of systems Aeyels and Peuteman (1999). Other recent work on exponential stability of nonlinear systems using discontinuous Liapunov functions was presented in Linh and Phat (2001). Both these works provide sufficient conditions for exponential stability of non-linear time-varying systems using Liapunov comparison functions. Converse Liapunov results for stability are classical for smooth systems Massera (1949, 1956); Kurzweil (1963). There is a body of work concerning converse Liapunov results for switched systems Liberzon and Morse (1999); Dayawansa and Martin (1999). In work by Lin et al. (1996) a general converse Liapunov result was proved that showed existence of a smooth Liapunov function for a system with bounded disturbances, a result with application to robust stability analysis of systems. Clarke et al. (1998a) showed existence of smooth Liapunov functions for strongly asymptotically stable systems with solutions of the Fillipov type. The existence of smooth Liapunov functions is somewhat counter intuitive in both these cases as the underlying system is non-smooth. More recent work in converse Liapunov results has considered non-uniform asymptotic stability Karafyllis and Tsinias (2003). A further perspective is obtained using the structure of skew product flows Grüne et al. (2007).

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In this paper, we provide a Liapunov function characterisation of uniform local exponential stability (ULES) for locally Lipschitz, non-linear, time-varying systems that admit Carathéodory solutions. The main result proves that a critical point of such a system is uniformly locally exponentially stable if and only if it admits a *ULES Liapunov* function, a Liapunov function that exhibits certain growth behavior and regularity (see Definition 2). The result is based on a generalisation of the Langenhop (1960) inequality that provides a lower bound on the rate of decrease of solutions. Our result requires only that a Carathéodory solution exists and that the vector field is uniformly Lipschitz around the origin. This appears to be a stronger result than other extensions of Langenhop that we have found in the literature Vidyasagar (2002). The Langenhop inequality is fundamental in the existence proof of the ULES Liapunov function for uniformly locally exponentially stable systems. The converse Liapunov result obtained dovetails with a theorem from Aeyels and Peuteman (1999) to provide the if and only if Liapunov characterisation of uniform local exponential stability.

The paper is organised into four sections including the present introduction, two main technical sections, and a short conclusion (Section 4). In the technical sections, Section 2 presents the problem formulation and proves the generalisation of the Langenhop inequality. Section 3 presents the converse Liapunov theorem for uniformly locally exponentially stable systems and states the main result.

2. A LANGENHOP INEQUALITY FOR TRAJECTORIES

Consider the time-varying system

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = f(t, x(t)) \tag{1}$$

where $I \subset \mathbb{R}$ is a non-empty interval, $U \subset \mathbb{R}^n$ is a non-empty open set and $f\colon I \times U \longrightarrow \mathbb{R}^n$. To begin with, we do not place any regularity conditions on f but we will introduce some conditions as we go.

The term *solution of* (1) will mean a Carathéodory solution, i.e. a function $x\colon J \longrightarrow U$ where $J \subset I$ is a non-empty interval such that $f(\cdot, x(\cdot))$ is locally (Lebesgue) integrable on J and such that $x(\cdot)$ fulfils the associated integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} f(\tau, x(\tau))\mathrm{d}\tau \tag{2}$$

for some $t_0 \in J$ and all $t \in J$. Since a Carathéodory solution is given by an integral of the form (2), it is automatically absolutely continuous. Hence, a Carathéodory solution is differentiable almost everywhere and fulfils equation (1) at the points where it is differentiable.

It is, of course, well known that Carathéodory solutions exist for all initial data $(t_0, x_0) \in I \times U$, $x(t_0) = x_0$ and that maximal [1] such solutions are unique if f fulfils the respective Carathéodory conditions [2]. To obtain our first result, we will not assume any of these conditions *a priori*, since the result is merely concerned with individual solutions (trajectories). However, the hypotheses of Theorem 1 below almost imply the usual Carathéodory conditions, at least "along" the trajectory under investigation. We refer the reader to the excellent book by Filippov (1988) for a thorough discussion of various notions of weak solutions and their respective properties. There is also a recent survey paper by Cortés (2008) that is available online.

The following is a version of Langenhop's inequality (Langenhop (1960)) for individual Carathéodory solutions. It is best applicable for locally uniformly Lipschitz systems in a neighborhood of a stationary solution. See the discussion below the theorem for the details.

Theorem 1. Let $x\colon J \longrightarrow U$ be a (Carathéodory) solution of (1) and let $\|x(t)\| > 0$ for some $t \in J$. Pick $t_0 \in J$ with $t_0 < t$. Assume that there exists $L > 0$ such that

$$\|f(\tau, x(\tau))\| \leq L \cdot \|x(\tau)\| \tag{3}$$

for almost all $\tau \in [t_0, t]$. Then

$$\|x(t)\| \geq \|x(t_0)\| \cdot \mathrm{e}^{-L(t-t_0)}\,.$$

Proof. Since $x(\cdot)$ is a solution of the integral equation (2) we have that

$$x(t) = x(\tau) + \int_{\tau}^{t} f(s, x(s))\mathrm{d}s$$

for all $t_0 \leq \tau \leq t$ and hence

$$\|x(t)\| \geq \|x(\tau)\| - \int_{\tau}^{t} \|f(s, x(s))\|\mathrm{d}s$$

for all $t_0 \leq \tau \leq t$. Defining

$$\phi(\tau) := \|x(t)\| + \int_{\tau}^{t} \|f(s, x(s))\|\mathrm{d}s$$

it follows that $\phi(\tau) \geq \|x(t)\| > 0$ and $\phi(\tau) \geq \|x(\tau)\|$ for all $t_0 \leq \tau \leq t$. But then $L \cdot \phi(\tau) \geq L \cdot \|x(\tau)\| \geq \|f(\tau, x(\tau))\|$ and hence

$$L - \frac{\|f(\tau, x(\tau))\|}{\phi(\tau)} \geq 0$$

for almost all $\tau \in [t_0, t]$. Integrating this inequality from t_0 to t yields

$$L(t - t_0) - \int_{t_0}^{t} \frac{\|f(\tau, x(\tau))\|}{\phi(\tau)}\mathrm{d}\tau \geq 0. \tag{4}$$

Next, we note that

$$\psi(s) := \begin{cases} 0 & , s < t_0 \\ \|f(s, x(s))\| & , t_0 \leq s \leq t \\ 0 & , s > t \end{cases}$$

is integrable over $\mathbb{R}$ and hence $\phi(\cdot)$ is absolutely continuous and in particular differentiable almost everywhere, namely at every Lebesgue point of $\psi(\cdot)$, and moreover $\phi'(\tau) = -\|f(\tau, x(\tau))\|$ at every such point $\tau \in [t_0, t]$, see e.g. (Rudin, 1987, Theorem 7.11). Hence (4) is equivalent to

$$L(t - t_0) + \int_{t_0}^{t} \frac{\phi'(\tau)}{\phi(\tau)}\mathrm{d}\tau \geq 0.$$

Since $\phi(\cdot)$ is absolutely continuous and bounded away from zero by $\|x(t)\| > 0$, the logarithm $\log(\phi(\cdot))$ is

[1] A solution $x\colon J \longrightarrow U$, $J \subset \tilde{J}$, is called *maximal* if it can not be continued beyond its existence interval J, i.e. if for every solution $\tilde{x}\colon \tilde{J} \longrightarrow U$ that coincides with x on J it follows that $\tilde{J} = J$.

[2] The usual Carathéodory conditions for existence of solutions are continuity in x for almost all t, measurability in t for all x and boundedness in x for almost all t where the bound is absolutely continuous in t. The usual condition for uniqueness of solutions is a local Lipschitz condition in x. For the details see e.g. Filippov (1988).

also absolutely continuous with derivative $\frac{\mathrm{d}}{\mathrm{d}\tau}\log(\phi(\tau)) = \phi'(\tau)/\phi(\tau)$ almost everywhere on $[t_0,t]$. Applying the fundamental theorem of calculus in a version for absolutely continuous functions, see e.g. (Rudin, 1987, Theorem 7.20), yields

$$\log(\phi(t)) \geq \log(\phi(t_0)) - L(t-t_0)$$

and hence using $\phi(t) = \|x(t)\|$ and $\phi(t_0) \geq \|x(t_0)\|$ that

$$\log(\|x(t)\|) \geq \log(\|x(t_0)\|) - L(t-t_0).$$

Taking exponentials on both sides completes the proof.

A few remarks are in order. For $r > 0$ and $x \in \mathbb{R}^n$ denote $B_r(x) := \{y \in \mathbb{R}^n \,|\, \|y-x\| < r\}$. The standard Carathéodory condition for uniqueness of solutions is that f is locally Lipschitz in x, or more precisely that for every $x \in U$ there exist an $r_x > 0$ and a locally integrable function $L_x\colon I \longrightarrow [0,\infty)$ such that $B_{r_x}(x) \subset U$ and

$$\|f(t,y) - f(t,z)\| \leq L_x(t)\|y-z\| \tag{5}$$

for all $y,z \in B_{r_x}(x)$ and all $t \in I$. The case where the local Lipschitz functions $L_x(\cdot)$ are bounded is of particular interest in the discussion of stability of stationary solutions, see e.g. Aeyels and Peuteman (1998).

For systems with $f(t,0) = 0$ almost everywhere, i.e. where $0 \in U$ is a stationary solution, our condition (3) follows from essential boundedness of the local Lipschitz functions $L_x(\cdot)$, where that bound is uniform in $x \in U$. In particular, if zero is a *stable* stationary solution then solutions that start close enough to zero remain in a compact set and hence (possibly non-uniform) essential boundedness of the local Lipschitz functions $L_x(\cdot)$ alone guarantees (3) for solutions that live close enough to zero. In the following, we will use Theorem 1 only in that latter context.

3. A CONVERSE LIAPUNOV THEOREM FOR UNIFORMLY LOCALLY EXPONENTIALLY STABLE SYSTEMS

In this section, we will assume that $[t_*,\infty) \subset I$ for some $t_* \in \mathbb{R}$ and that system (1) has unique maximal solutions [3] for all initial data $(t_0,x_0) \in I \times U$, $x(t_0) = x_0$. We will denote the corresponding maximal existence intervals by $J_\phi(t_0,x_0) \subset I$ and to avoid confusion, we will write $\phi(t;t_0,x_0)$ for the value of the solution with initial data (t_0,x_0) at time $t \in J_\phi(t_0,x_0)$.

Assume now that $f(t,0) = 0$ for almost all $t \in [t_*,\infty)$, i.e. that $0 \in U$ is a stationary solution of system (1). Recall that the zero solution is called *uniformly locally exponentially stable* if there exist positive numbers r, m and λ such that $J_\phi(t_0,x_0) \supset [t_0,\infty)$ and

$$\|\phi(t;t_0,x_0)\| \leq \|x_0\| \cdot m\,\mathrm{e}^{-\lambda(t-t_0)} \tag{6}$$

for all $x_0 \in B_r(0)$ and all $t \geq t_0 \geq t_*$. Note that this implies $m \geq 1$.

Definition 2. Let $\eta > 0$. A function $V\colon [t_*,\infty)\times B_\eta(0) \longrightarrow [0,\infty)$ is called a uniform local exponential stability Liapunov function, *ULES Liapunov function*, for system (1) if there exist positive numbers α, β, γ, δ, $\eta' < \eta$ and L such that

(1) $\alpha\|x\|^2 \leq V(t,x) \leq \beta\|x\|^2$,
(2) $|V(t,x) - V(t,y)| \leq L\cdot(\|x\|+\|y\|)\cdot\|x-y\|$ and
(3) $V(t+\delta,\phi(t+\delta;t,z)) - V(t,z) \leq -\gamma\|z\|^2$

for all $x,y \in B_\eta(0)$, all $z \in B_{\eta'}(0)$ and all $t \geq t_*$.

Theorem 4 below yields existence of ULES Liapunov functions for systems with essentially bounded local Lipschitz functions $L_x(\cdot)$ for which zero is a uniformly locally exponentially stable stationary solution. See the previous section for a short discussion of local Lipschitz functions. The proof of the theorem uses the following technical lemma.

Lemma 3. Assume that the local Lipschitz functions $L_x(\cdot)$ in (5) exist and are essentially bounded, i.e. for every $x \in U$ there exists $M_x > 0$ such that $L_x(t) \leq M_x$ for almost all $t \in I$. Let $K \subset U$ be a compact, convex set then there exists $M > 0$ such that $\|f(t,x) - f(t,y)\| \leq M\cdot\|x-y\|$ for almost all $t \in I$ and all $x,y \in K$. Furthermore, let $\phi(\cdot;t_0,x_0)$ and $\phi(\cdot;t_0,y_0)$, $x_0,y_0 \in K$ be two solutions of (1) that both exist and remain in K for some time interval $[t_0,t_1] \subset I$ with $t_1 \geq t_0$. Then

$$\|\phi(t;t_0,x_0) - \phi(t;t_0,y_0)\| \leq \|x_0-y_0\|\cdot\mathrm{e}^{M(t-t_0)}$$

for all $t_0 \leq t \leq t_1$.

Proof. Cover the compact set K with a finite number of local Lipschitz domains $B_{r_x}(x)$, cf. (5), and let $M > 0$ denote the maximum of the finitely many associated essential bounds M_x for the local Lipschitz functions $L_x(\cdot)$. A simple application of the triangle inequality shows that $\|f(t,x)-f(t,y)\| \leq M\cdot\|x-y\|$ for all $x,y \in K$ by breaking down the bounded line segment joining x and y within the convex set K into finitely many pieces, each covered by one of the local Lipschitz domains. This argument holds for almost all $t \in I$ since the union of finitely many sets of measure zero, namely the time sets where the individual bounds M_x may not hold, is again a set of measure zero.

Regarding the second statement, we have that

$$\phi(t;t_0,x_0) = x_0 + \int_{t_0}^{t} f(\tau,\phi(\tau;t_0,x_0))\mathrm{d}\tau$$

for all $t_0 \leq t \leq t_1$ and analogously for $\phi(t;t_0,y_0)$. Hence the difference $\psi(t) := \phi(t;t_0,x_0) - \phi(t;t_0,y_0)$ fulfils

$$\|\psi(t)\| \leq \|\psi(t_0)\| + \int_{t_0}^{t} M\cdot\|\psi(\tau)\|\mathrm{d}\tau$$

where we have used the first part of the lemma. Since $\|\psi(\cdot)\|$ is continuous, the statement now follows from an application of Gronwall's inequality, see e.g. (Hale, 1969, Corollary I.6.6).

Theorem 4. Let $[t_*,\infty) \subset I$ for some $t_* \in \mathbb{R}$ and let system (1) have unique maximal solutions for all initial data $(t_0,x_0) \in I\times U$, $x(t_0) = x_0$. Assume further that the local Lipschitz functions $L_x(\cdot)$ in (5) exist and are essentially bounded, i.e. for every $x \in U$ there exists $M_x > 0$ such that $L_x(t) \leq M_x$ for almost all $t \in I$. Let zero be a uniformly locally exponentially stable solution of (1). Then there exists a ULES Liapunov function for system (1).

Proof. Let $T > 0$ be arbitrary but fixed. There exists a constant $c \geq 1$ such that $\|x\|_2 \leq c\|x\|$ and $\|x\| \leq c\|x\|_2$ for all $x \in \mathbb{R}^n$, where $\|\cdot\|$ is our arbitrary norm and $\|\cdot\|_2$ is the Euclidean norm. Using the exponential bound (6),

[3] We choose to make this assumption directly in Theorems 4 and 5 as it is a key conceptual condition for the results presented. The results, however, could be stated without this requirement by noting for example that Lemma 3 implies uniqueness of maximal solutions.

$$\|\phi(\tau;t,x)\|_2 \leq \|x\| \cdot mc\,\mathrm{e}^{-\lambda(\tau-t)} \tag{7}$$

for all $x \in B_r(0)$ and all $\tau \geq t \geq t_*$ and hence

$$V(t,x) := \int_t^{t+T} \|\phi(\tau;t,x)\|_2^2 \mathrm{d}\tau$$

is well-defined for all $x \in B_r(0)$ and all $t \geq t_*$. Furthermore, $V(t,x) \leq \beta\|x\|^2$ for all $x \in B_r(0)$ and all $t \geq t_*$, where $\beta := (1 - \mathrm{e}^{-2\lambda T})(mc)^2/(2\lambda) > 0$.

From (7) it follows that $\phi(\tau;t,x) \in B_{r'}(0)$ for all $x \in B_r(0)$, where $r' := rmc^2 > 0$. Applying Lemma 3 to the compact, convex set $\overline{B_{r'}(0)}$ shows that $\|f(t,x) - f(t,y)\| \leq M \cdot \|x-y\|$ for all $x, y \in \overline{B_{r'}(0)}$. In particular, choosing $y = 0$ this implies $\|f(\tau,\phi(\tau;t,x))\| \leq M \cdot \|\phi(\tau;t,x)\|$ for all $x \in B_r(0)$ and all $\tau \geq t \geq t_*$. Now let $x \in B_r(0)$, $x \neq 0$. Applying Theorem 1 yields

$$\|\phi(\tau;t,x)\|_2 \geq \|x\| \cdot c\,\mathrm{e}^{-M(\tau-t)} \tag{8}$$

for all $\tau \geq t \geq t_*$ and hence $V(t,x) \geq \alpha\|x\|^2$ where $\alpha := (1 - \mathrm{e}^{-2MT})c^2/(2M) > 0$. The inequality is trivial for $x = 0$.

To deduce the second statement of the theorem, compute

$$\begin{aligned}
&|V(t,x) - V(t,y)| \\
&\quad = \left|\int_t^{t+T} \left(\|\phi_x\|_2^2 - \|\phi_y\|_2^2\right) \mathrm{d}\tau\right| \\
&\quad = \left|\int_t^{t+T} \left(\phi_x^\top \phi_x - \phi_y^\top \phi_y\right) \mathrm{d}\tau\right| \\
&\quad = \left|\int_t^{t+T} \left(\phi_x^\top(\phi_x - \phi_y) + (\phi_x - \phi_y)^\top \phi_y\right) \mathrm{d}\tau\right| \\
&\quad \leq c^2 \int_t^{t+T} (\|\phi_x\| + \|\phi_y\|) \cdot \|\phi_x - \phi_y\| \mathrm{d}\tau \\
&\quad \leq mc^2 (\|x\| + \|y\|) \cdot \|x-y\| \int_t^{t+T} \mathrm{e}^{(M-\lambda)(\tau-t)}\,\mathrm{d}\tau
\end{aligned}$$

for all $x \in B_r(0)$ and all $t \geq t_*$, where we have used the shorthand notation $\phi_x := \phi(\tau;t,x)$ and $\phi_y := \phi(\tau;t,y)$. Here, the last inequality follows by applying the exponential bound (6) and Lemma 3. The desired inequality now follows by setting $L := (\mathrm{e}^{(M-\lambda)T} - 1)mc^2/(M-\lambda) > 0$.

For the third and final statement of the theorem, we start by observing that (7) implies

$$\|\phi(\tau;t,x)\|_2 \leq \|x\|_2 \cdot m'\,\mathrm{e}^{-\lambda(\tau-t)} \tag{9}$$

where $m' := mc^2 \geq m \geq 1$. We compute

$$\begin{aligned}
&V(t+\delta, \phi(t+\delta;t,x)) - V(t,x) \\
&\quad = \int_{t+\delta}^{t+\delta+T} \|\phi(\tau;t+\delta,\phi(t+\delta;t,x))\|_2^2 \mathrm{d}\tau \\
&\qquad\qquad - \int_t^{t+T} \|\phi(\tau;t,x)\|_2^2 \mathrm{d}\tau \\
&\quad = \int_{t+\delta}^{t+\delta+T} \|\phi(\tau;t,x)\|_2^2 \mathrm{d}\tau - \int_t^{t+T} \|\phi(\tau;t,x)\|_2^2 \mathrm{d}\tau \\
&\quad = \int_t^{t+T} \left(\|\phi(\tau+\delta;t,x)\|_2^2 - \|\phi(\tau;t,x)\|_2^2\right) \mathrm{d}\tau \\
&\quad = \int_t^{t+T} \left(\|\phi(\tau+\delta;\tau,\phi(\tau;t,x))\|_2^2 - \|\phi(\tau;t,x)\|_2^2\right) \mathrm{d}\tau \\
&\quad \leq \left({m'}^2\,\mathrm{e}^{-2\lambda\delta} - 1\right) \int_t^{t+T} \|\phi(\tau;t,x)\|_2^2 \mathrm{d}\tau \\
&\quad \leq \left({m'}^2\,\mathrm{e}^{-2\lambda\delta} - 1\right) \beta\|x\|^2
\end{aligned}$$

for all $x \in B_r(0)$ and all $t \geq t_*$, where the second and fourth equalities follow from the uniqueness of maximal solutions, the first inequality follows from (9), and the last inequality follows from the first part of the theorem. The statement now follows by choosing $\delta > \log(m')/\lambda$ and $\gamma := (1 - {m'}^2\,\mathrm{e}^{-2\lambda\delta})\beta > 0$.

This leads to the main result of the paper.

Theorem 5. Let $[t_*, \infty) \subset I$ for some $t_* \in \mathbb{R}$ and let system (1) have unique maximal solutions for all initial data $(t_0, x_0) \in I \times U$, $x(t_0) = x_0$. Assume further that the local Lipschitz functions $L_x(\cdot)$ in (5) exist and are essentially bounded, i.e. for every $x \in U$ there exists $M_x > 0$ such that $L_x(t) \leq M_x$ for almost all $t \in I$. Let $0 \in U$ and $f(t,0) = 0$ for almost all $t \in I$.

Then zero is a uniformly locally exponentially stable solution of system (1) if and only if there exists a ULES Liapunov function for system (1) at zero, cf. Definition 2.

Proof. Existence of a ULES Liapunov function follows from Theorem 4. The reverse implication was proved by Aeyels and Peuteman (Aeyels and Peuteman, 1999, Theorem 1).

4. CONCLUSION

We have provided a characterization of uniform local exponential stability (ULES) for locally Lipschitz, nonlinear, time-varying, possibly non-smooth systems that admit Carathéodory solutions in terms of the existence of a certain type of local Liapunov function that we dubbed ULES Liapunov function. In future work this characterization will be used to provide a criterion for uniform local exponential stability in terms of exponential stability of the linearization.

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