A New Parametrization of Linear Observers

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***Abstract*—The estimation of unknown variables of a system, the plant, based on a set of known or measured variables is a fundamental and widely studied problem of system theory. We investigate observers that solve this task in the behavioral setting. Our framework includes the discrete and the continuous linear time-invariant cases, and various notions of when a signal is considered to be small. We characterize the observers that guarantee a small estimation error as those containing the "essential part" of the plant behavior. A constructive one-to-one parametrization of all such observers follows naturally from this result.**

I. Introduction

Estimating the values of state variables in a control system or, more generally, estimating the values of *some* subset of the system variables from measurements of some *other* subset of the system variables, is a fundamental problem in modern system theory and has therefore been very widely studied. In the context of deterministic (non-stochastic) system models the problem is known as the *observer problem* and, for the case of linear systems, can be traced back to the work of Kalman [10] and Luenberger [11]. More recently, the linear observer problem has been studied from the point of view of behavioral system theory by Valcher and co-authors [20], [1]. This paper is a contribution to behavioral observer theory although its conclusions apply equally well to classical state space systems.

From an applied point of view, it initially appears that a solution to the *observer construction problem* is all that is needed; i.e., given an observed system (the plant), find an observer that works for the given plant. Here, "working" typically means that the observer error is guaranteed to be small in some precise sense. For linear time-invariant finite-dimensional state space systems and asymptotically zero observer error this problem was already solved by Luenberger [12] and leads to an observer construction based on solving a certain Sylvester equation for an unknown stable matrix. From a more mathematical perspective, and also if one is concerned with additional observer properties such as optimality with respect to L^2-sensitivity or robustness, the additional problem of *characterizing all solutions* to the observer problem for a given plant arises. Much work has also been done on this problem, including in the setup of behavioral system theory, culminating in the characterization of finite-dimensional linear time-invariant differential (LTID) observers for finite-dimensional LTID plants through an *internal model principle* [18] that echoes the classical internal model principle for the regulator problem. See the discussion in [18] for a historic perspective on this topic and references to other previous results. Ideally, one would like to have a *parametrization* of all solutions to the observer problem to be able to efficiently search for those observers that have the desired additional properties. This paper focusses on this last problem and provides a new parametrization of observers for linear systems while also extending the internal model principle to a more general setting.

The results are derived in the framework of modern algebraic system theory [14], [5] and make extensive use of the concept of quotient signal modules due to Oberst. This means that the results apply to a wide range of linear systems, including the standard discrete-time and continuous-time cases, as well as a wide range of notions of "smallness" of the error, including the usual asymptotic and dead-beat cases. The resulting parametrization is fundamentally different from previously reported parametrizations (e.g. [20], [6], [18]) in that it only uses free parameters (in particular, no spectral, matrix invertibility or unimodularity conditions on the parameters) and that it allows to decide certain additional observer properties by inspection. As a side result, input/output-observers [2],[17] are characterized by a minimality condition with respect to one of the parameters. A preliminary version of the results in this paper has been reported at the MTNS 2012 [4].

This paper is organized as follows. Sections II and III cover necessary background material on behavioral observer theory and quotient signal modules, respectively. Section IV introduces the precise problem formulation for the observer parametrization problem. Section V extends the internal model principle to this paper's more general setup and Section VI contains additional results on the error behavior, including results on achievable error dynamics and on existence of observers. Section VII discusses the connection of these results to prior work while Section VIII states and proves the main new parametrization result. In Section IX the results are demonstrated by an example. A short set of conclusions is provided in Section X.

II. Behavioral observers

We assume a plant system with behavior $\mathscr{P}$, with components w_1, w_2, and w_3 (compare Figure 1a), where w_1 is known or can be measured, and we are interested in (an estimate for) w_2. The components w_3 are not known nor of

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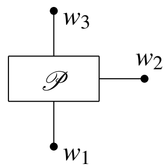
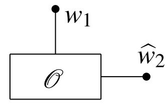
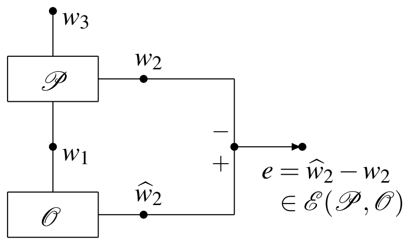


(a) The plant.



(b) The observer.



(c) The observer interconnection.

Fig. 1: The connection of plant and observer.

any interest, and are often referred to as irrelevant variables. In the standard case of Kalman state space systems, the known components would typically be the system's input and output variables, and the to-be-estimated components would be part of the state variable. An *observer* is a second behavior that can be connected with the plant via w_1 and produces an estimate $\widehat{w}_2$ for the desired components (compare Figure 1b). More precicely: plant and observer share the components w_1 and give rise to the *interconnection*

$$\mathscr{P} \wedge_{w_1} \mathscr{O} := \left\{ \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \widehat{w}_2 \end{pmatrix}; \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathscr{P}, \begin{pmatrix} w_1 \\ \widehat{w}_2 \end{pmatrix} \in \mathscr{O} \right\}.$$

Following Willems we use the intuitive notation $\mathscr{P} \wedge_{w_1} \mathscr{O}$ for the interconnection of $\mathscr{P}$ and $\mathscr{O}$ via the shared components w_1. The error behavior $\mathscr{E}(\mathscr{P}, \mathscr{O})$ (compare [20, p.2299]) is defined as the set of all estimation errors $\widehat{w}_2 - w_2$ that may occur in the connected behavior, i.e.,

$$\mathscr{E}(\mathscr{P}, \mathscr{O}) := \left\{ e = \widehat{w}_2 - w_2; \exists w_1, w_3 : \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathscr{P}, \begin{pmatrix} w_1 \\ \widehat{w}_2 \end{pmatrix} \in \mathscr{O} \right\},$$

compare Figure 1c. The task is to decide whether there exist observers such that the error behavior consists only of small signals, and additionally the interconnection with the observer does not essentially restrict the plant behavior ("nonintrusiveness" of the observer). If such observers do exist, all of them shall be constructed.

Remark 1. The following simplification is always possible: Denote by $\mathscr{P}_{(w_1, w_2)}$ the projection of $\mathscr{P}$ onto the variables w_1 and w_2, i.e.,

$$\mathscr{P}_{(w_1, w_2)} := \left\{ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}; \exists w_3 : \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathscr{P} \right\}.$$

In our framework (see Section III) $\mathscr{P}_{(w_1, w_2)}$ is then again a behavior which can be computed from the original behavior $\mathscr{P}$. Moreover,

$$\begin{aligned} \mathscr{E}(\mathscr{P}, \mathscr{O}) &= \mathscr{E}(\mathscr{P}_{(w_1, w_2)}, \mathscr{O}) \\ &= \left\{ e = \widehat{w}_2 - w_2; \exists w_1 : \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathscr{P}_{(w_1, w_2)}, \begin{pmatrix} w_1 \\ \widehat{w}_2 \end{pmatrix} \in \mathscr{O} \right\} \end{aligned}$$

and hence the task of finding observers $\mathscr{O}$ such that the error behavior is small can be conducted for $\mathscr{P}_{(w_1, w_2)}$ instead of $\mathscr{P}$. Therefore, we may and do always assume that no irrelevant variables w_3 are present in the plant, or that they have already been eliminated.

III. Quotient signal spaces

In the following we specify the signal space $\mathscr{F}$ and the class of behaviors we consider. Let F be a field (usually $F = \mathbb{R}$ or $F = \mathbb{C}$), and let $\mathscr{D} := F[s]$ denote the polynomial ring over F in one indeterminate s. We denote the action of an element $f \in \mathscr{D}$ on $w \in \mathscr{F}$ by $f \circ w \in \mathscr{F}$. The discrete standard signal space is the set of F-valued sequences, i.e.,

$$\mathscr{F} = F^{\mathbb{N}} = \{(w(0), w(1), w(2), \ldots); \forall n \in \mathbb{N} \colon w(n) \in F\}.$$

This is a module over $\mathscr{D} = F[s]$ by defining the action $(s \circ w)(n) := w(n+1)$ for $w \in \mathscr{F}$ and $n \in \mathbb{N}$, and accordingly

$$(\sum_{i=0}^{d} a_i s^i \circ w)(n) = \sum_{i=0}^{d} a_i w(n+i)$$

for $f = \sum_{i=0}^{d} a_i s^i \in \mathscr{D}$, $w \in \mathscr{F}$, and $n \in \mathbb{N}$. Hence the indeterminate s is interpreted as the left shift operator, and the ring $\mathscr{D}$ is the ring of (linear, time-invariant) difference operators acting on $\mathscr{F}$. In the continuous standard case we normally choose $F = \mathbb{R}$ or $F = \mathbb{C}$, and the signal space is the set of functions from $\mathbb{R}$ to F that are infinitely often differentiable, or alternatively the set of F-valued distributions, i.e.,

$$\mathscr{F} = \mathscr{C}^\infty(\mathbb{R}, F) \quad \text{or} \quad \mathscr{F} = \mathscr{D}'(\mathbb{R}, F).$$

Both of these sets are again modules over $\mathscr{D} = F[s]$ by defining $s \circ w$ as the derivative $w' = \frac{dw}{dt}$ for any signal $w \in \mathscr{F}$, hence for $f = \sum_{i=0}^{d} a_i s^i \in \mathscr{D}$ and $w \in \mathscr{F}$

$$f \circ w = \sum_{i=0}^{d} a_i \frac{d^i w}{dt^i} = f(\tfrac{d}{dt}) w.$$

Thus s is interpreted as the operator $\frac{d}{dt}$, and the ring $\mathscr{D}$ as the ring of (linear, time invariant) differential operators.

A *behavior* (more precisely, a ${}_{\mathscr{D}}\mathscr{F}$*-behavior*) is a set of the form

$$\mathscr{B} = \left\{ w \in \mathscr{F}^\ell; R \circ w = 0 \right\}$$

for some matrix $R \in \mathscr{D}^{k \times \ell}$ where $(R \circ w)_i := \sum_{j=1}^{\ell} R_{ij} \circ w_j$ for $i = 1, \ldots, k$. In the discrete resp. continuous standard cases this means that $\mathscr{B}$ is the solution set of the set of linear time-invariant homogeneous difference resp. differential equations given by the rows of R.

More generally, the signal space $\mathscr{F}$ may be any module over $\mathscr{D}$ that is an *injective cogenerator* over $\mathscr{D}$. We will briefly explain what this property signifies and why it is important. The injectivity of $\mathscr{F}$ over $\mathscr{D}$ ensures that images of behaviors are again behaviors, i.e., for any behavior $\mathscr{B} = \{w \in \mathscr{F}^\ell; R \circ w = 0\}$, $R \in \mathscr{D}^{k\times\ell}$, and for any operator matrix $P \in \mathscr{D}^{p\times\ell}$, there exists $Q \in \mathscr{D}^{m\times p}$ such that the image behavior $P \circ \mathscr{B} := \{P \circ w; w \in \mathscr{B}\}$ is equal to $\{y \in \mathscr{F}^p; Q \circ y = 0\}$. The matrix Q may be chosen as Y if $(X,Y) \in \mathscr{D}^{r\times(k+p)}$ is a *universal left annihilator* of $\binom{R}{P}$, i.e., if the rows of (X,Y) are a basis of the set of all $\xi \in \mathscr{D}^{1\times(k+p)}$ with $\xi\binom{R}{P} = 0$. In particular, the projection onto some components (or the "elimination" of the other components) of a given behavior yields again a behavior, compare also [14, Thm.2.34, p.25f], [15, Sec.6.2]. The additional cogenerator property guarantees essentially that all the information on the row space of the matrix R is contained in the behavior $\mathscr{B} = \{w \in \mathscr{F}^\ell; R \circ w = 0\}$. More precisely: If $\mathscr{F}$ is an injective cogenerator over $\mathscr{D}$ and $\mathscr{B}_i = \{w \in \mathscr{F}^\ell; R_i \circ w = 0\}$, $R_i \in \mathscr{D}^{k_i\times\ell}$, $i = 1,2$, are two behaviors, then

$$\mathscr{B}_1 \subseteq \mathscr{B}_2 \quad \Leftrightarrow \quad \mathscr{D}^{1\times k_2}R_2 \subseteq \mathscr{D}^{1\times k_1}R_1.$$

Summing up, the injective cogenerator property of $\mathscr{F}$ over $\mathscr{D}$ yields a complete duality between the equations (the row space of the polynomial matrix R) on the one hand and the behavior (the space of valid system trajectories) on the other hand.

In order to describe when a signal in $\mathscr{F}$ is considered to be *small* or *negligible* we introduce a set $T \subseteq \mathscr{D}\setminus\{0\}$ which is assumed to be *multiplicatively closed*, i.e., $1 \in T$ and $(f,g \in T \Rightarrow fg \in T)$, and *saturated*, i.e., $(fg \in T$ for $f,g \in \mathscr{D} \Rightarrow f \in T,\ g \in T)$. We call a signal $w \in \mathscr{F}^\ell$ *T-small* if it satisfies $h \circ w = 0$ for some element h of T. The set of all T-small signals in $\mathscr{F}$ is the T-torsion submodule $\mathrm{t}_T(\mathscr{F}) := \{w \in \mathscr{F}; \exists h \in T : h \circ w = 0\}$ of $\mathscr{F}$.

Example 2. The following are the most commonly used definitions for small signals.

1) The smallest possible choice for T is $T := F\setminus\{0\}$. Then the zero signal is the only T-small signal.
2) If $T := \mathscr{D}\setminus\{0\}$ in the continuous standard case the T-small signals are exactly the polynomial-exponential functions (Bohl functions).
3) The probably most important choice for T in the continuous standard case is the set of polynomials with (complex) zeroes only in the open left half complex plane. Then a signal is T-small if and only if it is a polynomial-exponential function converging to zero for the time tending to infinity. These are the classical "stable signals".
4) In the discrete standard case, choosing T as the set of polynomials with (complex) zeroes only within the open unit disc ensures that all T-small signals are asymptotically zero.
5) Another important choice in the discrete case is $T := \{us^\mu \in \mathscr{D}; u \in F\setminus\{0\}, \mu \in \mathbb{N}\}$. Then T-small signals are non-zero only at finitely many time instances. This is the classical "dead-beat" scenario.

We call a behavior *T-autonomous* if all its signals are T-small. Any T-autonomous behavior is in particular autonomous. In the standard cases 3 and 4 in the previous example T-autonomy signifies that all trajectories of the behavior are asymptotically zero, i.e., that the behavior is asymptotically stable.

The set T gives rise to the ring

$$\mathscr{D}_T := \left\{\tfrac{f}{h} \in F(s); f \in \mathscr{D}, h \in T\right\} \subseteq F(s) := \left\{\tfrac{f}{g}; f \in \mathscr{D}, g \in \mathscr{D}\setminus\{0\}\right\}$$

of *T-stable rational functions*, and to the quotient signal module

$$\mathscr{F}_T := \left\{\tfrac{w}{h}; w \in \mathscr{F}, h \in T\right\}.$$

Then $\mathscr{F}_T$ is a module over $\mathscr{D}_T$ in the natural fashion, and in fact even an injective cogenerator [2, Thm.1.6]. We summarize some further results from [2, Sec.1] that originate from Oberst.

The signal module $\mathscr{F}$ can be decomposed as

$$\mathscr{F} = \mathscr{F}' \oplus \mathrm{t}_T(\mathscr{F})$$

where $\mathscr{F}'$ denotes a (non-unique and non-constructive) submodule of $\mathscr{F}$. Hence $\mathscr{F}' \cong \mathscr{F}/\mathrm{t}_T(\mathscr{F})$. In the standard cases where $\mathrm{t}_T(\mathscr{F})$ is the set of polynomial-exponential functions that converge to zero this corresponds to the decomposition of signals $w \in \mathscr{F}$ into a steady state (in $\mathscr{F}'$) and a transient state (in $\mathrm{t}_T(\mathscr{F})$). $\mathscr{F}'$ contains exactly one representative of every equivalence class $[w] = w + \mathrm{t}_T(\mathscr{F}) \in \mathscr{F}/\mathrm{t}_T(\mathscr{F})$ of signals in $\mathscr{F}$ up to T-small signals. The non-unique module $\mathscr{F}' \subseteq \mathscr{F}$ is closely related to the quotient signal module $\mathscr{F}_T$ via the following isomorphism:

$$\begin{array}{ccccc} \mathscr{F}' & \cong & \mathscr{F}/\mathrm{t}_T(\mathscr{F}) & \cong & \mathscr{F}_T \\ w & \mapsto & [w] = w + \mathrm{t}_T(\mathscr{F}) & \mapsto & \frac{w}{1}. \end{array}$$

Hence signals in $\mathscr{F}_T$ may be interpreted as the "essential part" or "steady state" of a signal in $\mathscr{F}$, or as a signal in $\mathscr{F}$ "up to a T-small (and hence negligible) part".

For a behavior $\mathscr{B} = \{w \in \mathscr{F}^\ell; R \circ w = 0\}$, $R \in \mathscr{D}^{k\times\ell} \subseteq \mathscr{D}_T^{k\times\ell}$, the decomposition

$$\mathscr{B} = \mathscr{B}' \oplus \mathrm{t}_T(\mathscr{B})$$

holds where $\mathscr{B}' := \mathscr{B} \cap \mathscr{F}'^\ell$ and $\mathrm{t}_T(\mathscr{B}) := \mathscr{B} \cap \mathrm{t}_T(\mathscr{F})^\ell$. Moreover, the quotient behavior $\mathscr{B}_T := \left\{\frac{w}{h} \in \mathscr{F}_T^\ell; w \in \mathscr{B}, h \in T\right\}$ is isomorphic to $\mathscr{B}'$. Hence, the quotient behavior $\mathscr{B}_T$ can be interpreted as "$\mathscr{B}$ up to a T-small part". The equalities $\mathscr{B}_T = \{w \in \mathscr{F}_T^\ell; R \circ w = 0\}$ and $(P \circ \mathscr{B})_T = P \circ \mathscr{B}_T$ hold for a linear operator $P \in \mathscr{D}^{p\times\ell}$.

The transfer from a behavior $\mathscr{B}$ in $\mathscr{F}^\ell$ to its quotient (or localization) $\mathscr{B}_T \subseteq \mathscr{F}_T^\ell$ can be imagined as concentrating on the "essential part" of the behavior and not bothering about the T-small (and in the standard cases transient) part.

Example 3. Assume the continuous standard setting and the set T from Example 2.3. Let $\mathscr{B}$ be an *autonomous* behavior with the decomposition $\mathscr{B} = \mathscr{B}_{\text{antistab}} \oplus \mathscr{B}_{\text{stab}}$ into its antistable and stable part, compare e.g. [18]. Then $\mathscr{B}' = \mathscr{B}_{\text{antistab}}$ and $\mathrm{t}_T(\mathscr{B}) = \mathscr{B}_{\text{stab}}$.

If the behavior $\mathscr{B}$ is *not* autonomous, then the sets $\mathscr{B}'$ and $\mathrm{t}_T(\mathscr{B})$ are in general no ${}_{\mathscr{D}}\mathscr{F}$-behaviors.

The following characterization of T-autonomous behaviors demonstrates the applicability of the above results: The behavior $\mathscr{B} = \{w \in \mathscr{F}^\ell; R \circ w = 0\}$, $R \in \mathscr{D}^{k\times\ell}$, is by definition

T-autonomous if all its signals are T-small, i.e., if it is contained in $\mathrm{t}_T(\mathscr{F})^\ell$. By the above direct sum decomposition this signifies that $0=\mathscr{B}'\cong\mathscr{B}_T=\left\{w\in\mathscr{F}_T^\ell;\,R\circ w=0\right\}$. Since $\mathscr{F}_T$ is an injective cogenerator over $\mathscr{D}_T$ this is the case if and only if $\mathscr{D}_T^{1\times k}R=\mathscr{D}_T^{1\times\ell}$, i.e., if and only if R is left invertible over $\mathscr{D}_T$.

IV. Problem formulation

We consider a plant behavior

$$\mathscr{P}=\left\{\begin{pmatrix}w_1\\w_2\end{pmatrix}\in\mathscr{F}^{\mathtt{w}_1+\mathtt{w}_2};\,R_1\circ w_1=R_2\circ w_2\right\},\quad R:=(R_1,\,-R_2)\in\mathscr{D}^{\mathtt{k}\times(\mathtt{w}_1+\mathtt{w}_2)}.$$

We assume that the signal w_1 can be measured and we are interested in (an estimate for) w_2.

Definition and Lemma 4. *[2, Def.2.1, Ex.2.12] We say that w_2 is T-*observable *from w_1 in $\mathscr{P}$ if w_2 is determined by w_1 up to a T-small part, i.e., more precicely, if the following equivalent conditions are satisfied:*

1) *If two trajectories $\binom{w_1}{w_2}$ and $\binom{w_1}{v_2}$ are contained in $\mathscr{P}$, then the difference w_2-v_2 is T-small.*
2) *From $\binom{w_1}{w_2}\in\mathscr{P}$ and $w_1=0$, it follows that w_2 is T-small.*
3) *The* hidden behavior *of w_2 in $\mathscr{P}$ (terminology according to Willems)*

$$\mathscr{N}_{\mathtt{w}_2}(\mathscr{P}):=\left\{w_2\in\mathscr{F}^{\mathtt{w}_2};\,\begin{pmatrix}0\\w_2\end{pmatrix}\in\mathscr{P}\right\}=\{w_2\in\mathscr{F}^{\mathtt{w}_2};\,R_2\circ w_2=0\}$$

   is T-autonomous.
4) $\mathscr{N}_{\mathtt{w}_2}(\mathscr{P})_T=\left\{w_2\in\mathscr{F}_T^{\mathtt{w}_2};\,R_2\circ w_2=0\right\}=0.$
5) *The matrix R_2 admits a left inverse $X\in\mathscr{D}_T^{\mathtt{w}_2\times k}$.*

Proof: The equivalence of conditions 1 and 2 is due to linearity. Condition 3 is merely a reformulation of condition 2. Condition 3 is equivalent to conditions 4 and 5 by the arguments in the last paragraph of Section III. ∎

With T chosen according to item 1 resp. 2 resp. 3 or 4 resp. 5 in Example 2, T-observability coincides with observability resp. trackability resp. detectability resp. reconstructibility, compare [21, Def.VI.1][20, Def.2.1][19, Def.3.1][6, Def.4.1].

Definition 5. Consider the observer

$$\mathscr{O}=\left\{\begin{pmatrix}w_1\\\widehat{w}_2\end{pmatrix}\in\mathscr{F}^{\mathtt{w}_1+\mathtt{w}_2};\,\widehat{R}_1\circ w_1=\widehat{R}_2\circ\widehat{w}_2\right\},\quad \widehat{R}:=(\widehat{R}_1,\,-\widehat{R}_2)\in\mathscr{D}^{\widehat{\mathtt{k}}\times(\mathtt{w}_1+\mathtt{w}_2)}.$$

1) We say that the observer $\mathscr{O}$ is *nonintrusive* if the behavior of the plant variables (w_1,w_2) remains unchanged after interconnection with the observer, formally $(\mathscr{P}\wedge_{w_1}\mathscr{O})_{(w_1,w_2)}=\mathscr{P}$. In [20] nonintrusive observers are called *acceptors*.
2) We call the observer $\mathscr{O}$ *T-nonintrusive* if the plant remains *essentially* unchanged after interconnection with the observer, i.e., if $\left((\mathscr{P}\wedge_{w_1}\mathscr{O})_{(w_1,w_2)}\right)_T=\mathscr{P}_T$. This signifies that for any $\binom{w_1}{w_2}$ in $\mathscr{P}$ there exists $\binom{v_1}{v_2}\in(\mathscr{P}\wedge_{w_1}\mathscr{O})_{(w_1,w_2)}$ such that v_i-w_i is T-small for $i=1,2$.
3) We call the observer $\mathscr{O}$ a *T-observer* if the error behavior $\mathscr{E}(\mathscr{P},\mathscr{O})$ is T-autonomous.

Remark 6. The observer $\mathscr{O}$ is nonintrusive if and only if $\mathscr{P}_{w_1}\subseteq\mathscr{O}_{w_1}$. Likewise, $\mathscr{O}$ is T-nonintrusive if and only if $(\mathscr{P}_{w_1})_T\subseteq(\mathscr{O}_{w_1})_T$.

Nonintrusiveness (or at least T-nonintrusiveness) is clearly a sensible requirement for an observer. For example, the zero behavior is a T-observer for any plant where w_2 is T-observable from w_1. Restricting the plant, in this case even forcing it to consist only of T-small signals, is not what we would expect from a meaningful observer.

On the other hand, nonintrusiveness and T-nonintrusiveness are weaker requirements than an input/output structure on the observer with input w_1 and output $\widehat{w}_2$, which is the structure observers are traditionally assumed to have (e.g. in the state-space setting). We do not require the components w_1 to be completley free in $\mathscr{O}$, but we ensure that all signals w_1 *that can actually occur* are accepted by the observer (up to a T-small part, in the case of T-nonintrusiveness).

The goal of the subsequent sections is to characterize T-nonintrusive (and nonintrusive) T-observers for a given plant, to study their existence and to find a constructive parametrization of all such observers.

V. The internal model principle

The following theorem is a quotient signal space version of the internal model principle for observers. See the discussion in Section VII for how this relates to prior work, in particular [18].

Theorem 7 (Characterization of T-nonintrusive T-observers)**.** *Assume a plant $\mathscr{P}$ and an observer $\mathscr{O}$. Then the following two statements are equivalent:*

1) *The observer $\mathscr{O}$ is T-nonintrusive and the error behavior $\mathscr{E}(\mathscr{P},\mathscr{O})$ is T-autonomous.*
2) a) *The signal $\widehat{w}_2$ is T-observable from w_1 in $\mathscr{O}$, i.e., $\mathscr{N}_{\widehat{\mathtt{w}}_2}(\mathscr{O})$ is T-autonomous, and*
   b) *the inclusion $\mathscr{P}_T\subseteq\mathscr{O}_T$ holds.*

These conditions imply in particular that w_2 is T-observable from w_1 in $\mathscr{P}$, i.e., a necessary condition on the plant behavior.

The property $\mathscr{P}_T\subseteq\mathscr{O}_T$ signifies that “the essential part of $\mathscr{P}$” is contained in $\mathscr{O}$, or that $\mathscr{P}$ is contained in $\mathscr{O}$ “up to T-small signals”.

In order to prove this central result we use the following lemmas:

Lemma 8.

$$\mathscr{E}(\mathscr{P},\mathscr{O})=\mathscr{N}_{\mathtt{w}_2}(\mathscr{P}+\mathscr{O})$$

where $\mathscr{N}_{\mathtt{w}_2}(\mathscr{P}+\mathscr{O})=\left\{w_2\in\mathscr{F}^{\mathtt{w}_2};\,\binom{0}{w_2}\in\mathscr{P}+\mathscr{O}\right\}.$

Proof:

$$\mathscr{N}_{\mathrm{w}_2}(\mathscr{P}+\mathscr{O}) = \left\{ e \in \mathscr{F}^{\mathrm{w}_2};\ \begin{pmatrix} 0 \\ e \end{pmatrix} \in \mathscr{P}+\mathscr{O} \right\}$$
$$= \left\{ e \in \mathscr{F}^{\mathrm{w}_2};\ \exists \begin{pmatrix} -w_1 \\ -w_2 \end{pmatrix} \in \mathscr{P}\ \exists \begin{pmatrix} \widehat{w}_1 \\ \widehat{w}_2 \end{pmatrix} \in \mathscr{O} : \begin{pmatrix} 0 \\ e \end{pmatrix} = \begin{pmatrix} -w_1 \\ -w_2 \end{pmatrix} + \begin{pmatrix} \widehat{w}_1 \\ \widehat{w}_2 \end{pmatrix} \right\}$$
$$= \left\{ \widehat{w}_2 - w_2 \in \mathscr{F}^{\mathrm{w}_2};\ \exists w_1 \in \mathscr{F}^{\mathrm{w}_1} : \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathscr{P},\ \begin{pmatrix} w_1 \\ \widehat{w}_2 \end{pmatrix} \in \mathscr{O} \right\}$$
$$= \mathscr{E}(\mathscr{P},\mathscr{O}).$$

■

Lemma 9. *The error behavior $\mathscr{E}(\mathscr{P},\mathscr{O})$ contains the hidden behaviors $\mathscr{N}_{\mathrm{w}_2}(\mathscr{P})$ and $\mathscr{N}_{\widehat{\mathrm{w}}_2}(\mathscr{O})$.*

Proof: Follows from Lemma 8 since $\mathscr{N}_{\mathrm{w}_2}(\mathscr{P}) = \mathscr{N}_{\mathrm{w}_2}(\mathscr{P}+0) \subseteq \mathscr{N}_{\mathrm{w}_2}(\mathscr{P}+\mathscr{O}) = \mathscr{E}(\mathscr{P},\mathscr{O})$ by Lemma 8. The assertion for $\mathscr{N}_{\widehat{\mathrm{w}}_2}(\mathscr{O})$ follows likewise. ■

Proof of Theorem 7.

1⇒2: T-autonomy of $\mathscr{N}_{\widehat{\mathrm{w}}_2}(\mathscr{O})$ follows from Lemma 9. The relation $\mathscr{P}_T \subseteq \mathscr{O}_T$ holds by the following argument: Consider an arbitrary element $\binom{w_1}{w_2} \in \mathscr{P}_T$. The T-nonintrusiveness of $\mathscr{O}$ signifies that there exists $\widehat{w}_2 \in \mathscr{F}_T^{\mathrm{w}_1}$ such that $\binom{w_1}{\widehat{w}_2} \in \mathscr{O}_T$. Then $\widehat{w}_2 - w_2 \in \mathscr{E}(\mathscr{P}_T,\mathscr{O}_T) = (\mathscr{E}(\mathscr{P},\mathscr{O}))_T$ which is zero since $\mathscr{E}(\mathscr{P},\mathscr{O})$ is assumed to be T-autonomous. It follows that $w_2 = \widehat{w}_2$ in $\mathscr{F}_T^{\mathrm{w}_2}$ and hence that $\binom{w_1}{w_2} = \binom{w_1}{\widehat{w}_2} \in \mathscr{O}_T$.
2⇒1: The inclusion $\mathscr{P}_T \subseteq \mathscr{O}_T$ implies the inclusion $(\mathscr{P}_{w_1})_T \subseteq (\mathscr{O}_{w_1})_T$, i.e., T-nonintrusiveness of the observer $\mathscr{O}$, as well as the equality $\mathscr{P}_T + \mathscr{O}_T = \mathscr{O}_T$. Using Lemma 8 we deduce that $\mathscr{E}(\mathscr{P},\mathscr{O})_T = \mathscr{E}(\mathscr{P}_T,\mathscr{O}_T)$ is equal to $\mathscr{N}_{\mathrm{w}_2}(\mathscr{O}_T)$ which is zero due to condition 2a.

VI. The error behavior

In this section we give a characterization of the error behaviors that actually occur in observer interconnections with T-observers. As a direct consequence we get a simple necessary and sufficient condition for the existence of T-observers in terms of an observability condition on the plant. Moreover, we show that observers can always be chosen so that they are nonintrusive. All these results are straightforward generalizations of the respective results reported in [20], [2] and [18].

Definition 10. A T-autonomous behavior $\mathscr{E}$ is *achievable as error behavior for* $\mathscr{P}$ if there exists an observer $\mathscr{O}$ such that $\mathscr{E}(\mathscr{P},\mathscr{O}) = \mathscr{E}$.

Lemma 11. *A T-autonomous behavior $\mathscr{E}$ is achievable as error behavior for $\mathscr{P}$ if and only if $\mathscr{N}_{\mathrm{w}_2}(\mathscr{P}) \subseteq \mathscr{E}$. In this case there exists even a nonintrusive T-observer $\mathscr{O}$ with $\mathscr{E}(\mathscr{P},\mathscr{O}) = \mathscr{E}$.*

Proof: Assume a T-autonomous behavior $\mathscr{E} = \{e \in \mathscr{F}^{\mathrm{w}_2};\ V \circ e = 0\}$, $V \in \mathscr{D}^{\mathrm{v}\times\mathrm{w}_2}$. We show that there exists an observer $\mathscr{O}$ such that $\mathscr{E}(\mathscr{P},\mathscr{O}) = \mathscr{E}$ if and only if $\mathscr{N}_{\mathrm{w}_2}(\mathscr{P}) \subseteq \mathscr{E}$. The constructed observer will be nonintrusive.

⇒: According to Lemma 9 any observer $\mathscr{O}$ satisfies $\mathscr{N}_{\mathrm{w}_2}(\mathscr{P}) \subseteq \mathscr{E}(\mathscr{P},\mathscr{O})$.
⇐: By the cogenerator property of $\mathscr{F}$ over $\mathscr{D}$, the inclusion $\mathscr{N}_{\mathrm{w}_2}(\mathscr{P}) \subseteq \mathscr{E}$ is equivalent to the inclusion $\mathscr{D}^{1\times\mathrm{v}}V \subseteq \mathscr{D}^{1\times\mathrm{k}}R_2$, i.e., to the existence of $S \in \mathscr{D}^{\mathrm{v}\times\mathrm{k}}$ such that $V = SR_2$. Defining $\widehat{R} := SR$ and $\mathscr{O} := \{w \in \mathscr{F}^{\mathrm{w}};\ \widehat{R} \circ w = 0\}$ leads to the inclusion $\mathscr{P} \subseteq \mathscr{O}$, hence also $\mathscr{P}_{w_1} \subseteq \mathscr{O}_{w_1}$. This is equivalent to nonintrusiveness of the observer $\mathscr{O}$. Moreover, $\mathscr{E}(\mathscr{P},\mathscr{O}) = \mathscr{N}_{\mathrm{w}_2}(\mathscr{P}+\mathscr{O}) = \mathscr{N}_{\mathrm{w}_2}(\mathscr{O}) = \{w_2 \in \mathscr{F}^{\mathrm{w}_2};\ SR_2 \circ w_2 = 0\} = \mathscr{E}$. We conclude that $\mathscr{O}$ is a nonintrusive observer for $\mathscr{P}$ with T-autonomous error behavior $\mathscr{E}(\mathscr{P},\mathscr{O}) = \mathscr{E}$.

■

Lemma 12. *There exists a T-observer if and only if w_2 is T-observable from w_1 in $\mathscr{P}$, i.e., if and only if $\mathscr{N}_{\mathrm{w}_2}(\mathscr{P})$ is T-autonomous. In this case $\mathscr{O} := \mathscr{P}$ is a nonintrusive T-observer.*

Proof: T-autonomy of $\mathscr{E}(\mathscr{P},\mathscr{O})$ implies T-autonomy of $\mathscr{N}_{\mathrm{w}_2}(\mathscr{P})$ by Lemma 9. On the other hand, the choice $\mathscr{O} := \mathscr{P}$ yields $\mathscr{P}+\mathscr{O} = \mathscr{P}$ and hence $\mathscr{E}(\mathscr{P},\mathscr{O}) = \mathscr{N}_{\mathrm{w}_2}(\mathscr{P}+\mathscr{O}) = \mathscr{N}_{\mathrm{w}_2}(\mathscr{P})$. The observer $\mathscr{O} = \mathscr{P}$ is clearly nonintrusive. ■

The previous result may at first appear unintuitive and requires some discussion. The general part of behavioral observer theory is concerned with existence conditions and characterization results for observers that do not unduly influence the operation of the plant (hence the nonintrusiveness property) and produce sensible error behaviors (that contain only small signals). A completely separate consideration is whether such observers can be implemented with a particular predefined set of building blocks, or, more abstractly, allow representations of a particular kind.

The choice $\mathscr{O} := \mathscr{P}$ will only seem odd if we focus our attention on these latter questions and insist that an observer must correspond to a particular signal flow diagram, for example that it take w_1 as input and "produce" $\hat{w}_2$ as the corresponding output and, moreover, do so in some non-anticipatory or "causal" fashion [9], [15]. If we require all this *a priori*, the choice $\mathscr{O} := \mathscr{P}$ will obviously only work for invertible plants, where the role of inputs and outputs are interchangeable. Note, though, that even in this constrained scenario there are still (some) plants where this choice can make practical sense.

We argue that a more fruitful approach is to deal with implementability questions *a posteriori*, and to derive characterization and parametrization results in full generality. Firstly, depending on the application not all these implementability conditions will be required. For example, if we use an observer for post-processing of batch data (e.g. in econometric modelling), there is no need for a non-anticipatory condition such as properness. As another example, if the observer is realized as a directly connected mechanical component (rather than as a signal processor) there may not be a need for an input/output-structure. Secondly, as we will see in Section VIII, the general approach leads to a much cleaner parametrization result where implementability questions can easily be dealt with after the fact, sometimes even by mere inspection. We demonstrate this for the input/output-property.

Since most current applications of observers use the on-line signal processing paradigm for their implementation, the associated implementability questions warrant thorough investigation. We propose that the results reported in this paper provide a better starting point for this than some of the more cumbersome parametrization results that have been reported in the literature. For example, it turns out that every T-nonintrusive T-observer contains an embedded input/output observer, see Remark 23. An input/output structure can hence be assumed without loss of generality.

VII. Connection to prior work

Let again $\mathscr{P} := \{\binom{w_1}{w_2} \in \mathscr{F}^{\mathtt{w}_1+\mathtt{w}_2}; R_1 \circ w_1 = R_2 \circ w_2\}$, $R := (R_1, -R_2) \in \mathscr{D}^{\mathtt{k}\times(\mathtt{w}_1+\mathtt{w}_2)}$, be a plant, $\mathtt{w} := \mathtt{w}_1 + \mathtt{w}_2$. We compare the results derived in Section V with those from [2] and [18]. See the discussions in those papers for the relationship to classical results.

The situation in [2] is more restrictive than the one considered here insofar as only *input/output* T-observers that accept the measured signal as input and output an estimate of the desired signal are considered. On the other hand, in [2] the measured signal is any (linear) image $P \circ w$ of the plant signal w, and the to-be-estimated signal is another image $Q \circ w$. Putting $w = \binom{w_1}{w_2}$, $P := (\mathrm{id}_{\mathtt{w}_1}, 0)$ and $Q := (0, \mathrm{id}_{\mathtt{w}_2})$ yields $P \circ w = w_1$ and $Q \circ w = w_2$. By Lemma 2.5 and Theorem 2.6 in [2], the input/output behavior $\mathscr{O} := \{\binom{w_1}{\widehat{w}_2} \in \mathscr{F}^{\mathtt{w}_1+\mathtt{w}_2}; \widehat{R}_1 \circ w_1 = \widehat{R}_2 \circ \widehat{w}_2\}$, $\widehat{R} := (\widehat{R}_1, -\widehat{R}_2) \in \mathscr{D}^{\mathtt{w}_2\times(\mathtt{w}_1+\mathtt{w}_2)}$, $\det(\widehat{R}_2) \neq 0$, is an input/output T-overserver of w_2 from w_1 in $\mathscr{P}$ if and only if $\mathscr{O}$ is T-stable (i.e., the autonomous part $\mathscr{O}^0 := \{\widehat{w}_2 \in \mathscr{F}^{\mathtt{w}_2}; \widehat{R}_2 \circ \widehat{w}_2 = 0\} = \mathscr{N}_{\widehat{w}_2}(\mathscr{O})$ is T-autonomous) and there exists $X \in \mathscr{D}_T^{\mathtt{w}_2\times k}$ such that $\widehat{R}_2^{-1}\widehat{R}_1(\mathrm{id}_{\mathtt{w}_1}, 0) - (0, \mathrm{id}_{\mathtt{w}_2}) = XR$. The first condition is equivalent to invertibility of $\widehat{R}_2$ over $\mathscr{D}_T$ or to T-observability of $\widehat{w}_2$ from w_1 in $\mathscr{O}$, the second one to solvability of the equation $\widehat{R} = \widehat{R}_2 XR$ for some $X \in \mathscr{D}_T^{\mathtt{w}_2\times\mathtt{k}}$, or, equivalently, solvability of $\widehat{R} = YR$ for some $Y \in \mathscr{D}_T^{\mathtt{w}_2\times\mathtt{k}}$ since $\widehat{R}_2 \in \mathrm{Gl}_{\mathtt{w}_2}(\mathscr{D}_T)$. By the injective cogenerator property of $\mathscr{F}_T$ over $\mathscr{D}_T$, this last condition signifies that $\mathscr{P}_T \subseteq \mathscr{O}_T$. Hence the characterizations of T-observers from Theorem 7 in the present paper and from Lemma 2.5 and Theorem 2.6 in [2] are equivalent for the case of *input/output* T-observers.

In [18, Thm.5.6] it has been shown that – for the continuous standard case and T as in item 3 in Example 2 – a nonintrusive observer $\mathscr{O}$ is a T-observer for $\mathscr{P}$ if and only if $\widehat{w}_2$ is T-observable (i.e., detectable) from w_1 in $\mathscr{O}$ and $\mathscr{P}_{\mathrm{cont}} \oplus \mathscr{P}_{\mathrm{antistab}} \subseteq \mathscr{O}$ where $\mathscr{P} = \mathscr{P}_{\mathrm{cont}} \oplus \mathscr{P}_{\mathrm{aut}}$ is a decomposition of $\mathscr{P}$ in the controllable part and an autonomous subsystem, and $\mathscr{P}_{\mathrm{stab}} \oplus \mathscr{P}_{\mathrm{antistab}}$ is the decomposition of $\mathscr{P}_{\mathrm{aut}}$ into the stable and the antistable subbehavior. Similar results for other standard choices of T have been derived in Theorem 5.3 and Theorem 5.7 in [18].

We will now establish the relationship between this characterization and the one from Theorem 7. Let $R^{(\mathrm{T})} \in \mathscr{D}^{\mathtt{k}\times\mathtt{w}}$ be such that

$$\mathscr{D}_T^{1\times\mathtt{k}} R \cap \mathscr{D}^{1\times\mathtt{w}} = \mathscr{D}^{1\times\mathtt{k}} R^{(\mathrm{T})}. \tag{1}$$

This matrix can be computed using Algorithm 17 below. By $\mathscr{P}^{(\mathrm{T})}$ we denote the behavior defined by $R^{(\mathrm{T})}$, i.e., $\mathscr{P}^{(\mathrm{T})} = \{w \in \mathscr{F}^{\mathtt{w}}; R^{(\mathrm{T})} \circ w = 0\}$.

Remark 13 (Interpretation of $R^{(\mathrm{T})}$ and $\mathscr{P}^{(\mathrm{T})}$)**.** The row module $U = \mathscr{D}^{1\times\mathtt{k}} R^{(\mathrm{T})}$ is by definition the largest submodule of $\mathscr{D}^{1\times\mathtt{w}}$ whose localization, U_T, coincides with $\mathscr{D}_T^{1\times\mathtt{k}} R$. The cogenerator property of $\mathscr{F}$ over $\mathscr{D}$ and of $\mathscr{F}_T$ over $\mathscr{D}_T$ implies that $\mathscr{P}^{(\mathrm{T})} \subseteq \mathscr{F}^{\mathtt{w}}$ is the smallest ${}_{\mathscr{D}}\mathscr{F}$-behavior whose localization, $(\mathscr{P}^{(\mathrm{T})})_T$, coincides with $\mathscr{P}_T$.

Behaviors $\mathscr{P}^{(\mathrm{T})}$ have already been considered in [16, Thm.4, Cor.6] where $\mathscr{P}$ is identified with $\mathrm{Hom}_{\mathscr{D}}(M, \mathscr{F}) \cong \mathscr{P}$ for $M := \mathscr{D}^{1\times\mathtt{w}}/\mathscr{D}^{1\times\mathtt{k}}R$ (Malgrange isomorphism), and $\mathscr{P}^{(\mathrm{T})}$ is defined as $\mathrm{Hom}_{\mathscr{D}}(M/\mathrm{t}_T(M), \mathscr{F})$.

The following lemma shows that the quotient behavior inclusion relation in Theorem 7 can be re-cast as a behavior inclusion. Hence it allows to associate the unique and constructible ${}_{\mathscr{D}}\mathscr{F}$-behavior $\mathscr{P}^{(\mathrm{T})}$ with the "abstract" quotient behavior $\mathscr{P}_T$.

Lemma 14. *The following statements are equivalent:*

1) $\mathscr{P}_T \subseteq \mathscr{O}_T$, *i.e.,* $\mathscr{D}_T^{1\times\widehat{\mathtt{k}}} \widehat{R} \subseteq \mathscr{D}_T^{1\times\mathtt{k}} R$.
2) $\mathscr{P}^{(\mathrm{T})} \subseteq \mathscr{O}$, *i.e.,* $\mathscr{D}^{1\times\widehat{\mathtt{k}}} \widehat{R} \subseteq \mathscr{D}^{1\times k} R^{(\mathrm{T})}$.

Proof: By the injective cogenerator property of $\mathscr{F}_T$ over $\mathscr{D}_T$, the inclusion $\mathscr{P}_T \subseteq \mathscr{O}_T$ is equivalent to the inclusion $\mathscr{D}_T^{1\times\widehat{\mathtt{k}}} \widehat{R} \subseteq \mathscr{D}_T^{1\times\mathtt{k}} R$. This signifies, since $\widehat{R} \in \mathscr{D}^{\widehat{\mathtt{k}}\times\mathtt{w}}$, that $\mathscr{D}^{1\times\widehat{\mathtt{k}}} \widehat{R} \subseteq \mathscr{D}_T^{1\times\mathtt{k}} R \cap \mathscr{D}^{1\times\mathtt{w}} \overset{(1)}{=} \mathscr{D}^{1\times\mathtt{k}} R^{(\mathrm{T})}$ as asserted. ∎

The next lemma shows that the behavior $\mathscr{P}^{(\mathrm{T})}$ is in fact the behavior that appears in the internal model principle in [18]. Let $\mathscr{P} = \mathscr{P}_{\mathrm{cont}} \oplus \mathscr{P}_{\mathrm{aut}}$ be the decomposition of $\mathscr{P}$ into the controllable part $\mathscr{P}_{\mathrm{cont}}$ and a (non-unique) autonomous part $\mathscr{P}_{\mathrm{aut}}$. With

$$T' := \left\{h' \in \mathscr{D} \setminus \{0\}; \forall h \in T : \gcd(h, h') = 1\right\},$$

the autonomous part $\mathscr{P}_{\mathrm{aut}}$ of $\mathscr{P}$ can be decomposed as $\mathscr{P}_{\mathrm{aut}} = \mathrm{t}_T(\mathscr{P}_{\mathrm{aut}}) \oplus \mathrm{t}_{T'}(\mathscr{P}_{\mathrm{aut}})$ as a consequence of the Chinese Remainder Theorem. For $T = F \setminus \{0\}$ the identities $\mathrm{t}_T(\mathscr{P}_{\mathrm{aut}}) = 0$ and $\mathrm{t}_{T'}(\mathscr{P}_{\mathrm{aut}}) = \mathscr{P}_{\mathrm{aut}}$ hold, for $T = \mathscr{D} \setminus \{0\}$ we get $\mathrm{t}_T(\mathscr{P}_{\mathrm{aut}}) = \mathscr{P}_{\mathrm{aut}}$, $\mathrm{t}_{T'}(\mathscr{P}_{\mathrm{aut}}) = 0$. For the standard choices for T from item 3 and 4 in Example 2, $\mathrm{t}_T(\mathscr{P}_{\mathrm{aut}})$ is the stable part $\mathscr{P}_{\mathrm{stab}}$ of $\mathscr{P}_{\mathrm{aut}}$ and $\mathrm{t}_{T'}(\mathscr{P}_{\mathrm{aut}})$ is the antistable part $\mathscr{P}_{\mathrm{antistab}}$.

Lemma 15. *The behavior* $\mathscr{P}^{(\mathrm{T})}$ *is equal to* $\mathscr{P}_{\mathrm{cont}} \oplus \mathrm{t}_{T'}(\mathscr{P}_{\mathrm{aut}})$.

Proof: The proof proceeds in four steps.

1) Let $R_{\mathrm{cont}} \in \mathscr{D}^{\mathtt{k}\times\mathtt{w}}$ such that the controllable part of $\mathscr{P}$ is $\mathscr{P}_{\mathrm{cont}} = \{w \in \mathscr{F}^{\mathtt{w}}; R_{\mathrm{cont}} \circ w = 0\}$. Then $\mathscr{D}^{1\times\mathtt{k}} R_{\mathrm{cont}} = F(s)^{1\times\mathtt{k}} R \cap \mathscr{D}^{1\times\mathtt{w}}$, hence the matrix $(R_{\mathrm{cont}})^{(\mathrm{T})}$ defining the behavior $(\mathscr{P}_{\mathrm{cont}})^{(\mathrm{T})}$ satisfies $\mathscr{D}^{1\times\mathtt{k}} (R_{\mathrm{cont}})^{(\mathrm{T})} = \mathscr{D}_T^{1\times\mathtt{k}} R_{\mathrm{cont}} \cap \mathscr{D}^{1\times\mathtt{w}} = (F(s)^{1\times\mathtt{k}} R \cap \mathscr{D}_T^{1\times\mathtt{w}}) \cap \mathscr{D}^{1\times\mathtt{w}} = F(s)^{1\times\mathtt{k}} R \cap \mathscr{D}^{1\times\mathtt{w}} = \mathscr{D}^{1\times\mathtt{k}} R_{\mathrm{cont}}$, and consequently $(\mathscr{P}_{\mathrm{cont}})^{(\mathrm{T})} = \mathscr{P}_{\mathrm{cont}}$.
2) We show that $(\mathscr{P}_{\mathrm{aut}})^{(\mathrm{T})} = \mathrm{t}_{T'}(\mathscr{P}_{\mathrm{aut}})$, i.e., according to Remark 13, that $(\mathrm{t}_{T'}(\mathscr{P}_{\mathrm{aut}}))_T = (\mathscr{P}_{\mathrm{aut}})_T$ and that any $\mathscr{C} \subseteq \mathrm{t}_{T'}(\mathscr{P}_{\mathrm{aut}})$ with $\mathscr{C}_T = (\mathscr{P}_{\mathrm{aut}})_T$ coincides with $\mathrm{t}_{T'}(\mathscr{P}_{\mathrm{aut}})$. The decomposition $\mathscr{P}_{\mathrm{aut}} = \mathrm{t}_T(\mathscr{P}_{\mathrm{aut}}) \oplus \mathrm{t}_{T'}(\mathscr{P}_{\mathrm{aut}})$ implies that $(\mathscr{P}_{\mathrm{aut}})_T = (\mathrm{t}_T(\mathscr{P}_{\mathrm{aut}}))_T \oplus (\mathrm{t}_{T'}(\mathscr{P}_{\mathrm{aut}}))_T = 0 \oplus (\mathrm{t}_{T'}(\mathscr{P}_{\mathrm{aut}}))_T$. Now assume a behavior $\mathscr{C} \subseteq \mathrm{t}_{T'}(\mathscr{P}_{\mathrm{aut}})$ with $\mathscr{C}_T = (\mathscr{P}_{\mathrm{aut}})_T$. Let $w \in \mathrm{t}_{T'}(\mathscr{P}_{\mathrm{aut}})$. From $(\mathrm{t}_{T'}(\mathscr{P}_{\mathrm{aut}}))_T = (\mathscr{P}_{\mathrm{aut}})_T = \mathscr{C}_T$ we deduce that there exists $h \in T$ such that $h \circ w \in \mathscr{C}$. On the other hand, $w \in \mathrm{t}_{T'}(\mathscr{P}_{\mathrm{aut}})$ implies the

existence of $h' \in T'$ such that $h' \circ w = 0$. Since h and h' are coprime, there exist $a, a' \in \mathscr{D}$ such that $1 = ah + a'h'$. It follows that $w = ah \circ w + a'h' \circ w \in \mathscr{C} + 0 = \mathscr{C}$. We deduce that $\mathscr{C} = \mathrm{t}_{T'}(\mathscr{P}_{\text{aut}})$.
3) For any behaviors $\mathscr{B} = \{w \in \mathscr{F}^{\mathtt{w}}; B \circ w = 0\}$, $\mathscr{B}_i = \{w \in \mathscr{F}^{\mathtt{w}}; B_i \circ w = 0\}$, $i = 1, 2$, the relationship $\mathscr{B} = \mathscr{B}_1 \oplus \mathscr{B}_2$ implies that $\mathscr{B}^{(\mathrm{T})} = \mathscr{B}_1{}^{(\mathrm{T})} \oplus \mathscr{B}_2{}^{(\mathrm{T})}$. To see this, use the fact that $\mathscr{B} = \mathscr{B}_1 \oplus \mathscr{B}_2$ if and only if $\mathscr{D}^{1\times \mathtt{k}_1} B_1 \cap \mathscr{D}^{1\times \mathtt{k}_2} B_2 = \mathscr{D}^{1\times \mathtt{k}} B$ and $\mathscr{D}^{1\times \mathtt{k}_1} B_1 + \mathscr{D}^{1\times \mathtt{k}_2} B_2 = \mathscr{D}^{1\times \mathtt{w}}$.
4) The decomposition $\mathscr{P} = \mathscr{P}_{\text{cont}} \oplus \mathscr{P}_{\text{aut}}$ implies, by the previous item, that $\mathscr{P}^{(\mathrm{T})} = \mathscr{P}_{\text{cont}}{}^{(\mathrm{T})} \oplus \mathscr{P}_{\text{aut}}{}^{(\mathrm{T})}$. Substituting the behaviors derived above, we conclude that $\mathscr{P}^{(\mathrm{T})} = \mathscr{P}_{\text{cont}} \oplus \mathrm{t}_{T'}(\mathscr{P}_{\text{aut}})$.

■

The above derivations allow to recover the main results from [18]:

Corollary 16. *The observer $\mathscr{O}$ is a T-nonintrusive observer with T-autonomous error behavior $\mathscr{E}(\mathscr{P}, \mathscr{O})$ if and only if $\widehat{w}_2$ is T-observable from w_1 in $\mathscr{O}$ and $\mathscr{O}$ contains $\mathscr{P}^{(\mathrm{T})} = \mathscr{P}_{\text{cont}} \oplus \mathrm{t}_{T'}(\mathscr{P}_{\text{aut}})$.*

For the continuous or discrete standard signal module $\mathscr{F}$ and standard choices of T this implies:
1) *For a nonintrusive observer $\mathscr{O}$, the error behavior $\mathscr{E}(\mathscr{P}, \mathscr{O})$ is zero if and only if $\widehat{w}_2$ is observable from w_1 in $\mathscr{O}$ and $\mathscr{P} \subseteq \mathscr{O}$, compare [18, Thm.5.7].*
2) *For a nonintrusive observer $\mathscr{O}$, the error behavior $\mathscr{E}(\mathscr{P}, \mathscr{O})$ is autonomous if and only if $\widehat{w}_2$ is trackable from w_1 in $\mathscr{O}$ and $\mathscr{P}_{\text{cont}} \subseteq \mathscr{O}$, compare [18, Thm.5.3].*
3) *For a nonintrusive observer $\mathscr{O}$, the error behavior $\mathscr{E}(\mathscr{P}, \mathscr{O})$ is asymptotically stable if and only if $\widehat{w}_2$ is detectable from w_1 in $\mathscr{O}$ and $\mathscr{P}_{\text{cont}} \oplus \mathscr{P}_{\text{antistab}} \subseteq \mathscr{O}$, compare [18, Thm.5.6].*

Proof: Follows directly from Theorem 7, Lemma 14, Lemma 15, and the choices $T = F \setminus \{0\}$, $T = \mathscr{D} \setminus \{0\}$, and T as in item 3 or 4 in Example 2, respectively. ■

Algorithm 17 (Computation of $R^{(\mathrm{T})}$, compare [3, Alg. 4.16])**.** *For given $R \in \mathscr{D}^{k\times w}$ we construct $R^{(\mathrm{T})} \in \mathscr{D}^{k\times w}$ such that $\mathscr{D}_T^{1\times k} R \cap \mathscr{D}^{1\times w} = \mathscr{D}^{1\times k} R^{(\mathrm{T})}$. Then $\mathscr{D}^{1\times k} R^{(\mathrm{T})}$ is the largest submodule of $\mathscr{D}^{1\times w}$ whose localization w.r.t. T is equal to $\mathscr{D}_T^{1\times k} R$. Let*

$$XRY = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} e_1 & & 0 \\ & \ddots & \\ 0 & & e_r \end{pmatrix} \in \mathscr{D}^{r\times r},$$
$$r := \operatorname{rank}(R), \quad e_1 | \dots | e_r \in \mathscr{D}, \quad X \in \mathrm{Gl}_k(\mathscr{D}),\ Y \in \mathrm{Gl}_w(\mathscr{D}),$$

be the Smith form of R w.r.t. $\mathscr{D}$. For each elementary divisor e_i we consider the prime factor decomposition

$$e_i = u_i \prod_{p\in\mathscr{P}} p^{\mu(p)} = u_i \underbrace{\prod_{p\in\mathscr{P}_1} p^{\mu(p)}}_{=:h_i} \underbrace{\prod_{p\in\mathscr{P}_2} p^{\mu(p)}}_{=:f_i} = u_i h_i f_i,$$
$$0 \neq u_i \in F,\ \mu(p) \geq 0,$$

where $\mathscr{P}$ denotes the set of all monic irreducible polynomials, $\mathscr{P}_1 := \mathscr{P} \cap T$ are the primes in T, and $\mathscr{P}_2 := \mathscr{P} \setminus \mathscr{P}_1$. Define

$$F := \begin{pmatrix} f_1 & & 0 \\ & \ddots & \\ 0 & & f_r \end{pmatrix} \quad \text{and} \quad R^{(\mathrm{T})} := X^{-1} \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix} Y^{-1} \in \mathscr{D}^{k\times w}.$$

Then $\mathscr{D}_T^{1\times k} R \cap \mathscr{D}^{1\times w} = \mathscr{D}^{1\times k} R^{(\mathrm{T})}$.

Proof:

$$\begin{aligned}
\mathscr{D}_T^{1\times \mathtt{k}} R &= \mathscr{D}_T^{1\times \mathtt{k}} \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} Y^{-1} = \mathscr{D}_T^{1\times \mathtt{r}} (E, 0) Y^{-1} \\
&= \bigoplus_{i=1}^{\mathtt{r}} \mathscr{D}_T e_i (Y^{-1})_{i-} = \bigoplus_{i=1}^{\mathtt{r}} \mathscr{D}_T u_i h_i f_i (Y^{-1})_{i-} \\
&= \bigoplus_{i=1}^{\mathtt{r}} \mathscr{D}_T f_i (Y^{-1})_{i-} \\
&\supseteq \bigoplus_{i=1}^{\mathtt{r}} \mathscr{D} f_i (Y^{-1})_{i-} = \mathscr{D}^{1\times \mathtt{r}} (F, 0) Y^{-1} = \mathscr{D}^{1\times \mathtt{k}} R^{(\mathrm{T})},
\end{aligned}$$

where $(Y^{-1})_{i-}$ denotes the i-th row of the matrix Y^{-1}. It follows that $\mathscr{D}^{1\times \mathtt{k}} R^{(\mathrm{T})} \subseteq \mathscr{D}_T^{1\times \mathtt{k}} R \cap \mathscr{D}^{1\times \mathtt{w}}$. If on the other hand

$$\xi = \sum_{i=1}^{\mathtt{r}} \xi_i f_i (Y^{-1})_{i-} \in \mathscr{D}_T^{1\times \mathtt{k}} R \cap \mathscr{D}^{1\times \mathtt{w}}, \quad \xi_i \in \mathscr{D}_T,$$

then $\xi' := \xi Y = \sum_{i=1}^{\mathtt{r}} \xi_i f_i \delta_i$ (where $\delta_i \in \mathscr{D}^{1\times \mathtt{w}}$, $(\delta_i)_j = \delta_{ij}$ the Kronecker delta) is also contained in $\mathscr{D}^{1\times \mathtt{w}}$ because $Y \in \mathrm{Gl}_{\mathtt{w}}(\mathscr{D})$. Since the f_i have no prime factors in T by construction we deduce that $\xi_i \in \mathscr{D}$, i.e., $\xi \in \mathscr{D}^{1\times \mathtt{k}} R^{(\mathrm{T})}$. ■

VIII. Observer parametrization

The characterization of T-nonintrusive T-observers in Theorem 7 and Corollary 16 naturally leads to a constructive parametrization of all such observers. The three theorems in this section are the main results of this paper.

In the following we assume that the given plant

$$\mathscr{P} = \left\{ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathscr{F}^{\mathtt{w}_1 + \mathtt{w}_2}; R_1 \circ w_1 = R_2 \circ w_2 \right\},$$

$R := (R_1, -R_2) \in \mathscr{D}^{\mathtt{k}\times(\mathtt{w}_1+\mathtt{w}_2)} = \mathscr{D}^{\mathtt{k}\times \mathtt{w}}$, admits observers with T-autonomous error behavior, i.e., that w_2 is T-observable from w_1 in $\mathscr{P}$ according to Lemma 12. This signifies that $\mathscr{N}_{\mathtt{w}_2}(\mathscr{P})$ is T-autonomous or, equivalently, that $R_2 \in \mathscr{D}^{\mathtt{k}\times \mathtt{w}_2}$ admits a left inverse matrix over $\mathscr{D}_T$. We assume furthermore that the matrix $R \in \mathscr{D}^{\mathtt{k}\times \mathtt{w}}$ has full row rank $\mathtt{k}$ (otherwise, it can be replaced by a full row rank matrix $R' \in \mathscr{D}^{\mathtt{k}'\times \mathtt{w}}$ which is *row equivalent* to R, i.e., which satisfies $\mathscr{D}^{1\times \mathtt{k}} R = \mathscr{D}^{1\times \mathtt{k}'} R'$). By $R^{(\mathrm{T})} =: (R_1^{(\mathrm{T})}, -R_2^{(\mathrm{T})}) \in \mathscr{D}^{\mathtt{k}\times(\mathtt{w}_1+\mathtt{w}_2)} = \mathscr{D}^{\mathtt{k}\times \mathtt{w}}$ we denote the matrix with the property $\mathscr{D}_T^{1\times \mathtt{k}} R \cap \mathscr{D}^{1\times \mathtt{w}} = \mathscr{D}^{1\times \mathtt{k}} R^{(\mathrm{T})}$ that can be computed using Algorithm 17.

Theorem 18 (Constructive parametrization of T-nonintrusive T-observers)**.** *Let*

$$\begin{pmatrix} E \\ 0 \end{pmatrix} = U R_2^{(\mathrm{T})} V \tag{2}$$

be the Smith form of $R_2^{(\mathrm{T})}$ over $\mathscr{D}$.

1) *A matrix $\widehat{R} = (\widehat{R}_1, -\widehat{R}_2) \in \mathscr{D}^{\widehat{\mathtt{k}}\times(\mathtt{w}_1+\mathtt{w}_2)}$ of (w.l.o.g.) full row rank $\widehat{\mathtt{k}}$ defines a T-nonintrusive T-observer $\mathscr{O} = \{w \in \mathscr{F}^{\mathtt{w}}; \widehat{R} \circ w = 0\}$ if and only if*

$$\mathtt{w}_2 \leq \widehat{\mathtt{k}} \leq \mathtt{k} \quad \text{and} \quad \widehat{R} = XUR^{(\mathrm{T})}$$

for some matrix $X = (X_I, X_{II}) \in \mathscr{D}^{\widehat{\mathrm{k}}\times(\mathrm{w}_2+(\mathrm{k}-\mathrm{w}_2))}$ of full row rank $\widehat{\mathrm{k}}$ whose first component X_I is left invertible over $\mathscr{D}_T$.

2) *Any T-nonintrusive T-observer $\mathscr{O}$ for the plant $\mathscr{P}$ can be constructed using the following steps:*
   - *Choose $\widehat{\mathrm{k}} \in \mathbb{N}$ with $\mathrm{w}_2 \le \widehat{\mathrm{k}} \le \mathrm{k}$.*
   - *Choose $D \in \mathscr{D}^{\mathrm{w}_2\times\mathrm{w}_2} \cap \mathrm{Gl}_{\mathrm{w}_2}(\mathscr{D}_T)$ in Hermite form (see e.g. [13, pp.15-18], [9, pp.375f], [8, 23-6]) i.e., choose monic diagonal elements $D_{jj} \in T$ for $j = 1,\dots,\mathrm{w}_2$ and then define $D_{ij} \in \mathscr{D}$ such that $\deg(D_{ij}) < \deg(D_{jj})$ for $i < j$, $D_{ij} := 0$ for $i > j$.*
     Choose a full row rank matrix $G_2 \in \mathscr{D}^{(\widehat{\mathrm{k}}-\mathrm{w}_2)\times(\mathrm{k}-\mathrm{w}_2)}$ in Hermite form, and choose $G_1 \in \mathscr{D}^{\mathrm{w}_2\times(\mathrm{k}-\mathrm{w}_2)}$ such that $X := \begin{pmatrix} D & G_1 \\ 0 & G_2 \end{pmatrix} \in \mathscr{D}^{(\mathrm{w}_2+(\widehat{\mathrm{k}}-\mathrm{w}_2))\times(\mathrm{w}_2+(\mathrm{k}-\mathrm{w}_2))} = \mathscr{D}^{\widehat{\mathrm{k}}\times\mathrm{k}}$ is in Hermite form.
   - *Compute $\widehat{R} := XUR^{(\mathrm{T})}$ and define $\mathscr{O} := \{w \in \mathscr{F}^{\mathrm{w}}; \widehat{R} \circ w = 0\}$.*

   Every possible choice for the parameters $\widehat{\mathrm{k}}$ and $X = \begin{pmatrix} D & G_1 \\ 0 & G_2 \end{pmatrix}$ yields a different observer.

Proof: According to Theorem 7 and Lemma 14, $\mathscr{O} = \{w \in \mathscr{F}^{\mathrm{w}}; \widehat{R} \circ w = 0\}$, $\widehat{R} = (\widehat{R}_1, -\widehat{R}_2) \in \mathscr{D}^{\widehat{\mathrm{k}}\times\mathrm{w}}$ (where we assume w.l.o.g. that $\mathrm{rank}(\widehat{R}) = \widehat{\mathrm{k}}$) is a T-nonintrusive T-observer for $\mathscr{P}$ if and only if $\mathscr{P}^{(\mathrm{T})} \subseteq \mathscr{O}$ and $\mathscr{N}_{\widehat{\mathrm{w}}_2}(\mathscr{O}_T) = 0$. By the injective cogenerator property of $\mathscr{F}$ over $\mathscr{D}$, the inclusion $\mathscr{P}^{(\mathrm{T})} \subseteq \mathscr{O}$ is equivalent to the inclusion $\mathscr{D}^{1\times\widehat{\mathrm{k}}}\widehat{R} \subseteq \mathscr{D}^{1\times\mathrm{k}}R^{(\mathrm{T})}$, i.e., to the existence of $\widetilde{X} \in \mathscr{D}^{\widehat{\mathrm{k}}\times\mathrm{k}}$ such that $\widehat{R} = \widetilde{X}R^{(\mathrm{T})}$. Since $R^{(\mathrm{T})}$ has by construction full row rank $\mathrm{k} = \mathrm{rank}(R)$, the requirement $\mathrm{rank}(\widehat{R}) = \widehat{\mathrm{k}}$ is equivalent to $\mathrm{rank}(\widetilde{X}) = \widehat{\mathrm{k}}$, and implies in particular that $\widehat{\mathrm{k}} \le \mathrm{k}$.

The condition $\mathscr{N}_{\widehat{\mathrm{w}}_2}(\mathscr{O}_T) = 0$ is satisfied iff the matrix $\widehat{R}_2 = \widetilde{X}R_2^{(\mathrm{T})} \in \mathscr{D}^{\widehat{\mathrm{k}}\times\mathrm{w}_2}$ is left invertible over $\mathscr{D}_T$ (and hence in particular $\mathrm{w}_2 \le \widehat{\mathrm{k}}$). Since V is invertible over $\mathscr{D}$ this is the case if and only if $\widetilde{X}R_2^{(\mathrm{T})}V = \widetilde{X}U^{-1}UR_2^{(\mathrm{T})}V = \widetilde{X}U^{-1}\begin{pmatrix} E \\ 0 \end{pmatrix}$ is left invertible over $\mathscr{D}_T$. With $\widetilde{X}U^{-1} =: X =: (X_I, X_{II}) \in \mathscr{D}^{\widehat{\mathrm{k}}\times(\mathrm{w}_2+(\mathrm{k}-\mathrm{w}_2))}$ this means that $X\begin{pmatrix} E \\ 0 \end{pmatrix} = X_I E$ is left invertible over $\mathscr{D}_T$, i.e., that X_I is left invertible over $\mathscr{D}_T$ since $E \in \mathrm{Gl}_{\mathrm{w}_2}(\mathscr{D}_T)$ (because by assumption R_2 and hence also $R_2^{(\mathrm{T})}$ is left invertible over $\mathscr{D}_T$).

Summing up, the observer $\mathscr{O}$ described by the full row rank matrix $\widehat{R}$ is a T-nonintrusive T-observer for $\mathscr{P}$ if and only if $\widehat{R} = XUR^{(\mathrm{T})}$ for some full row rank matrix $X = (X_I, X_{II}) \in \mathscr{D}^{\widehat{\mathrm{k}}\times(\mathrm{w}_2+(\mathrm{k}-\mathrm{w}_2))}$ where X_I is left invertible over $\mathscr{D}_T$.

Since the behavior $\mathscr{O}$ is determined by the row space $\mathscr{D}^{1\times\widehat{\mathrm{k}}}\widehat{R}$ and not by the matrix $\widehat{R}$ itself, and matrices X that are row equivalent over $\mathscr{D}$ yield matrices $\widehat{R} = XUR^{(\mathrm{T})}$ that are row equivalent over $\mathscr{D}$, it is sufficient to consider only those full row rank matrices X that are in Hermite form. Then left invertibility of X_I over $\mathscr{D}_T$ signifies that X is of the asserted form.

Now assume that two choices $(\widehat{\mathrm{k}}, X)$ and $(\widehat{\mathrm{k}}', X')$ give rise to the same observer $\mathscr{O}$, i.e., $\mathscr{D}^{1\times\widehat{\mathrm{k}}}XUR^{(\mathrm{T})} = \mathscr{D}^{1\times\widehat{\mathrm{k}}'}X'UR^{(\mathrm{T})}$. Since $UR^{(\mathrm{T})}$ is of full row rank k it follows that $\mathscr{D}^{1\times\widehat{\mathrm{k}}}X = \mathscr{D}^{1\times\widehat{\mathrm{k}}'}X'$. Both X and X' being full row rank matrices in Hermite form, we deduce that $\widehat{\mathrm{k}} = \widehat{\mathrm{k}}'$ and $X = X'$. ∎

Remark 19. The above parametrization is significantly different from the parametrizations previously reported in the literature [20], [1], [7], [6], [2], [18], [17] in that it is one-to-one and all parameters are free, provided that we can parametrize the set T. The requirements for certain submatrices to be in Hermite form merely constrain the degrees of certain polynomial entries and hence the *number* of free parameters in the parametrization.

A nice property of the above parametrization is that it allows to easily distinguish those observers that are nonintrusive and not only T-nonintrusive.

Theorem 20 (Nonintrusive T-observers)**.** *Let $\mathscr{O} = \{w \in \mathscr{F}^{\mathrm{w}}; \widehat{R} \circ w = 0\}$ with $\widehat{R} = \begin{pmatrix} D & G_1 \\ 0 & G_2 \end{pmatrix} UR^{(\mathrm{T})}$ be a T-observer for $\mathscr{P}$ constructed according to Theorem 18. Furthermore, let $C \in \mathscr{D}^{\mathrm{k}\times\mathrm{k}} \cap \mathrm{Gl}_{\mathrm{k}}(\mathscr{D}_T)$ be such that $R = CR^{(\mathrm{T})}$ and let $A^{-1}Z$ be a left coprime factorization of U_2C^{-1} over $\mathscr{D}$ where $U =: \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \in \mathscr{D}^{(\mathrm{w}_2+(\mathrm{k}-\mathrm{w}_2))\times\mathrm{k}}$. Then the observer $\mathscr{O}$ is nonintrusive (and not only T-nonintrusive) if and only if $G_2 \in \mathscr{D}^{(\widehat{\mathrm{k}}-\mathrm{w}_2)\times(\mathrm{k}-\mathrm{w}_2)}A$.*

Proof: From the relations $\mathscr{D}^{1\times\mathrm{k}}R \subseteq \mathscr{D}^{1\times\mathrm{k}}R^{(\mathrm{T})}$ and $\mathscr{D}_T^{1\times\mathrm{k}}R = \mathscr{D}_T^{1\times\mathrm{k}}R^{(\mathrm{T})}$ and the fact that R and $R^{(\mathrm{T})}$ have full row rank k we infer that there exists $C \in \mathscr{D}^{\mathrm{k}\times\mathrm{k}} \cap \mathrm{Gl}_{\mathrm{k}}(\mathscr{D}_T)$ such that $R = CR^{(\mathrm{T})}$ as required.

According to Remark 6 the observer $\mathscr{O}$ is nonintrusive if and only if $\mathscr{P}_{w_1} \subseteq \mathscr{O}_{w_1}$. In order to compute these behaviors (i.e., eliminate the variables w_2 resp. $\widehat{w}_2$ in $\mathscr{P}$ resp. $\mathscr{O}$) we need to find universal left annihilators of R_2 and $\widehat{R}_2$ over $\mathscr{D}$.

Let $Z' \in \mathscr{D}^{(\mathrm{k}-\mathrm{w}_2)\times\mathrm{k}}$ denote a universal left annihilator of R_2 over $\mathscr{D}$. We show that Z and Z' are row equivalent over $\mathscr{D}$, and hence that also Z is a universal left annihilator of R_2 over $\mathscr{D}$. Let $\xi \in \mathscr{D}^{1\times\mathrm{k}}$. Then $\xi R_2 = \xi CR_2^{(\mathrm{T})} = 0$ if and only if $\xi C \in \mathscr{D}^{1\times(\mathrm{k}-\mathrm{w}_2)}U_2$ since U_2 is a universal left annihilator of $R_2^{(\mathrm{T})}$ over $\mathscr{D}$. Hence

$$\begin{aligned}\mathscr{D}^{1\times(\mathrm{k}-\mathrm{w}_2)}U_2C^{-1} \cap \mathscr{D}^{1\times\mathrm{k}} &= \left\{\xi \in \mathscr{D}^{1\times\mathrm{k}}; \xi R_2 = 0\right\} \\ &= \mathscr{D}^{1\times(\mathrm{k}-\mathrm{w}_2)}Z'.\end{aligned}$$

It follows that Z' is contained in $\mathscr{D}^{(\mathrm{k}-\mathrm{w}_2)\times(\mathrm{k}-\mathrm{w}_2)}U_2C^{-1}$, and hence that there exists a matrix $A' \in \mathscr{D}^{(\mathrm{k}-\mathrm{w}_2)\times(\mathrm{k}-\mathrm{w}_2)}$ such that $Z' = A'U_2C^{-1}$. Moreover, the equality $\mathscr{D}_T^{1\times(\mathrm{k}-\mathrm{w}_2)}A'U_2C^{-1} = \mathscr{D}_T^{1\times(\mathrm{k}-\mathrm{w}_2)}Z' = \mathscr{D}_T^{1\times(\mathrm{k}-\mathrm{w}_2)}U_2C^{-1}$ and the fact that U_2C^{-1} has full row rank imply that $A' \in \mathrm{Gl}_{\mathrm{k}-\mathrm{w}_2}(\mathscr{D}_T)$. The factorization $(A')^{-1}Z'$ of U_2C^{-1} is a left coprime factorization over $\mathscr{D}$ since Z' is a universal left annihilator over $\mathscr{D}$ and hence admits a right inverse over $\mathscr{D}$, and consequently also $(A', -Z')$ is right invertible over $\mathscr{D}$. Since also $A^{-1}Z$ is a left coprime factorization of U_2C^{-1} over $\mathscr{D}$, it follows that $(A, -Z)$ and $(A', -Z')$ are row equivalent over $\mathscr{D}$, and hence in particular that Z, like Z', is a universal left annihilator of R_2 over $\mathscr{D}$.

We compute a universal left annihilator $\widehat{Z}$ of $\widehat{R}_2$: Let $\xi \in \mathscr{D}^{1\times\widehat{\mathrm{k}}}$. Then $\xi\widehat{R}_2 = \xi XUR_2^{(\mathrm{T})} = 0$ if and only if $\xi XUR_2^{(\mathrm{T})}V = 0$ since $V \in \mathrm{Gl}_{\mathrm{w}_2}(\mathscr{D})$. Substituting (2) yields $\xi X\begin{pmatrix} E \\ 0 \end{pmatrix} = 0$, i.e., $\xi X_I E = 0$. From $E \in \mathrm{Gl}_{\mathrm{w}_2}(\mathscr{D}_T)$ we infer that this is equivalent to $\xi X_I = 0$. Summing up, a matrix is a universal left annihilator of $\widehat{R}_2$ over $\mathscr{D}$ if and only if it is a universal left annihilator of $X_I = \begin{pmatrix} D \\ 0 \end{pmatrix}$ over $\mathscr{D}$. The matrix $\widehat{Z} := (0, \mathrm{id}_{\widehat{\mathrm{k}}-\mathrm{w}_2}) \in \mathscr{D}^{(\widehat{\mathrm{k}}-\mathrm{w}_2)\times\widehat{\mathrm{k}}}$ clearly has this property.

The elimination theorem [15, Ch.6], [14, Cor.38 on p.26] yields that $\mathscr{P}_{w_1} = \{w_1 \in \mathscr{F}^{\mathrm{w}_1}; ZR_1 \circ w_1 = 0\}$ and $\mathscr{O}_{w_1} = \{w_1 \in$

$\mathscr{F}^{\mathtt{w}_1}; \widehat{Z}\widehat{R}_1 \circ w_1 = 0\}$. The observer $\mathscr{O}$ is nonintrusive if and only if $\mathscr{P}_{w_1} \subseteq \mathscr{O}_{w_1}$, i.e., since $\mathscr{F}$ is an injective cogenerator over $\mathscr{D}$, if and only if $\mathscr{D}^{1\times(\widehat{\mathtt{k}}-\mathtt{w}_2)}(\widehat{Z}\widehat{R}_1) \subseteq \mathscr{D}^{1\times(\mathtt{k}-\mathtt{w}_2)}(ZR_1)$, i.e., $\widehat{Z}\widehat{R}_1 \in \mathscr{D}^{(\widehat{\mathtt{k}}-\mathtt{w}_2)\times(\mathtt{k}-\mathtt{w}_2)}(ZR_1)$. Substituting $\widehat{Z} = (0, \mathrm{id}_{\widehat{\mathtt{k}}-\mathtt{w}_2})$, $\widehat{R}_1 = XUR_1^{(\mathrm{T})}$ where $X = \begin{pmatrix} D & G_1 \\ 0 & G_2 \end{pmatrix}$, $Z = AU_2C^{-1}$, and $R_1 = CR_1^{(\mathrm{T})}$ we get $G_2U_2R_1^{(\mathrm{T})} \in \mathscr{D}^{(\widehat{\mathtt{k}}-\mathtt{w}_2)\times(\mathtt{k}-\mathtt{w}_2)}AU_2R_1^{(\mathrm{T})}$. This is equivalent to $G_2 \in \mathscr{D}^{(\widehat{\mathtt{k}}-\mathtt{w}_2)\times(\mathtt{k}-\mathtt{w}_2)}A$ because $U_2R_1^{(\mathrm{T})}$ has full row rank $\mathtt{k}-\mathtt{w}_2$ since $R^{(\mathrm{T})}$ and hence also the product $UR^{(\mathrm{T})} = \begin{pmatrix} U_1R_1^{(\mathrm{T})} & U_1R_2^{(\mathrm{T})} \\ U_2R_1^{(\mathrm{T})} & 0 \end{pmatrix}$ has full row rank $\mathtt{k}$. ∎

Remark 21. According to the preceding theorem all nonintrusive T-observers $\mathscr{O}$ for the given plant $\mathscr{P}$ can be constructed by the following algorithm:

- Choose $\widehat{\mathtt{k}} \in \mathbb{N}$ with $\mathtt{w}_2 \le \widehat{\mathtt{k}} \le \mathtt{k}$.
- Choose a matrix $D \in \mathscr{D}^{\mathtt{w}_2\times\mathtt{w}_2} \cap \mathrm{Gl}_{w_2}(\mathscr{D}_T)$ in Hermite form. Choose a full row rank matrix $\widetilde{G}_2 \in \mathscr{D}^{(\widehat{\mathtt{k}}-\mathtt{w}_2)\times(\mathtt{k}-\mathtt{w}_2)}$ in Hermite form, and define G_2 as the Hermite form of $\widetilde{G}_2A$.
  Finally choose $G_1 \in \mathscr{D}^{\mathtt{w}_2\times(\mathtt{k}-\mathtt{w}_2)}$ such that $X := \begin{pmatrix} D & G_1 \\ 0 & G_2 \end{pmatrix}$ is in Hermite form.
- Compute $(\widehat{R}_1, -\widehat{R}_2) := \widehat{R} := XUR^{(\mathrm{T})}$ and define $\mathscr{O} := \{w \in \mathscr{F}^{\mathtt{w}}; \widehat{R} \circ w = 0\}$.

Finally, input/output observers are easily characterized in terms of a minimal choice for the observer parameter $\widehat{\mathtt{k}}$.

Theorem 22 (Input/output observers)**.** *A T-observer $\mathscr{O}$ for $\mathscr{P}$ constructed according to Theorem 18 is an input/output behavior with input w_1 and output $\widehat{w}_2$ if and only if $\widehat{\mathtt{k}} = \mathtt{w}_2$.*

Proof: The T-observer $\mathscr{O}$ is an input/output behavior with input w_1 and output $\widehat{w}_2$ if and only if $\mathrm{rank}(\widehat{R}_1, -\widehat{R}_2) = \mathrm{rank}(\widehat{R}_2) = \mathtt{w}_2$, compare [15, Sec.3.3], [14, Thm.2.69 on p.27]. The equality $\mathrm{rank}(\widehat{R}_2) = \mathtt{w}_2$ is satisfied for all $\widehat{R}$ constructed in Theorem 18. Moreover, $\mathrm{rank}(\widehat{R}) = \mathrm{rank}(X) = \widehat{\mathtt{k}}$ by construction, compare also the proof of Theorem 18. It follows that $\mathscr{O}$ has the required input/output structure if and only if $\widehat{\mathtt{k}} = \mathtt{w}_2$. ∎

Remark 23. In fact, if

$$\mathscr{O} = \left\{ \begin{pmatrix} w_1 \\ \widehat{w}_2 \end{pmatrix} \in \mathscr{F}^{\mathtt{w}_1+\mathtt{w}_2}; \widehat{R}_1 \circ w_1 = \widehat{R}_2 \circ \widehat{w}_2 \right\}$$

and

$$(\widehat{R}_1, -\widehat{R}_2) =: \begin{pmatrix} \widehat{R}_1^1 & -\widehat{R}_2^1 \\ \widehat{R}_1^2 & -\widehat{R}_2^2 \end{pmatrix} \in \mathscr{D}^{((\mathtt{w}_2+(\widehat{\mathtt{k}}-\mathtt{w}_2))\times(\mathtt{w}_1+\mathtt{w}_2)}$$

is constructed according to Theorem 18 (or Remark 21), then $\widehat{R}_2^2 = 0$ and $\mathscr{O}$ can be interpreted as the interconnection of

$$\mathscr{O}_{\mathrm{io}} := \left\{ \begin{pmatrix} w_1 \\ \widehat{w}_2 \end{pmatrix} \in \mathscr{F}^{\mathtt{w}_1+\mathtt{w}_2}; \widehat{R}_1^1 \circ w_1 = \widehat{R}_2^1 \circ \widehat{w}_2 \right\} \quad \text{and}$$
$$\mathscr{O}_1 := \left\{ w_1 \in \mathscr{F}^{\mathtt{w}_1}; \widehat{R}_1^2 \circ w_1 = 0 \right\}$$

via w_1 where $\mathscr{O}_{\mathrm{io}}$ is an input/output behavior with input w_1 and output $\widehat{w}_2$, and $\mathscr{O}_1 = \mathscr{O}_{w_1}$ is a behavior restricting the variable w_1. In accordance with the T-nonintrusiveness (resp. the nonintrusiveness) of the observer $\mathscr{O}$, it holds that $(\mathscr{P}_{w_1})_T \subseteq (\mathscr{O}_1)_T$ (resp. $\mathscr{P}_{w_1} \subseteq \mathscr{O}_1$). This is an illustration of the "curious" result reported in [18, Proposition 4.5] and allows to choose input/output observers without loss of generality.

IX. Example

We assume the continuous standard situation from Example 2.3, i.e., the T-small signals are the polynomial-exponential functions which are asymptotically zero. Consider the plant

$$\mathscr{P} = \left\{ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathscr{F}^{3+1}; R_1 \circ w_1 = R_2 \circ w_2 \right\}, \quad R := (R_1, -R_2)$$

where

$$R_1 = \begin{pmatrix} s^3+s^2+s+2 & s+1 & 2s \\ -s^2+1 & 0 & -3s^3-s^2+3s+1 \end{pmatrix},$$
$$R_2 = \begin{pmatrix} -s^2-s \\ s^4-2s^2+1 \end{pmatrix}.$$

The Smith form of R_2 is $\binom{s+1}{0}$, hence R_2 is left invertible over $\mathscr{D}_T$, which is equivalent to T-observability of w_2 from w_1 in $\mathscr{P}$, compare Definition and Lemma 4. With Lemma 12 we conclude that $\mathscr{P}_1$ admits (nonintrusive) T-observers.

The matrix $R^{(\mathrm{T})} = (R_1^{(\mathrm{T})}, -R_2^{(\mathrm{T})})$ can be computed by means of Algorithm 17 which yields the result displayed in equation (M1) in Figure 2.

The Smith form of $R_2^{(\mathrm{T})}$ is $UR_2^{(\mathrm{T})}V = \binom{1}{0}$ where

$$U = \begin{pmatrix} \frac{1}{2}s^3+s^2-\frac{1}{2}s-\frac{3}{2} & \frac{1}{2}s+1 \\ -s^4-s^3+s^2+2s-1 & -s^2-s \end{pmatrix}, \quad V = (1).$$

We construct a T-nonintrusive T-observer by means of Theorem 18. We choose $\widehat{\mathtt{k}} := 2$, hence $\mathtt{w}_2 = 1 \le \widehat{\mathtt{k}} \le 2 = \mathtt{k}$. A possible choice for the matrix X is

$$X := \begin{pmatrix} D & G_1 \\ 0 & G_2 \end{pmatrix} := \begin{pmatrix} s+2 & 1 \\ 0 & s \end{pmatrix} \in \mathscr{D}^{(1+1)\times(1+1)}.$$

This yields the matrix $\widehat{R} := (\widehat{R}_1, -\widehat{R}_2) := XUR^{(\mathrm{T})}$ displayed in (M2) in Figure 2 and the T-nonintrusive T-observer $\mathscr{O} = \left\{ \begin{pmatrix} w_1 \\ \widehat{w}_2 \end{pmatrix} \in \mathscr{F}^{3+1}; \widehat{R}_1 \circ w_1 = \widehat{R}_2 \circ \widehat{w}_2 \right\}$. Computation of the error behavior $\mathscr{E}(\mathscr{P}, \mathscr{O})$ yields

$$\begin{aligned} \mathscr{E}(\mathscr{P}, \mathscr{O}) &= (0, -\mathrm{id}_{\mathtt{w}_2}, \mathrm{id}_{\mathtt{w}_2}) \circ (\mathscr{P} \wedge_{w_1} \mathscr{O}) \\ &= \{e \in \mathscr{F}^{\mathtt{w}_2}; V \circ e = 0\} \quad \text{with} \\ V &= \begin{pmatrix} -s^3-4s^2-5s-2 \\ 8s^3+30s^2+34s+12 \end{pmatrix} \end{aligned}$$

which is T-autonomous since the Smith form of V is

$$\begin{pmatrix} (s+2)(s+1) \\ 0 \end{pmatrix}.$$

It can also be checked that $\mathscr{O}$ is really a T-nonintrusive observer, that $\widehat{w}_2$ is T-observable from w_1 in $\mathscr{O}$, that $\mathscr{P}_T \subseteq \mathscr{O}_T$ (but not $\mathscr{P} \subseteq \mathscr{O}$), and that $\mathscr{P}^{(\mathrm{T})} \subseteq \mathscr{O}$.

The observer is however not nonintrusive. The matrix A from Theorem 20 (which is uniquely determined up to row equivalence over $\mathscr{D}$) is

$$A = \begin{pmatrix} s+1 \end{pmatrix}.$$

$$R^{(\mathrm{T})} = \begin{pmatrix} s^3+s^2+s+2 & s+1 & 2s & s^2+s \\ -s^5-s^4+s^3-2s^2+2s+1 & -s^3-s^2+2s & -2s^3+s+1 & -s^4-s^3+s^2+2s-1 \end{pmatrix} \tag{M1}$$

$$\widehat{R}_1 = \begin{pmatrix} -\frac{1}{2}s^5+2s^4-s^3-2s^2-6 & -\frac{1}{2}s^3+2s^2-\frac{1}{2}s-4 & \frac{1}{2}s^3-\frac{1}{2}s^2-5s+2 \\ -s^6+s^5+2s^2-2s & -s^4+s^3+s^2-s & s^4+2s^3-3s^2 \end{pmatrix}, \quad \widehat{R}_2 = \begin{pmatrix} s+2 \\ 0 \end{pmatrix} \tag{M2}$$

$$\widehat{R}_1 = \begin{pmatrix} -\frac{1}{2}s^5+2s^4-s^3-2s^2-6 & -\frac{1}{2}s^3+2s^2-\frac{1}{2}s-4 & \frac{1}{2}s^3-\frac{1}{2}s^2-5s+2 \\ -s^6+s^4+2s^2-2 & -s^4+2s^2-1 & s^4+3s^3-s^2-3s \end{pmatrix}, \quad \widehat{R}_2 = \begin{pmatrix} s+2 \\ 0 \end{pmatrix} \tag{M3}$$

$$\widehat{R}_1 = \begin{pmatrix} -\frac{1}{2}s^5+2s^4-s^3-2s^2-6 & -\frac{1}{2}s^3+2s^2-\frac{1}{2}s-4 & \frac{1}{2}s^3-\frac{1}{2}s^2-5s+2 \end{pmatrix}, \quad \widehat{R}_2 = \begin{pmatrix} s+2 \end{pmatrix}. \tag{M4}$$

Fig. 2: The matrices appearing in the example.

According to this theorem, the observer is nonintrusive iff G_2 is chosen in $\mathscr{D}^{(\widehat{\mathtt{k}}-\mathtt{w}_2)\times(\mathtt{k}-\mathtt{w}_2)}A$. For example the choice

$$\widetilde{G}_2 := \begin{pmatrix} 1 \end{pmatrix}, \quad G_2 := \widetilde{G}_2 A = \begin{pmatrix} s+1 \end{pmatrix}, \quad G_1 := \begin{pmatrix} 1 \end{pmatrix},$$

hence

$$X = \begin{pmatrix} D & G_1 \\ 0 & G_2 \end{pmatrix} = \begin{pmatrix} s+2 & 1 \\ 0 & s+1 \end{pmatrix} \in \mathscr{D}^{(1+1)\times(1+1)}$$

yields the nonintrusive T-observer $\mathscr{O} = \left\{ \binom{w_1}{\widehat{w}_2} \in \mathscr{F}^{3+1};\ \widehat{R}_1 \circ w_1 = \widehat{R}_2 \circ \widehat{w}_2 \right\}$ with the matrix $\widehat{R} = (\widehat{R}_1, -\widehat{R}_2)$ from (M3) in Figure 2. Again, all properties of the observer $\mathscr{O}$ can be checked and verified.

In order to construct observers with input w_1 and output $\widehat{w}_2$, the parameter $\widehat{\mathtt{k}}$, i.e., the rank of the matrix $\widehat{R}$, has to be chosen as $\widehat{\mathtt{k}} = \mathtt{w}_2 = 1$. Then the parameter G_2 does not appear. If we choose

$$D := \begin{pmatrix} s+2 \end{pmatrix} \quad \text{and} \quad G_1 := \begin{pmatrix} 1 \end{pmatrix}$$

as above, we get the input/output T-observer $\mathscr{O} = \left\{ \binom{w_1}{\widehat{w}_2} \in \mathscr{F}^{3+1};\ \widehat{R}_1 \circ w_1 = \widehat{R}_2 \circ \widehat{w}_2 \right\}$ with $\widehat{R} = (\widehat{R}_1, -\widehat{R}_2)$ as displayed in (M4) in Figure 2. These matrices contain the first row of the respective matrices defining the previous (nonintrusive) observer (compare (M3)), an illustration of Remark 23.

X. Conclusions

We have provided a novel characterization of observers for linear systems in the behavioral framework. This characterization takes the form of a generalized internal model principle, stating in essence that T-nonintrusive T-observers are exactly those observer systems that include the "undesirable" plant dynamics. Building on this characterization, we were able to derive a constructive, one-to-one parametrization of all T-nonintrusive T-observers for a given plant where this parametrization has only free parameters and nicely embeds a parametrization of all input/output-observers. The results reported in this paper generalize and extend all similar results known to the authors, including the classical results for linear time-invariant state space systems and the authors' own recent work in the area.

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