Observers for systems with invariant outputs

C. Lageman, J. Trumpf and R. Mahony

***Abstract*— In this paper we introduce a general design approach for observers for left-invariant systems on a Lie group with right-invariant outputs on a manifold. This situation is encountered for example in attitude estimation problems where only partial information of the state can be measured. We propose a gradient-like observer design on the output space which can be lifted to a gradient-like observer on the state space. The observer on the output space converges for generic initial conditions and has local exponential convergence properties. The lifted observer retains the same convergence properties with respect to the fibers of the output map. Our observer constructions are illustrated on the attitude estimation problem for a rigid body.**

I. INTRODUCTION

Nonlinear observer design for invariant systems on Lie groups has received a growing amount of interest in the last years, driven mainly by applications in state estimation for autonomouos robotic vehicles. The interest in nonlinear observers is motivated by their potential lower computational complexity and their increased robustness compared to standard filtering approaches. Invariant systems on Lie groups have been studied in a series of works starting in the seventies, focusing mainly on general, theoretical observability and controllablity questions [1], [2], [3], [4], [5]. Nonlinear observer design approaches for invariant systems have been generally more closely connected to specific applications. A large body of works considers nonlinear attitude observers for rigid body dynamics [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17] or the full pose estimation problem [18], [19], [20], [21]. However, theoretical aspects of observer design for invariant systems have received little attention in the past. Only recent works have begun to tackle such questions and consider the general structure of observers for invariant systems and propose general approaches to observer design [22], [23], [14], [24], [25], [26], [27].

In this paper, we extend the gradient-like observer construction for full state observations proposed in [25], [27] to systems with partial state observations given by a right-invariant output map. This system set up has been considered in [26], where the authors discuss a general observer structure which has autonomous error dynamics and yields a local convergence result for a suitable choice of gain coefficients.

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In this paper, we approach the observer construction problem by first designing a gradient-like observer on the output space. This is justified by the fact that the system projects to a system with full state observations on the output space. For this projected system a modification of the gradient-like error construction for full state observations on a Lie group [25], [27] can be applied, which yields an output observer for the system on the group. To obtain an observer for the group state space we use a lift of this output observer. This lift can be interpreted as a gradient-like observer on the Lie group. This links our approach both to [26] and the gradient-like observers for full state observation [25], [27] as the lifted observer fits the observer structures discussed in these papers. We show that under mild conditions the errors for the output observer and the group state observer converge to a reference output or the stabilizer of this reference output, respectively. Furthermore, this convergence is locally exponential. To illustrate our concepts, we discuss an attitude estimation problem for a rigid body with partial state measurements.

In Section II we introduce our notation and the system set up. Section III discusses the observer construction on the output space and the convergence properties. The lift to the group and the related convergence results are considered in Section IV. Section V illustrates the methods on the attitude estimation problem. The paper ends with conclusions in Section VI.

II. INVARIANT SYSTEMS WITH OBSERVATIONS ON A HOMOGENEOUS SPACE

A. Notation

We start with the notation used in the paper. We denote by G a connected, finite-dimensional Lie group with Lie algebra $\mathfrak{g}$ and identity e. M will denote a manifold on which G acts by a right action $h\colon G \times M \to M$, i.e. $h(e, y) = y$ for all $y \in M$, $e \in G$ the identity and $h(X, h(\hat{X}, y)) = h(\hat{X}X, y)$ for all $X, \hat{X} \in G$ and $y \in M$. The action h is transitive if for all $y, \hat{y} \in M$ there exists an $X \in G$ with $h(y, X) = \hat{y}$. The group acts effectively on M if for all $X \in G, X \neq e$ exists a $y \in M$ (possibly depending on X) with $\tilde{h}_X(y) \neq y$. To simplify our notation we use h_y for the map $h_y\colon G \to M$, $h_y(X) = h(X, y)$ and $\tilde{h}_X$ for the map $\tilde{h}_X\colon M \to M$, $\tilde{h}_X(y) = h(X, y)$. We denote by $\mathrm{stab}(y)$ for $y \in M$ the stabilizer group $\mathrm{stab}(y) = \{X \in G \mid h(X, y) = y\}$. We use the notation $T_X G$ and $T_y M$ for the tangent spaces of G and M at X and y, respectively. The tangent map for a function $\phi\colon G \to M$ at $X \in G$ is denoted by $T_X\phi$. A Riemannian metric $\langle \cdot, \cdot \rangle$ on G is called right-invariant if for any $X, Y \in G$, $v, w \in \mathfrak{g}$ and the right-multiplication map $R_Y: G \to G$, $R_Y(X) = XY$, one has $\langle T_e R_Y v, T_e R_Y w\rangle_G = \langle v, w\rangle_G$. A Riemannian metric on M is called invariant (under h) if for all $X \in G$, $y \in M$, $v, w \in T_y M$ one has $\langle v, w\rangle = \langle T_y \tilde{h}_X v, T_y \tilde{h}_X\rangle$. A distribution $H(X)$ on G is called right-invariant if for all $X, Y \in G$ the equation $H(XY) = T_X R_Y H(X)$ holds. We say that a right-invariant Riemannian metric on G induces an invariant Riemannian metric on M if there is a right-invariant horizontal distribution $H(X)$ such that for all $y \in M$, $X \in G$, $v, w \in T_X G$ we have $\langle T_X h_y(Xv), T_X h_y(Xw)\rangle = \langle v, w\rangle_G$. We will use the notation $(v)^H$ for the horizontal lift of $v \in T_y M$, $y \in M$ to $T_X G$, $X \in \mathrm{stab}(y)$ with respect to a horizontal distribution $H(X)$, i.e. the unique $w \in H(X)$ with $T_X h_y w = v$. At last, we recall that the gradient of a function f on G or M with respect to a Riemannian metric $\langle\cdot,\cdot\rangle$ is given by the identity $\langle \mathrm{grad}\, f(x), v\rangle = df(x)(v)$ for all $v \in T_x G$ (or all $v \in T_x M$, respectively).

B. System set-up and assumptions

We introduce now the systems discussed in this paper. It is assumed that we are given a connected Lie group G and a smooth manifold M on which G acts by a transitive right action. We consider a left-invariant system on G with right-invariant outputs on M, i.e. a system of the form

$$\begin{aligned} \dot{X} &= Xu, \quad X \in G \\ y &= h(X, y_0), \quad y \in M \end{aligned} \tag{1}$$

with $u: \mathbb{R} \to \mathfrak{g}$ a smooth input and y_0 a fixed reference output. We will refer to M as the output space. Since our observer construction will be gradient-based, we assume that there is a right-invariant metric on G which induces an invariant metric on M. This assumption is satisfied e.g. for all reductive homogeneous spaces M (cf. [28] for a definition) on which G acts effectively, see [29, Proposition 3.16][1]. This includes in particular $G = SO(n)$ with $M = S^{n-1}$ and the action $h(R, y) = R^\top y$ for $R \in SO(n)$ and $y \in S^{n-1}$, cf. Section V. The right-invariant horizontal distribution associated with the pair of Riemannian metrics is denoted by $H(X)$.

In this paper we consider the problem of constructing an observer for systems of the form (1) given only measurements of the input u and the output y.

III. Gradient-like observers on the homogeneous space

Our observer design approach is based on the fact that the invariant system (1) can be reduced to a system on the output space with full state observations. We will propose an observer for this projected system similar to the gradient-like observers for invariant systems with full state observations on the group proposed in [25], [27].

Let us begin with the reduction of the system to the output space.

[1]Usually only left actions on M are considered in the literature but the results extend analogously to right actions by interchanging left- and right-invariance.

Theorem 1: System (1) projects to the system

$$\dot{y} = T_X h_{y_0}(Xu) \tag{2}$$

with $X \in h_{y_0}^{-1}(y)$. This system is well defined and a minimal realization of (1).

This result is mainly based on the symmetry reduction methods as introduced in [30]. We omit a detailed proof of the theorem.

As in the Lie group case with full state observations [25], [27], the observer on the homogeneous space consists of a copy of the system dynamics and a gradient-like innovation which will drive a suitably chosen error term into a "no error" state. For that, we assume that we are given a smooth cost function $f: M \times M \to \mathbb{R}$ on M. The proposed observer has the form:

$$\dot{\hat{y}} = T_{\hat{X}} h_{y_0}(\hat{X}u) - \mathrm{grad}_1 f(\hat{y}, y). \tag{3}$$

To discuss the convergence properties of the observer for the projected system, i.e. the output, we have to introduce a suitable error function which compares the observer state with the state of the original system on the group.

Definition 2: We define the *output error* E as the function $E: M \times G \to M$,

$$E(\hat{y}, X) := h(\hat{X}X^{-1}, y_0) = h(X^{-1}, \hat{y}) \quad \hat{X} \in h_{y_0}^{-1}(\hat{y}).$$

Note that this error is well-defined and a smooth function. The error value y_0 corresponds to the "no error" states, i.e. that the state of the observer is the output corresponding to the state X. The use of this error is justified by the fact that any other smooth error function $\hat{E}: M \times M \to N$ which is constant on two copies of the system (2) can be lifted to $M \times G$ as $\hat{E}(\hat{y}, h_{y_0}(X))$ and this lift is the composition $g \circ E$ of a smooth, $\mathrm{stab}(y_0)$-invariant function $g: M \to N$ with the output error.

Under suitable invariance conditions on the cost, the output error has autonomous gradient dynamics.

Theorem 3: Assume that f is invariant under the action of G on $M \times M$, i.e. $f(h(S, \hat{y}), h(S, y)) = f(\hat{y}, y)$ and that the conditions on the Riemannian metrics of Section II hold. Then we have the following autonomous dynamics for the output error.

$$\frac{d}{dt} E(\hat{y}, X) = -\mathrm{grad}_1 f(E(\hat{y}, X), y_0).$$

Proof: In this proof we will denote the left and right multiplication in G by $Y \in G$ as L_Y and R_Y. Straightforward calculations using the equivariance of h show that $h_y \circ L_X = h_{h(X,y)}$ and $\tilde{h}_X \circ h_y = h_y \circ R_X$. Thus we have

$$\begin{aligned} &\frac{d}{dt} E(\hat{y}, X) \\ &\quad= -T_{X^{-1}} h_{\hat{y}} T_e L_{X^{-1}} T_X R_{X^{-1}} X u \\ &\qquad + T_{\hat{y}} \tilde{h}_{X^{-1}} T_{\hat{X}} h_{y_0} \hat{X} u - T_{\hat{y}} \tilde{h}_{X^{-1}} \mathrm{grad}_1 f(\hat{y}, y) \\ &\quad= -T_{X^{-1}} h_{\hat{y}} T_e R_{X^{-1}} u + T_{\hat{X}X^{-1}} h_{y_0} T_{\hat{X}} R_{X^{-1}} \hat{X} u \\ &\qquad - T_{\hat{y}} \tilde{h}_{X^{-1}} \mathrm{grad}_1 f(\hat{y}, y) \\ &\quad= -T_{\hat{X}X^{-1}} h_{y_0} T_{X^{-1}} L_{\hat{X}} T_e R_{X^{-1}} u \\ &\qquad + T_{\hat{X}X^{-1}} h_{y_0} T_{\hat{X}} R_{X^{-1}} \hat{X} u - T_{\hat{y}} \tilde{h}_{X^{-1}} \mathrm{grad}_1 f(\hat{y}, y) \\ &\quad= -T_{\hat{y}} \tilde{h}_{X^{-1}} \mathrm{grad}_1 f(\hat{y}, y). \end{aligned}$$

Since the Riemannian metric on M is invariant under the action of h and f is invariant under the action of G on $M \times M$, we get

$$\begin{aligned} T_{\hat{y}}\tilde{h}_{X^{-1}} \operatorname{grad}_1 f(\hat{y}, y) &= \operatorname{grad}_1 f(\tilde{h}_{X^{-1}}(\hat{y}), \tilde{h}_{X^{-1}}(y)) \\ &= \operatorname{grad}_1 f(E(\hat{y}, X), y_0). \end{aligned}$$

This yields the formula for the output error dynamics. ■

The gradient error dynamics yields the following convergence result for the output error on M.

Theorem 4: Assume that f is invariant under the action of G and that the conditions on the Riemannian metrics of Section II hold. Furthermore let $y \mapsto f(y, y_0)$ be a Morse-Bott function with one global minimum and no further local minima. Then the output error on M converges to y_0 for generic initial conditions and the convergence is locally exponential.

Proof: By Theorem 3 the output error has gradient dynamics of the function $y \mapsto f(y, y_0)$. As the latter function is Morse-Bott, the gradient system converges for generic initial conditions to the set of local minima and the convergence at a manifold of local minima is exponential [31]. Since y_0 is the only local minimum, we get generic convergence to y_0 at a locally exponential rate. ■

Remark 5: Note that for all results in this section, it is sufficient that we are given an invariant Riemannian metric on M. The Riemannian metric on G will only play a role for the observer construction on G.

IV. Gradient-like observers on the group

We turn now to the construction of an observer on the group. Since the projected system on the output space is a minimal realization, a natural approach is to lift the observer equation for the projected system to the Lie group. Here, we use a direct copy of the system dynamics Xu and lift the innovation term on M to the group by a horizontal lift with respect to the horizontal distribution H. This yields the observer

$$\dot{\hat{X}} = \hat{X}u - \left(\operatorname{grad}_1 f(h_{y_0}(\hat{X}), y)\right)^H, \tag{4}$$

where the term $(v)^H \in T_{\hat{X}}G$ denotes the horizontal lift of $v \in T_{h(\hat{X}, y_0)}M$, $\hat{X} \in G$.

Straightforward calculations and the invariance properties of $H(X)$ yield the following projection result for the lifted observer.

Proposition 6: Assume that the conditions on the Riemannian metrics of Section II hold. The observer (4) on G projects via the map h_{y_0} to the observer (3) on M, i.e.

$$\begin{aligned} T_{\hat{X}}h_{y_0} &\left(\hat{X}u - \left(\operatorname{grad}_1 f(h_{y_0}(\hat{X}), y)\right)^H\right) \\ &= T_{\hat{X}}h_{y_0}(\hat{X}u) - \operatorname{grad}_1 f(\hat{y}, y). \end{aligned}$$

Furthermore, for any fixed initial values $X(0)$, $\hat{X}(0) \in G$ and any smooth input u the error $E = \hat{X}X^{-1}$ of the observer on G projects via the map h_{y_0} to the output error of the observer on M for initial values $X(0)$, $\hat{y}(0) = h_{y_0}(\hat{X}(0))$ and the same input u.

Proof: It follows from the definition of the horizontal lift that (4) projects onto (3). For the statement on the observer error note that $E(\hat{y}(t), X(t)) = h_{y_0}(\hat{X}(t)X(t)^{-1})$ by definition of the output error, i.e. $\hat{X}X^{-1}$ projects onto the output error. ■

The observer (4) can be alternatively obtained by the design method introduced in [25] for gradient-like observers with full state observations. Our conditions on the Riemannian metrics on M and G imply that the horizontal lift of the gradient of the cost function is the gradient of a lifted cost function.

Proposition 7: Assume that the conditions on the Riemannian metrics of Section II hold. Let $\tilde{F}\colon G \times G \to \mathbb{R}$ be the lift of the cost function $f\colon M \times M \to \mathbb{R}$, i.e. $\tilde{F}(\hat{X}, X) = f(h(\hat{X}, y_0), h(X, y_0))$. The gradient $\operatorname{grad}_1 \tilde{F}$ of the lifted cost function is the horizontal lift of $\operatorname{grad}_1 f$.

This property of the lift allows the interpretation of the lifted innovation term as a gradient on G.

Corollary 8: Under the conditions of Proposition 7, the observer (4) can be written as

$$\dot{\hat{X}} = \hat{X}u - \operatorname{grad}_1 \tilde{F}(\hat{X}, X). \tag{5}$$

In particular, the observer (5) projects to the observer (3) on the output manifold.

Note that (5) is implementable using only observations of y and u since one has $\tilde{F}(\hat{X}, X) = f(h(\hat{X}, y_0), y)$ and thus the gradient term does only depend on y and $\hat{X}$.

Since the observer and the error $\hat{X}X^{-1}$ project to the observer and the output error on M, it inherits the convergence properties of the observer on M relative to the fibers of the map $X \mapsto h_{y_0}(X)$. Here, convergence to a fiber $h_{y_0}^{-1}(y)$ means that

$$\inf\{\operatorname{dist}(\hat{X}(t)X^{-1}(t), Z) \mid Z \in h_{y_0}^{-1}(y)\}$$

converges to 0 with dist the Riemannian distance on G.

Theorem 9: Assume that the conditions on $f\colon M \times M \to \mathbb{R}$ as in Theorem 4 hold. Then the error $\hat{X}X^{-1}$ on G of the observer (4) converges to $\operatorname{stab}(y_0)$ for generic initial conditions and the convergence is locally exponential with respect to the Riemannian distance on G.

Proof: Note that our conditions on the Riemannian metrics imply that $\operatorname{dist}(h_{y_0}(X), y_0) = \inf\{\operatorname{dist}(X, Z) \mid Z \in \operatorname{stab}(y_0)\}$. Since the error $\hat{E} = \hat{X}X^{-1}$ projects to the output error on M, it converges to the fiber $h_{y_0}^{-1}(y_0) = \operatorname{stab}(y_0)$ if and only if the output error converges to y_0. By definition $h_{y_0}\colon G \to M$ is the projection map of the fiber bundle G over M and hence any open, dense subset of $M \times M$ whose complement has measure[2] 0 lifts by the inverse of $(\hat{X}, X) \mapsto (h_{y_0}(\hat{X}), h_{y_0}(X))$ to an open, dense set in $G \times G$ whose complement has measure 0. Thus the generic convergence on M for the canonical error yields the generic convergence on G for the error $\hat{X}X^{-1}$. ■

[2]with respect to the Riemannian measure

Note, that since the minimal realization of (1) is the projected system on the output space, the subgroup $\operatorname{stab}(y_0)$ is unobservable and the error $\hat{X}X^{-1}$ for any observer of (1) can only converge to the subgroup $\operatorname{stab}(y_0)$. Hence our observer provides the best possible convergence result for generic initial conditions.

V. Example

In this section we will illustrate our observer construction on the attitude estimation problem.

The attitude of a rigid body, with body-fixed-frame $\{B\}$, measured with respect to an inertial frame $\{A\}$, can be identified with an element R of $SO(3)$. The left invariant dynamics

$$\dot{R} = R\omega_\times \tag{6}$$

on $SO(3)$ correspond to the natural body-fixed-frame kinematics of the system. Here $\omega = (\omega_1, \omega_2, \omega_3)^\top \in \{B\}$ is the angular velocity of the rigid-body expressed in the body-fixed-frame and

$$\omega_\times = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

Consider the case where only partial information on the rigid-body attitude is measured. The most common situation is when an inertial direction, such as the magnetic field, or gravitational field, is measured by sensor systems such as magnetometers or accelerometers on board a vehicle. The inertial direction $y_0 \in \{A\}$ of the measured direction is known *a-priori*, for example, the gravitational direction lies in the z-axis of the inertial frame. The measured direction in the body-fixed-frame,

$$y = R^\top y_0 \in \{B\},$$

is obtained by using the attitude matrix to transform the inertial direction y_0 from the inertial frame into the body fixed frame. The corresponding right action of $SO(3)$ on S^2 is

$$h : SO(3) \times S^2 \to S^2, \quad h(R, y) = R^\top y.$$

This yields the system

$$\dot{R} = R\omega_\times \tag{7a}$$

$$y = R^\top y_0 \tag{7b}$$

on $SO(3)$ with outputs on S^2. For the observer constructions proposed in this paper, we have to choose a Riemannian metric on $SO(3)$ and S^2. We equip the unit sphere with the Riemannian metric induced by the Euclidean scalar product in $\mathbb{R}^3$. To choose a Riemannian metric on $SO(3)$ recall that the tangent spaces $T_R SO(3)$ are given by $\{R\Omega \mid \Omega^\top = -\Omega\}$. We use the Riemannian metric $\langle R\Omega, R\Theta\rangle = -\frac{1}{2}\operatorname{tr}(\Omega\Theta)$ on $SO(3)$. The right-invariant horizontal distribution $H(R)$ can be obtained as the orthogonal complement of $\ker T_e h_{y_0}$, i.e. $H(R) = T_e R_R (\ker T_e h_{y_0})^\perp$. Straightforward calculations show that these Riemannian metrics satisfy the conditions from Section II.

By Theorem 1, the system (7) projects to a system on S^2. A straighforward calculation shows that the projected system can be written as

$$\dot{y} = -\omega_\times y, \quad y(0) = h(X(0), y_0) \tag{8}$$

on S^2. First we design an observer for the projected system by the methods introduced in Section III.

For constructing a gradient-like observer on S^2 we have to choose a cost function. The most natural choice is the Euclidean distance, i.e. $f(\hat{y}, y) = (k/2)\|\hat{y} - y\|^2$ with $k > 0$ a positive scaling constant. The constant k is introduced to provide a tunable gain for the observer. The gradient of a function restricted to an embedded submanifold of the Euclidean space with respect to the induced metric is given by the Euclidean orthogonal projection of the Euclidean gradient to the (embedded) tangent space of the submanifold. For the unit sphere this projection is given by $(I - \hat{y}\hat{y}^\top)$. This yields the gradient

$$\operatorname{grad}_1 f(\hat{y}, y) = k(I - \hat{y}\hat{y}^\top)(y - \hat{y}) = -k\left(I - \hat{y}\hat{y}^\top\right) y.$$

Thus we get from (3) the observer

$$\dot{\hat{y}} = -\omega_\times \hat{y} + k(I - \hat{y}\hat{y}^\top) y \tag{9}$$

on S^2. It's easy to see that the cost function f satisfies the conditions of Theorem 4, i.e. that $\hat{y} \mapsto f(\hat{y}, y)$ is a Morse function with only one global minimum (y_0) and no further local minima, and is invariant under the action $(R, (\hat{y}, y)) \mapsto (R^\top \hat{y}, R^\top y)$ of $SO(3)$ on $S^2 \times S^2$. Thus the canonical error on S^2 for this observer converges to y_0 for generic initial conditions.

Some calculations show that

$$k(I - \hat{y}\hat{y}^\top) y = k(\hat{y} \times y) \times \hat{y}.$$

Hence, the observer (9) can be written as

$$\dot{\hat{y}} = -\omega_\times \hat{y} + k(\hat{y} \times y) \times \hat{y}. \tag{10}$$

This observer was originally studied by Metni *et al.* [10], [13]. The derivation given in earlier work is based on a Lyapunov analysis and includes an integral term for compensation of gyro bias that does not fit the analysis framework presented in this paper, however, the main proportional terms in [10], [13] are exactly (10).

To obtain an observer on $SO(3)$ we have to lift (9) to the group. Since the Riemmiann metrics on S^2 and $SO(3)$ satisfy the conditions of Section II, we can apply Corollary 8 and construct the observer from a lift of the cost function to $SO(3)$. The lifted cost function $\tilde{F}$ of f is given by

$$\tilde{F}(\hat{R}, R) = \frac{k}{2}\|\hat{R}^\top y_0 - R^\top y_0\|^2.$$

To calculate the gradient we need the differential d_1 of $\tilde{F}$ with respect to the first variable. Some matrix calculations yield

$$\begin{aligned}\mathrm{d}_1\, \tilde{F}(\hat{R},R)(\hat{R}\Omega) =& k\,\mathrm{tr}\left(\hat{R}\Omega(\hat{R}-R)^\top y_0 y_0^\top\right)\\ &- k\,\mathrm{tr}\left((\hat{R}-R)\Omega\hat{R}^\top y_0 y_0^\top\right)\\ =& k\,\mathrm{tr}\left(\Omega(\hat{y}-y)\hat{y}^\top\right)\\ &- k\,\mathrm{tr}\left((\hat{R}-R)\Omega\hat{y}(\hat{y}-y)^\top\right)\\ =& 2k\,\mathrm{tr}\left(\Omega(\hat{y}y^\top - y\hat{y}^\top)\right).\end{aligned}$$

From the identity

$$\begin{aligned}\langle \mathrm{grad}_1\, \tilde{F}(\hat{R},R), R\Omega\rangle = 2\,\mathrm{tr}\left((\mathrm{grad}_1\, \tilde{F}(\hat{R},R))^\top \hat{R}^\top \Omega\right)\\ = \mathrm{d}_1\, \tilde{F}(R)(R\Omega),\end{aligned}$$

$\Omega^\top = -\Omega$, and the fact that $\hat{R}\,\mathrm{grad}_1\, \tilde{F}(\hat{R},R)$ is a skew symmetric matrix, one obtains the gradient

$$\mathrm{grad}_1\, \tilde{F}(\hat{R},R) = -k\hat{R}(\hat{y}y^\top - y\hat{y}^\top) = -k\hat{R}(\hat{y}\times y)_\times.$$

Thus the gradient-like observer on the Lie group is

$$\frac{d}{dt}\hat{R} = \hat{R}u - k\hat{R}(\hat{y}\times y)_\times.$$

Some further calculations show that the observer has the error dynamics

$$\frac{d}{dt}(\hat{R}R^\top) = k\hat{R}R^\top\,\mathrm{sk}(R\hat{R}^\top y_0 y_0^\top)$$

with $\mathrm{sk}(A) = \frac{1}{2}(A - A^\top)$, which is the negative gradient of $f(E) = \frac{k}{2}\|Ey_0 - y_0\|^2$. The observer is the explicit complementary filter proposed in [14] excluding the integral introduced in that paper to compensate gyro bias. In particular, our observer design approach illustrates the connection between the filters in [10], [13] and the complementary filter in [14].

VI. Conclusions

In this paper we introduced a design method for constructing an observer for a left-invariant system on a Lie group with right invariant outputs on a manifold. We proposed a gradient-like observer for the projected system on the output space which can be lifted to a gradient-like observer on the group. The error for the observer for the projected system converges to the reference output for generic initial states, with a locally exponential convergence rate. This lifts to convergence to the stabilizer of the reference output for the lifted observer on the Lie group, again with a locally exponential convergence rate. We have demonstrated this design method on the attitude estimation problem with partial state measurements for a rigid body. In particular, this gave us new insights into the filters proposed in [10], [13], [14].

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