A NEW PARAMETRIZATION OF OBSERVERS

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Abstract. This paper extends and generalizes recent results on the characterization and parametrization of observers for linear systems in the behavioral framework. We formulate the results in the language of quotient signal modules that was developed by Oberst and first used in the context of observer theory by the first author. The resulting characterization of observers in terms of a generalized internal model principle is both elegant and concise. It includes all such results known to the authors as special cases, including the classical results for linear time-invariant state space systems. Moreover, this new characterization of observers leads to a clean and simple one-to-one parametrization result with only free parameters. This new parametrization allows to decide certain additional observer properties (such as input/output structure or nonintrusiveness) purely by inspection.

1. Problem formulation. We study *plants* (observed systems) of the form

$$\mathscr{P} = \left\{ \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathscr{F}^{\mathtt{w}_1+\mathtt{w}_2+\mathtt{w}_3};\, R_1 \circ w_1 = R_2 \circ w_2 - R_3 \circ w_3 \right\},$$

where w_1 denotes the *measured* variable, w_2 the *to be estimated* variable and w_3 an *irrelevant* variable, respectively. As usual in the behavioral context [2], the w_i should be thought of as vector-valued trajectories of time (vectors with $\mathtt{w}_i$ entries, each of which is a scalar function of time) and the equation given in terms of operators R_i holds in some function space $\mathscr{F}^{\mathtt{k}}$.

More precisely, let F be a field and let $\mathscr{D} := F[s]$ denote the polynomial ring in one indeterminate over F. We interpret $\mathscr{D}$ as a ring of operators that act on (scalar) trajectories of time, turning the set $\mathscr{F}$ of these into a $\mathscr{D}$-module with operation $\circ$. For example, we could choose $F = \mathbb{R}$ and $\mathscr{F} = \mathscr{C}^\infty(\mathbb{R}, F)$ with the operator $s = \frac{\mathrm{d}}{\mathrm{d}t}$, the usual continuous time derivative. The $R_i \in \mathscr{D}^{\mathtt{k}\times\mathtt{w}_i}$ are then just constant coefficient (higher order, matrix) differential operators and the plant $\mathscr{P}$ is a linear time-invariant continuous time system. Another choice would be $F = \mathbb{R}$, $\mathscr{F} = F^{\mathbb{N}}$ and $(s \circ w)(t) = w(t+1)$, i.e., s is the left shift operator. This yields standard linear discrete time systems. Many more choices are possible, all we require is that the *signal module* $\mathscr{F}$ is an injective cogenerator in the category of $\mathscr{D}$-modules, cf. [1].

Following the ideas developed by Valcher and co-authors [3, 4], an *observer* in our context is just another system

$$\mathscr{O} = \left\{ \begin{pmatrix} w_1 \\ \widehat{w}_2 \end{pmatrix} \in \mathscr{F}^{\mathtt{w}_1+\mathtt{w}_2};\, \widehat{R}_1 \circ w_1 = \widehat{R}_2 \circ \widehat{w}_2 \right\},$$

that is interconnected to the plant through the measured variable w_1. We interpret $\widehat{w}_2$ as an estimate for w_2. Here, $\widehat{R}_i \in \mathscr{D}^{\widehat{\mathtt{k}}\times\mathtt{w}_i}$.

The interconnection of the plant $\mathscr{P}$ and the observer $\mathscr{O}$ gives rise to the *error behavior* [3]

$$\mathscr{E}(\mathscr{P},\mathscr{O}) = \left\{ \widehat{w}_2 - w_2 \in \mathscr{F}^{\mathtt{w}_2};\, \begin{matrix} \exists w_1 \in \mathscr{F}^{\mathtt{w}_1}, \\ \exists w_3 \in \mathscr{F}^{\mathtt{w}_3} \end{matrix} : \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathscr{P},\, \begin{pmatrix} w_1 \\ \widehat{w}_2 \end{pmatrix} \in \mathscr{O} \right\},$$

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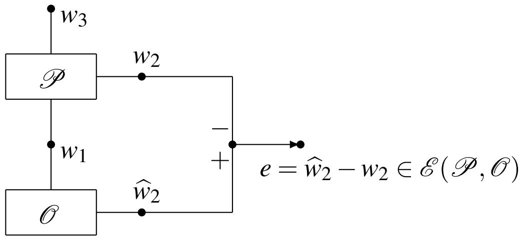

FIG. 1.1. *The observer interconnection.*

cf. Figure 1.1. A typical design goal in observer design is to make this error behavior *stable*, such that the observer error tends to zero for time to infinity. More generally, we consider a multiplicatively closed saturated subset (monoid) T of $\mathscr{D} \setminus \{0\}$ that defines our notion of stability. A scalar trajectory is considered stable (or T-*small*) if it lies in $\mathrm{t}_T(\mathscr{F}) := \{w \in \mathscr{F};\, \exists t \in T : t \circ w = 0\}$, the T-torsion submodule of $\mathscr{F}$. For example, in the continuous time standard case T could be the set of all Hurwitz polynomials. Then the T-torsion submodule of $\mathscr{F} = \mathscr{C}^\infty(\mathbb{R}, F)$ consists precisely of those polynomial-exponential functions (Bohl functions) that go to zero for time to infinity, yielding the usual notion of *asymptotic observers*. Another choice would be $T = F \setminus \{0\}$ whence a "stable" trajectory is equal to zero, yielding *exact observers*. The latter case motivates the alternative name "T-small" trajectories. Following [7, 8] we call an observer $\mathscr{O}$ a T-*observer* for a given plant $\mathscr{P}$ if the resulting error behavior $\mathscr{E}(\mathscr{P}, \mathscr{O})$ is T-small.

Another common design goal in observer design is the requirement for the observer not to "disturb" the plant in its operation. Following [9], we call an observer *nonintrusive* if the behavior of the plant variables (w_1, w_2, w_3) remains unchanged after interconnection with the observer, formally $(\mathscr{P} \wedge_{w_1} \mathscr{O})_{(w_1,w_2,w_3)} = \mathscr{P}$ where $\wedge_{w_1}$ denotes interconnection through w_1, cf. Figure 1.1, and the variable subscript notation indicates projection on the variables (w_1, w_2, w_3), eliminating the variable $\widehat{w}_2$. A special case of a nonintrusive observer is an *i/o-observer*, where the measurement w_1 enters the observer as an input and the estimate $\widehat{w}_2$ is the corresponding output. Classical state space observers fall into this latter category.

We introduce a slight relaxation of the notion of nonintrusiveness and call an observer T-*nonintrusive* if the plant behavior remains *essentially* unchanged after interconnection with the observer, meaning that if there are changes then these changes are T-small. Formally, for any $(w_i) \in \mathscr{P}$ there exists a $(\tilde{w}_i) \in (\mathscr{P} \wedge_{w_1} \mathscr{O})_{(w_1,w_2,w_3)}$ such that $\tilde{w}_i - w_i$ is T-small for $i = 1, 2, 3$. Obviously, a nonintrusive observer is T-nonintrusive for any T.

We are interested in characterizing and parametrizing T-observers for a given plant, and to do this in a way that other desirable observer properties such as nonintrusiveness can be easily decided based on the corresponding parameter values.

2. Main results. Our first main result is the following general *internal model principle* for observers.

THEOREM 2.1. *Given a plant $\mathscr{P}$, an observer $\mathscr{O}$ is a T-nonintrusive T-observer if and only if $\widehat{w}_2$ is T-observable from w_1 in $\mathscr{O}$ and*

$$(\mathscr{P}_T)_{(w_1,w_2)} \subseteq \mathscr{O}_T.$$

Here, $\widehat{w}_2$ is T-observable from w_1 in $\mathscr{O}$ if

$$\begin{pmatrix} w_1 \\ \widehat{w}_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ \widehat{w}_2' \end{pmatrix} \in \mathscr{O} \quad \Longrightarrow \quad \widehat{w}_2' - \widehat{w}_2 \in (\mathrm{t}_T(\mathscr{F}))^{\mathtt{w}_2},$$

i.e., if w_1 determines $\widehat{w}_2$ in $\mathscr{O}$ up to T-small trajectories. In our context, this is equivalent to the implication $\binom{0}{\widehat{w}_2} \in \mathscr{O} \;\Rightarrow\; \widehat{w}_2$ is T-small. In the standard continuous time case this notion corresponds to the usual notion of observability if $T = F \setminus \{0\}$ and to the usual notion of detectability if T is the set of Hurwitz polynomials. The *quotient behaviors* $\mathscr{P}_T$ and $\mathscr{O}_T$ are defined via the quotient signal module $\mathscr{F}_T$ that has a natural $\mathscr{D}_T$-module structure over the *localized ring* $\mathscr{D}_T$. The decomposition $\mathscr{F} \cong \mathscr{F}_T \oplus \mathrm{t}_T(\mathscr{F})$ allows the interpretation of a signal in $\mathscr{F}_T$ as an equivalence class of signals with respect to the equivalence relation defined by $\mathrm{t}_T(\mathscr{F})$, i.e. as a signal in $\mathscr{F}$ that is specified "up to a T-small part". The corresponding decomposition $\mathscr{B} \cong \mathscr{B}_T \oplus \left(\mathscr{B} \cap \mathrm{t}_T(\mathscr{F})^\ell\right)$ for a behavior $\mathscr{B} \subseteq \mathscr{F}^\ell$ allows a similar interpretation for quotient behaviors $\mathscr{B}_T$. For the details see [8]. The name "internal model principle" is justified by the (nontrivial) fact that the inclusion can equivalently be formulated as $(\mathscr{P}_{(w_1,w_2)})^{(T)} \subseteq \mathscr{O}$, recovering and generalizing the results of [9]. Here, $(\mathscr{P}_{(w_1,w_2)})^{(T)}$ is the largest subbehavior of $\mathscr{P}_{(w_1,w_2)}$ that has no nontrivial T-small ("stable") autonomous part. See Corollary 2.3 below for a precise statement and proof of this result.

Proof of Theorem 2.1. We use the proof technique via quotient signal modules as in [8].

Let $\mathscr{O}$ be a T-nonintrusive T-observer for $\mathscr{P}$. Assume $\binom{0}{\widehat{w}_2} \in \mathscr{O}$. Since the zero signal is contained in $\mathscr{P}$, it follows that $\widehat{w}_2 = \widehat{w}_2 - 0$ is contained in $\mathscr{E}(\mathscr{P},\mathscr{O})$ and is hence T-small. We deduce that $\widehat{w}_2$ is T-observable from w_1 in $\mathscr{O}$. Now consider a (quotient) trajectory $\binom{w_1}{w_2} \in (\mathscr{P}_T)_{(w_1,w_2)}$. T-nonintrusiveness of $\mathscr{O}$ implies that there exists $\widehat{w}_2 \in \mathscr{F}_T^{\mathtt{w}_2}$ such that $\binom{w_1}{\widehat{w}_2} \in \mathscr{O}_T$. It follows that $\widehat{w}_2 - w_2$ is contained in $\mathscr{E}(\mathscr{P}_T,\mathscr{O}_T) = (\mathscr{E}(\mathscr{P},\mathscr{O}))_T$ which is zero since all elements of $\mathscr{E}(\mathscr{P},\mathscr{O})$ are assumed to be T-small. We deduce that $\binom{w_1}{w_2} = \binom{w_1}{\widehat{w}_2}$ and hence $\binom{w_1}{w_2} \in \mathscr{O}_T$, proving that $(\mathscr{P}_T)_{(w_1,w_2)} \subseteq \mathscr{O}_T$.

Conversely, let $\widehat{w}_2$ be T-observable from w_1 in $\mathscr{O}$ and let $(\mathscr{P}_T)_{(w_1,w_2)} \subseteq \mathscr{O}_T$. Projection onto w_1 in this inclusion yields $(\mathscr{P}_T)_{w_1} \subseteq (\mathscr{O}_T)_{w_1}$. This is equivalent to $(\mathscr{P}_T \wedge_{w_1} \mathscr{O}_T)_{(w_1,w_2,w_3)} = \mathscr{P}_T$, and hence $\mathscr{O}$ is T-nonintrusive. Finally we show the T-observer property by showing that all signals in $\mathscr{E}(\mathscr{P},\mathscr{O})$ are T-small or, equivalently, that $\mathscr{E}(\mathscr{P}_T,\mathscr{O}_T) = (\mathscr{E}(\mathscr{P},\mathscr{O}))_T = 0$. To this end consider arbitrary signals $\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathscr{P}_T$ and $\binom{w_1}{\widehat{w}_2} \in \mathscr{O}_T$. Then $\binom{w_1}{w_2} \in (\mathscr{P}_T)_{(w_1,w_2)} \subseteq \mathscr{O}_T$. Now T-observability of $\widehat{w}_2$ from w_1 in $\mathscr{O}$ is equivalent to observability of $\widehat{w}_2$ from w_1 in $\mathscr{O}_T$ and hence $\binom{w_1}{w_2}, \binom{w_1}{\widehat{w}_2} \in \mathscr{O}_T$ implies that $\widehat{w}_2 - w_2 = 0$ as required. □

The previous theorem immediately implies the following existence result for T-nonintrusive T-observers.

COROLLARY 2.2. *The plant $\mathscr{P}$ admits a T-nonintrusive T-observer if and only if w_2 is T-observable from w_1 in $\mathscr{P}$. In this case there always exists a nonintrusive T-observer.*

Proof. Let w_2 be T-observable from w_1 in $\mathscr{P}$. The choice $\mathscr{O} := \mathscr{P}_{(w_1,w_2)}$ satisfies the requirements and is even nonintrusive.

Conversely, let $\mathscr{O}$ be a T-nonintrusive T-observer. Assume $\binom{0}{w_2} \in \mathscr{P}_{(w_1,w_2)}$. Since the zero signal is contained in $\mathscr{O}$, it follows that $w_2 = w_2 - 0$ is contained in $\mathscr{E}(\mathscr{P}_{(w_1,w_2)},\mathscr{O}) = \mathscr{E}(\mathscr{P},\mathscr{O})$ and is hence T-small. We deduce that w_2 is T-observable from w_1 in $\mathscr{P}_{(w_1,w_2)}$ and hence in $\mathscr{P}$. □

The above results are completely independent of the irrelevant variables w_3, as would be expected. They only depend on the projection of the plant behavior on the variables (w_1,w_2), i.e. on the behavior obtained after eliminating the variable w_3. In the following, we assume that no irrelevant variables are present, or that they have already been eliminated, in order to simplify notation. We consider a given plant

$$\mathscr{P} = \left\{ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathscr{F}^{\mathtt{w}_1+\mathtt{w}_2}; R_1 \circ w_1 = R_2 \circ w_2 \right\}$$

where $R_i \in \mathscr{D}^{\mathtt{k}\times\mathtt{w}_i}$, $\mathtt{w} := \mathtt{w}_1 + \mathtt{w}_2$, and $R := (R_1, -R_2) \in \mathscr{D}^{\mathtt{k}\times\mathtt{w}}$ is of full row rank $\mathtt{k}$.

We define the behavior $\mathscr{P}^{(T)}$ as the largest subbehavior of $\mathscr{P}$ that does not itself contain a nontrivial T-small autonomous subbehavior. More precisely:

$$\mathscr{P}^{(T)} = \mathscr{P}_{\text{cont}} \oplus \mathscr{P}_{\text{antistab}}$$

where $\mathscr{P} = \mathscr{P}_{\text{cont}} \oplus \mathscr{P}_{\text{aut}}$ is a decomposition of $\mathscr{P}$ in its controllable part and an autonomous complement and $\mathscr{P}_{\text{aut}} = \mathscr{P}_{\text{stab}} \oplus \mathscr{P}_{\text{antistab}}$, $\mathscr{P}_{\text{stab}} = \mathscr{P}_{\text{aut}} \cap (\mathrm{t}_T(\mathscr{F}))^{\mathtt{w}}$, meaning that $\mathscr{P}_{\text{stab}}$ consists of those signals in $\mathscr{P}_{\text{aut}}$ that are T-small. It can be shown that $\mathscr{P}^{(T)}$ is the smallest behavior such that its quotient $(\mathscr{P}^{(T)})_T$ is equal to the quotient behavior $\mathscr{P}_T$ of $\mathscr{P}$, and hence in particular $\mathscr{P}^{(T)}$ does not depend on the (non-unique) choice of the autonomous part $\mathscr{P}_{\text{aut}}$ of $\mathscr{P}$. For any kernel representation $\mathscr{P}^{(T)} = \{\binom{w_1}{w_2} \in \mathscr{F}^{\mathtt{w}_1+\mathtt{w}_2}; R^{(T)} \circ \binom{w_1}{w_2} = 0\}$ where $R^{(T)} \in \mathscr{D}^{\mathtt{k}'\times\mathtt{w}}$, the injective cogenerator property yields that the module of equations $\mathscr{D}^{1\times\mathtt{k}'}R^{(T)}$ is the largest submodule U of $\mathscr{D}^{1\times\mathtt{w}}$ with $U_T = (\mathscr{D}^{1\times\mathtt{k}}R)_T = \mathscr{D}_T^{1\times\mathtt{k}}R$, i.e.,

$$\mathscr{D}^{1\times\mathtt{k}'}R^{(T)} = \mathscr{D}_T^{1\times\mathtt{k}}R \cap \mathscr{D}^{1\times\mathtt{w}}.$$

It follows that $\operatorname{rank}(R^{(T)}) = \operatorname{rank}(R) = \mathtt{k}$, and hence we may w.l.o.g. choose $R^{(T)} \in \mathscr{D}^{\mathtt{k}\times\mathtt{w}}$, i.e., $\mathtt{k}' = \mathtt{k}$.

COROLLARY 2.3. *Given a plant $\mathscr{P} \subseteq \mathscr{F}^{\mathtt{w}_1+\mathtt{w}_2}$, an observer $\mathscr{O}$ is a T-nonintrusive T-observer if and only if $\widehat{w}_2$ is T-observable from w_1 in $\mathscr{O}$ and*

$$\mathscr{P}^{(T)} \subseteq \mathscr{O}.$$

Proof. It is intuitively clear that the inclusions $\mathscr{P}_T \subseteq \mathscr{O}_T$ and $\mathscr{P}^{(T)} \subseteq \mathscr{O}$ are equivalent since both signify that $\mathscr{P}$ is contained in $\mathscr{O}$ "up to a T-small part". More precisely, by the injective cogenerator property of $\mathscr{F}_T$ over $\mathscr{D}_T$, an observer $\mathscr{O} = \{\binom{w_1}{\widehat{w}_2} \in \mathscr{F}^{\mathtt{w}_1+\mathtt{w}_2}; \widehat{R} \circ \binom{w_1}{\widehat{w}_2} = 0\}$, $\widehat{R} \in \mathscr{D}^{\widehat{\mathtt{k}}\times\mathtt{w}}$, satisfies $\mathscr{P}_T \subseteq \mathscr{O}_T$ if and only if $\mathscr{D}_T^{1\times\widehat{\mathtt{k}}}\widehat{R} \subseteq \mathscr{D}_T^{1\times\mathtt{k}}R$, i.e. if and only if $\widehat{R} \in \mathscr{D}_T^{\widehat{\mathtt{k}}\times\mathtt{k}}R \cap \mathscr{D}^{\widehat{\mathtt{k}}\times\mathtt{w}} = \mathscr{D}^{\widehat{\mathtt{k}}\times\mathtt{k}}R^{(T)}$. The injective cogenerator property of $\mathscr{F}$ over $\mathscr{D}$ implies that this is the case if and only if $\mathscr{P}^{(T)} \subseteq \mathscr{O}$. □

The significance of the previous characterization result for observers owes to the fact that it is easy to compute a kernel representation of the behavior $\mathscr{P}^{(T)}$ from the kernel representation of $\mathscr{P}$ given by the R_i by means of Algorithm 2.6 stated below. The inclusion in the above theorem then corresponds to a factorization equation between this and the observer representation, yielding a novel parametrization of all T-nonintrusive T-observers.

Our second main result is the following constructive parametrization of all T-nonintrusive T-observers for a given plant. We assume again that the irrelevant variables w_3 have already been eliminated in order to simplify notation.

THEOREM 2.4. *Assume a plant $\mathscr{P} \subseteq \mathscr{F}^{\mathtt{w}_1+\mathtt{w}_2}$ where w_2 is T-observable from w_1. Then any T-nonintrusive T-observer $\mathscr{O}$ for $\mathscr{P}$ can be constructed by the following steps:*

- *Choose $\widehat{\mathtt{k}} \in \mathbb{N}$ with $\mathtt{w}_2 \le \widehat{\mathtt{k}} \le \mathtt{k}$.*
- *Choose $D \in \mathscr{D}^{\mathtt{w}_2\times\mathtt{w}_2} \cap \mathrm{Gl}_{\mathtt{w}_2}(\mathscr{D}_T)$ in Hermite form, i.e., choose monic diagonal elements $D_{jj} \in T$ for $j = 1,\dots,\mathtt{w}_2$ and then choose $D_{ij} \in \mathscr{D}$ such that* $\deg(D_{ij}) < \deg(D_{jj})$ *for $j = 1,\dots,\mathtt{w}_2$, $i = 1,\dots,j-1$, $D_{ij} := 0$ for $i > j$.*

Choose a full row rank matrix $G_2 \in \mathscr{D}^{(\widehat{\mathtt{k}}-\mathtt{w}_2)\times(\mathtt{k}-\mathtt{w}_2)}$ *in Hermite form, and choose* $G_1 \in \mathscr{D}^{\mathtt{w}_2\times(\mathtt{k}-\mathtt{w}_2)}$ *such that* $X := \begin{pmatrix} D & G_1 \\ 0 & G_2 \end{pmatrix} \in \mathscr{D}^{(\mathtt{w}_2+(\widehat{\mathtt{k}}-\mathtt{w}_2))\times(\mathtt{w}_2+(\mathtt{k}-\mathtt{w}_2))} = \mathscr{D}^{\widehat{\mathtt{k}}\times\mathtt{k}}$ *is in Hermite form.*

- *Compute* $\widehat{R} := X\widetilde{U}R^{(\mathrm{T})}$ *and define* $\mathscr{O} := \left\{w \in \mathscr{F}^{\mathtt{w}};\, \widehat{R}\circ w = 0\right\}$ *where*

$$\begin{pmatrix} \widetilde{E} \\ 0 \end{pmatrix} = \widetilde{U}R_2^{(\mathrm{T})}\widetilde{V}$$

denotes the Smith form of $R_2^{(\mathrm{T})}$ *over* $\mathscr{D}$ *where* $R^{(T)} = (R_1^{(T)}, -R_2^{(T)}) \in \mathscr{D}^{\mathtt{k}\times(\mathtt{w}_1+\mathtt{w}_2)}$.

Every possible choice for the parameters $\widehat{\mathtt{k}}$ *and* $X = \begin{pmatrix} D & G_1 \\ 0 & G_2 \end{pmatrix}$ *yields a different observer.*

Proof. According to Corollary 2.3, $\mathscr{O} = \{w \in \mathscr{F}^{\mathtt{w}};\, \widehat{R}\circ w = 0\}$ is a T-nonintrusive T-observer for $\mathscr{P}$ if and only if $\widehat{w}_2$ is T-observable from w_1 in $\mathscr{O}$ and $\mathscr{P}^{(T)} \subseteq \mathscr{O}$. Here we use the shortcut notation $\widehat{R} = (\widehat{R}_1, -\widehat{R}_2) \in \mathscr{D}^{\widehat{\mathtt{k}}\times\mathtt{w}}$ with $\mathtt{w} = \mathtt{w}_1 + \mathtt{w}_2$, and we assume w.l.o.g. that $\mathrm{rank}(\widehat{R}) = \widehat{\mathtt{k}}$. But $\mathscr{P}^{(T)} \subseteq \mathscr{O}$ is equivalent to $\widehat{R} \in \mathscr{D}^{\widehat{\mathtt{k}}\times\mathtt{k}}R^{(T)}$ since $\mathscr{F}$ is an injective cogenerator over $\mathscr{D}$. This signifies that there exists $\widetilde{X} \in \mathscr{D}^{\widehat{\mathtt{k}}\times\mathtt{k}}$ such that $\widehat{R} = \widetilde{X}R^{(\mathrm{T})}$. Since $R^{(\mathrm{T})}$ has full row rank $\mathtt{k}$ by construction, the requirement $\mathrm{rank}(\widehat{R}) = \widehat{\mathtt{k}}$ is equivalent to $\mathrm{rank}(\widetilde{X}) = \widehat{\mathtt{k}}$, and implies in particular that $\widehat{\mathtt{k}} \leq \mathtt{k}$.

$\widehat{w}_2$ is T-observable from w_1 in $\mathscr{O}$ if and only if the matrix $\widehat{R}_2 = \widetilde{X}R_2^{(\mathrm{T})} \in \mathscr{D}^{\widehat{\mathtt{k}}\times\mathtt{w}_2}$ is left invertible over $\mathscr{D}_T$ (and hence in particular $\mathtt{w}_2 \leq \widehat{\mathtt{k}}$). Since $\widetilde{V}$ is invertible over $\mathscr{D}$, this is the case if and only if $\widehat{R}_2\widetilde{V} = \widetilde{X}\widetilde{U}^{-1}\widetilde{U}R_2^{(\mathrm{T})}\widetilde{V} = \widetilde{X}\widetilde{U}^{-1}\begin{pmatrix} \widetilde{E} \\ 0 \end{pmatrix}$ is invertible over $\mathscr{D}_T$. With $X := (X_I, X_{II}) := \widetilde{X}\widetilde{U}^{-1} \in \mathscr{D}^{\widehat{\mathtt{k}}\times(\mathtt{w}_2+(\mathtt{k}-\mathtt{w}_2))}$ this signifies that $X_I\widetilde{E}$ is left invertible over $\mathscr{D}_T$, i.e. that X_I is so, since $\widetilde{E} \in \mathrm{Gl}_{\mathtt{w}_2}(\mathscr{D}_T)$ as a consequence of the assumed T-observability of w_2 from w_1 in $\mathscr{P}$.

Since the behavior $\mathscr{O}$ is determined by the row space $\mathscr{D}^{1\times\widehat{\mathtt{k}}}\widehat{R}$ and not by the matrix $\widehat{R}$ itself, and since different matrices X that are row equivalent over $\mathscr{D}$ yield matrices $\widehat{R}$ that are row equivalent over $\mathscr{D}$, it is sufficient to consider only those full row rank matrices X that are in Hermite form. Then left invertibility of X_I over $\mathscr{D}_T$ signifies that X is of the form asserted in the theorem.

Now assume that two choices $(\widehat{\mathtt{k}}, X)$ and $(\widehat{\mathtt{k}}', X')$ give rise to the same observer $\mathscr{O}$, i.e. that $\mathscr{D}^{1\times\widehat{\mathtt{k}}}X\widetilde{U}R^{(\mathrm{T})} = \mathscr{D}^{1\times\widehat{\mathtt{k}}'}X'\widetilde{U}R^{(\mathrm{T})}$. Since $\widetilde{U}R^{(\mathrm{T})}$ is of full row rank $\mathtt{k}$, it follows that $\mathscr{D}^{1\times\widehat{\mathtt{k}}}X = \mathscr{D}^{1\times\widehat{\mathtt{k}}'}X'$. Both X and X' being full row rank matrices in Hermite form, we deduce that $\widehat{\mathtt{k}} = \widehat{\mathtt{k}}'$ and $X = X'$. □

The above parametrization is significantly different from the parametrizations previously reported in the literature [3, 4, 5, 6, 7, 8, 9] in that it is one-to-one and all parameters are free, provided that we can parametrize the set T. The requirements for certain submatrices to be in Hermite form merely constrain the degrees of certain polynomial entries and hence the *number* of free parameters in the parametrization.

Those T-observers that are in addition i/o-observers appear naturally in the above parametrization; they correspond precisely to the (minimal) choice $\widehat{\mathtt{k}} = \mathtt{w}_2$. The above theorem hence also provides a full one-to-one parametrization of i/o-observers with only free parameters.

COROLLARY 2.5. *A T-observer $\mathscr{O}$ for $\mathscr{P}$ constructed according to Theorem 2.4 is an input/output behavior with input w_1 and output $\widehat{w}_2$ if and only if $\widehat{\mathtt{k}} = \mathtt{w}_2$.*

Proof. The T-observer $\mathscr{O}$ is an input/output behavior with input w_1 and output $\widehat{w}_2$ if and only if $\mathrm{rank}(\widehat{R}_1, -\widehat{R}_2) = \mathrm{rank}(\widehat{R}_2) = \mathtt{w}_2$, cf. [2, Section 3.3] or [1, Theorem 2.69 on page 27]. The equality $\mathrm{rank}(\widehat{R}_2) = \mathtt{w}_2$ is satisfied for all $\widehat{R}$ constructed in Theorem 2.4. Moreover, $\operatorname{rank}(\widehat{R}) = \operatorname{rank}(X) = \widehat{\mathtt{k}}$ by construction. It follows that $\mathscr{O}$ has the required input/output structure if and only if $\widehat{\mathtt{k}} = \mathtt{w}_2$. □

Another important property of the parametrization in Theorem 2.4 is that it allows to characterize and construct all T-observers that are nonintrusive (rather than only T-nonintrusive). We only need to restrict the choice of G_2 in the above theorem. Specifically, we need to choose a full row rank matrix $\widetilde{G}_2 \in \mathscr{D}^{(\widehat{\mathtt{k}}-\mathtt{w}_2)\times(\mathtt{k}-\mathtt{w}_2)}$ in Hermite form, and define G_2 as the Hermite form of $\widetilde{G}_2 A$, where A can be computed from $\widetilde{U}$, R and $R^{(T)}$. The details of these computations will be published in a forthcoming paper.

We conclude this section with the following algorithm for the computation of $R^{(T)}$. Together with any of the well known algorithms for the computation of Smith forms, this algorithm allows to turn the parametrization provided in Theorem 2.4 into a constructive procedure.

ALGORITHM 2.6 (Construction of $R^{(T)}$). *For given* $R \in \mathscr{D}^{\mathtt{k}\times\mathtt{w}}$, $\operatorname{rank}(R) = \mathtt{k}$, *we construct* $R^{(\mathrm{T})} \in \mathscr{D}^{\mathtt{k}\times\mathtt{w}}$, $\operatorname{rank}(R^{(\mathrm{T})}) = \mathtt{k}$, *such that* $\mathscr{D}_T^{1\times\mathtt{k}} R \cap \mathscr{D}^{1\times\mathtt{w}} = \mathscr{D}^{1\times\mathtt{k}} R^{(\mathrm{T})}$.

Let

$$URV = (E, 0), \quad E = \begin{pmatrix} e_1 & & 0 \\ & \ddots & \\ 0 & & e_{\mathtt{k}} \end{pmatrix} \in \mathscr{D}^{\mathtt{k}\times\mathtt{k}}, \quad e_1 | \ldots | e_{\mathtt{k}} \in \mathscr{D},$$

be the Smith form of R w.r.t. $\mathscr{D}$. For each elementary divisor e_i we consider the prime factor decomposition

$$e_i = u_i \prod_{p\in\mathscr{P}} p^{\mu(p)} = u_i \underbrace{\prod_{p\in\mathscr{P}_1} p^{\mu(p)}}_{=:t_i} \underbrace{\prod_{p\in\mathscr{P}_2} p^{\mu(p)}}_{=:f_i} = u_i t_i f_i, \quad 0 \neq u_i \in F,\ \mu(p) \geq 0,$$

where $\mathscr{P}$ denotes the set of all monic irreducible polynomials, $\mathscr{P}_1 := \mathscr{P} \cap T$ are the primes in T, and $\mathscr{P}_2 := \mathscr{P} \setminus \mathscr{P}_1$. Define

$$F := \begin{pmatrix} f_1 & & 0 \\ & \ddots & \\ 0 & & f_{\mathtt{k}} \end{pmatrix} \quad \text{and} \quad R^{(\mathrm{T})} := U^{-1}(F, 0)V^{-1} \in \mathscr{D}^{\mathtt{k}\times\mathtt{w}}.$$

Then $R^{(\mathrm{T})}$ has the required properties.

3. Conclusions. We have provided a novel characterization of observers for linear systems in the behavioral framework. This characterization takes the form of a generalized internal model principle, stating in essence that T-nonintrusive T-observers are exactly those observer systems that include the "undesirable" plant dynamics. Building on this characterization, we were able to derive a constructive, one-to-one parametrization of all T-nonintrusive T-observers for a given plant where this parametrization has only free parameters and nicely embeds a parametrization of all i/o-observers. The results reported in this paper generalize and extend all similar results known to the authors, including the classical results for linear time-invariant state space systems and the authors' own recent work in the area.

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