2. Assignment
ENGN4226/6226 System Theory, Semester 1, 2007
due: 21/03, 10am (end of lecture)

1. (25%) Let \( s > 0 \) and consider the function \( \phi_s : \mathbb{R} \to \mathbb{R}, \)
\[
t \mapsto \phi_s(t) = \begin{cases} 
0, & t \leq 0 \\
e^{-\frac{t}{s-t}}, & 0 < t < s \\
0, & t \geq s
\end{cases}
\]
Draw a graph of \( \phi_s \) and show that it is infinitely often differentiable. This implies that the function \( \psi_s : \mathbb{R} \to \mathbb{R}, \)
\[
t \mapsto \psi_s(t) = \frac{\int_{-\infty}^{t} \phi_s(\tau) \, d\tau}{\int_{-\infty}^{\infty} \phi_s(\tau) \, d\tau}
\]
is also infinitely often differentiable. Draw a graph of \( \psi_s \). What is its decisive feature?

2. (25%) Consider two infinitely often differentiable functions \( f, g : \mathbb{R} \to \mathbb{R} \). Construct an infinitely often differentiable function \( h : \mathbb{R} \to \mathbb{R} \) that “switches” from \( f \) to \( g \) in the time interval \([0, s], s > 0\), i.e.
\[
h(t) = \begin{cases} 
f(t), & t \leq 0 \\
g(t), & t \geq s
\end{cases}
\]
**Hint:** Try this first with \( g(\cdot) \equiv 0 \) and use the function \( \psi_s \).

3. (25%) Consider the forced undamped harmonic oscillator
\[
\frac{d^2}{dt^2} x(t) = -kx(t) + F(t)
\]
and show that the Smith form of its standard kernel representation
\[
R(\frac{d}{dt}) \begin{pmatrix} x(t) \\ F(t) \end{pmatrix} = 0
\]
is \( R(z)V(z) = (1 \ 0) \).

4. (25%) How do the functions
\[
\begin{pmatrix} \ddot{x}(t) \\ \dot{F}(t) \end{pmatrix} = V(\frac{d}{dt})^{-1} \begin{pmatrix} x(t) \\ F(t) \end{pmatrix}
\]
look like, where \( (x(t) F(t))^\top \) is a legal trajectory of the forced undamped harmonic oscillator? Describe why it is possible to switch between two such legal trajectories in an arbitrary time interval \([0, s], s > 0\).
**Hint:** Do the switch for \( (\ddot{x}(t) \dot{F}(t))^\top \) and apply \( V(\frac{d}{dt}) \).