Passivity-Preserving Model Reduction by Analytic Interpolation

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Passive systems

\[
\begin{align*}
    \dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

\[
\begin{align*}
    \lambda(A) &\in \mathbb{C}_- \\
    D + D^T &> 0 \\
    (A, B) &\text{ reachable} \\
    (C, A) &\text{ observable}
\end{align*}
\]

\[
\int_0^T u(t)^T y(t) dt \geq 0 \quad \text{for all } T \geq 0 \quad \text{passive}
\]

\[
G(s) = C(sI - A)^{-1}B + D
\]

\[
G(i\omega) + G(-i\omega)^T \geq 0, \quad \omega \in \mathbb{R}
\]

\[
G(s) \sim \begin{bmatrix}
    A & B \\
    C & D
\end{bmatrix}
\]

positive real
Passivity-preserving model reduction

\((A, B, C) \rightarrow (\hat{A}, \hat{B}, \hat{C})\)
\[\dim = n \quad \dim = r < n\]
\[\hat{G}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + D \sim \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & D \end{bmatrix} \text{ positive real}\]

Projection method: Find \(U, V \in \mathbb{R}^{n \times r}\) such that \(U^TV = I_r\) and

\[\hat{A} = U^TAV, \quad \hat{B} = U^TB, \quad \hat{C} = CV\]
Ex: Stochastically balanced model reduction

(ARE) \[ AP + PA^T - (B + PC^T)(D + D^T)^{-1}(B + PC^T)^T = 0 \]

Solution set: \[ P_- \leq P \leq P_+ \]

Stochastic balancing: \[ TP_- T^T = \Sigma = T^{-T} P_+^{-1} T^{-1} \]

\( \Sigma \) = diagonal matrix consisting of the singular values of \( P_- P_+^{-1} \)

Truncation: \[ U^T = \begin{bmatrix} I_k & 0 \end{bmatrix} T \quad V = T^{-1} \begin{bmatrix} I_k \\ 0 \end{bmatrix} \]

\[ \hat{A} = U^T A V, \quad \hat{B} = U^T B, \quad \hat{C} = C V \]
Antoulas’ observation

Consider the class of approximants \( \hat{G}(s) \) that retains \( r \) stable spectral zeros of the original function; i.e., \( r \) stable zeros of

\[
G(s) + G(-s)^T
\]

Then, if \( s_1, s_2, \ldots, s_r \in \mathbb{C}_+ \) are the mirror images (in the imaginary axis) of these spectral zeros, the interpolant

\[
\hat{G}(s_j) = G(s_j), \quad j = 1, 2, \ldots, r
\]

is positive real. In other words, the passivity property is preserved in such a model reduction procedure.
**Spectral zeros**

Recall notation:

\[ G(s) \sim \begin{bmatrix} A & B \\ C & D \end{bmatrix} \]

\[
G(s) + G(-s)^T \sim \begin{bmatrix}
A & B \\
-A^T & -C^T \\
C & B^T \\
C & B^T & D + D^T
\end{bmatrix}
\]

The **spectral zeros** are the \( \lambda \) for which the matrix \( \mathcal{A} - \lambda \mathcal{E} \) is singular, where

\[
\mathcal{A} := \begin{bmatrix} A & B \\ -A^T & -C^T \\ C & B^T & D + D^T \end{bmatrix}, \quad \mathcal{E} = \begin{bmatrix} I_n & I_n \\ & 0_m \end{bmatrix}
\]
Sorensen’s algorithm

Partial real Schur decomposition:

\[
\begin{bmatrix}
A & B \\
-C^T & A^T \\
C & B^T & D + D^T
\end{bmatrix}
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}
= 
\begin{bmatrix}
X \\
Y \\
0
\end{bmatrix}
R
\begin{bmatrix}
X^T \\
Y \\
Z
\end{bmatrix}
= I_r
\]

Singular value decomposition:  
\[
Q_x \Sigma^2 Q_y^T = X^T Y
\]

\[
\hat{A} = U^T A V, \quad \hat{B} = U^T B, \quad \hat{C} = CV
\]

\[
\hat{G}(s) = \hat{C}(sI - \hat{A})^{-1} \hat{B} + D \quad \text{positive real}
\]

\[
V := X Q_x \Sigma^{-1}
\]

\[
U := Y Q_y \Sigma^{-1}
\]
Interpolation in the matrix case

Sorensen’s solution satisfies

$$\hat{G}(s_j)z_j = G(s_j)z_j, \quad j = 1, \ldots, r$$

where $z_j := Zr_j \neq 0$ with $r_j$ is the right eigenvector of $R$ corresponding to $s_j$

Moreover, $\hat{G}$ satisfies

$$z_j^T \hat{G}(-s_j) = z_j^T G(-s_j)$$

for each $j$ such that $(-s_j I_k - \hat{A})$ is invertible.
Analytic interpolation with degree constraint

Given:  \( \{(s_j, w_j) : s_j \in \mathbb{C}_+ \}_{j=0}^r \)

- \( s_j \neq s_k \) if \( j \neq k \), \( s_0 \) real
- \( w_j = \bar{w}_k \) if \( s_j = \bar{s}_k \)

Find: Positive real function

\[
\left[ \frac{w_j + \bar{w}_k}{s_j + \bar{s}_k} \right]_{j,k=0}^r > 0
\]

such that

(i) \( f(s_k) = w_k \), \( k = 0, 1, \ldots, r \)

(ii) \( f \) rational of degree at most \( r \)

- \( f \) analytic for \( \text{Re}\{z\} > 0 \)
- \( \text{Re}\{f(z)\} > 0 \) for \( \text{Re}\{z\} > 0 \)
Complete parameterization

**THEOREM.** To each real monic Hurwitz polynomial $\rho$ of degree $n$ there is a unique pair $(\alpha, \beta)$ of real monic Hurwitz polynomials of degree $n$ such that

(i) $f := \beta/\alpha$ is positive real,

(ii) $f(s_j) = w_j$, $j = 0, 1, \ldots, n$, and

(iii) $\alpha(s)\beta(-s) + \alpha(-s)\beta(s) = \rho(s)\rho(-s)$.

Conversely, any pair $(\alpha, \beta)$ of real polynomials of degree $n$ satisfying (i) and (ii) determines, via (iii), a unique (modulo sign) Hurwitz polynomial $\rho$ of degree $n$. The map $\rho \mapsto (\alpha, \beta)$ is a diffeomorphism.

(i) & (iii) $\quad \iff \quad f(s) + f(-s) = \frac{\rho(s)\rho(-s)}{\alpha(s)\alpha(-s)}$  

roots of $\rho =$ spectral zeros
Non-linear coordinates

The manifold of all \((\alpha, \beta)\) such that \(f = \beta/\alpha\) is positive real has two foliations:

A foliation with one leaf for each choice of spectral zeros (Kalman filtering)

Another foliation with one leaf for each choice of \(w_0, w_1, \ldots, w_n\)

**THEOREM.** The two foliations intersect transversely so that each leaf in one meets each leaf in the other in exactly one point.
Optimization approach

Given \((\rho, w)\), maximize

\[
\int_{-\infty}^{\infty} \left| \frac{\rho(i\omega)}{\tau(i\omega)} \right|^2 \log[f(i\omega) + f(-i\omega)] \frac{d\omega}{\omega^2 + s_0^2}
\]

over all positive real \(f\) subject to

\[f(s_j) = w_j, \quad j = 0, 1, \ldots, n\]

This optimization problem has a unique solution, which has the form

\[f = \frac{\beta}{\alpha} \quad \text{where} \quad \alpha(s)\beta(-s) + \alpha(-s)\beta(s) = \rho(s)\rho(-s)\]

determined via the dual problem
Dual problem

Given $\rho$ and any $w \in H^\infty$ such that $w(s_k) = w_k$, $k = 0, 1, \ldots, r$, minimize

$$\int_{-\infty}^{\infty} \left( [w(i\omega) + w(-i\omega)]Q(i\omega) - \left| \frac{\rho(i\omega)}{\tau(i\omega)} \right|^2 \log Q(i\omega) \right) \frac{d\omega}{\omega^2 + s_0^2}$$

over all $Q \in \mathcal{Q}$, where

$$\mathcal{Q} = \left\{ Q(i\omega) = \operatorname{Re} \sum_{k=0}^{r} \frac{q_k}{i\omega + s_k} \mid Q(i\omega) \geq 0 \right\}.$$

Convex optimization problem with a unique solution

optimal $\alpha$ and

$$\alpha(s)\beta(-s) + \alpha(-s)\beta(s) = \rho(s)\rho(-s) \quad \Rightarrow \quad f = \frac{\beta}{\alpha}$$
Maximum entropy solution

Choose the spectral zeros in the mirror image of the interpolation points; i.e.,

$$\rho(s) = \tau(s)^*.$$ 

Then the primal problem amounts to maximize

$$\int_{-\infty}^{\infty} \log[f(i\omega) + f(-i\omega)] \frac{d\omega}{\omega^2 + s_0^2}$$

over all positive real $f$ satisfying the interpolation constraints.

linear problem  Cf. Mustafa-Glover
Back to the dual problem

\[ J_P(Q) := \int_{-\infty}^{\infty} [\Phi Q - P \log Q] \frac{d\omega}{\omega^2 + s_0^2} \rightarrow \min \]

\[ \Phi(s) := \frac{1}{2} [G(s) + G(-s)^T], \quad P(i\omega) := \left| \frac{\rho(i\omega)}{\tau(i\omega)} \right|^2 \]

- Maximum entropy solution for \( \rho(s) = \tau(s)^* \)

- The Antoulas-Sorensen solution also requires choosing interpolation points in zeros of \( \Phi \) plus \( s = \infty \)

The Antoulas-Sorensen method as the maximum entropy solution

\[ \hat{G}(\infty) = w_0 := D \]
\[ \hat{G}(s_k) = w_k := G(s_k), \quad k = 1, 2, \ldots, r \]

where \( s_1, s_2, \ldots, s_r \) chosen in the mirror image of a self-conjugate set of spectral zeros.

**THEOREM.** For \( s_0 > 0 \) sufficiently large, let \( f_{s_0} \) be the maximum entropy solution corresponding to the interpolation conditions

\[ f_{s_0}(s_0) = w_0, \quad f_{s_0}(s_k) = w_k, \quad k = 1, \ldots, r. \]

Then, as \( s_0 \to \infty \), \( f_{s_0} \to \hat{G} \) pointwise (except in the poles of \( \hat{G} \)).
\[
\hat{G}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + D
\]

\[
\hat{C} = (\tilde{P}^{-1}w)^\top \\
\tilde{P} = \left[\frac{w_j + \bar{w}_\ell}{s_j + \bar{s}_\ell}\right]_{j,\ell=1}^r \\
w = \begin{bmatrix}
  w_1 - w_0 \\
  w_2 - w_0 \\
  \vdots \\
  w_r - w_0
\end{bmatrix}
\]

\[
\hat{A} = -\Lambda + h\hat{C} \\
\Lambda := \text{diag}(s_1, s_2, \ldots, s_r)
\]

\[
\hat{B} = 2w_0(Q\hat{C}^* + h)
\]

\[
\hat{A}Q + Q\hat{A}^* + hh^* = 0
\]
Global-analysis approach

Since we have a smooth parameterization, the reduced-order solution obtained by the (numerically efficient) Sorensen algorithm (or some other method) can be tuned to specifications by moving the

- spectral zeros
- interpolation points

while passivity and degree are preserved.
A benchmark problem

Original system:

\[ G(s) = \frac{s^5 + 3s^4 + 6s^3 + 9s^2 + 7s + 3}{s^5 + 7s^4 + 14s^3 + 21s^2 + 23s + 7} \]

Antoulas-Sorensen:

\[ \hat{G}(s) = \frac{s^3 + 2.553s^2 + 2.906s + 1.173}{s^3 + 6.681s^2 + 8.459s + 3.07} \]

Global-analysis approach:

\[ f(s) = \frac{1.002s^3 + 2.84s^2 + 1.927s + 0.8978}{s^3 + 7.298s^2 + 6.084s + 2.099} \]
A large-scale problem: A CD player

Model reduction: \( \deg G = 120 \rightarrow \deg \hat{G} = 12 \)

Antoulas-Sorensen solutions:
Global-analysis solution

Antoulas-Sorensen solutions
Discrete-time NP interpolation with degree constraint

Given:
\[ z_0, z_1, \ldots, z_n \text{ such that } |z_k| < 1 \text{ (distinct)} \]
\[ w_0, w_1, \ldots, w_n \text{ such that } \text{Re}\{w_k\} > 0 \]

Find: Carathéodory function \( f \) such that Pick matrix

\[
P_n = \left[ \frac{w_k + \bar{w}_\ell}{1 - z_k \bar{z}_\ell} \right]_{k,\ell=0}^n > 0
\]

such that
- \( f \) analytic for \( |z| \leq 1 \)
- \( f \in \mathcal{C}_+ \)
- \( \text{Re}\{f(z)\} > 0 \) for \( |z| \leq 1 \)

such that

(i) \( f(z_k) = w_k, \quad k = 0, 1, \ldots, n \)

(ii) \( f \) rational of degree at most \( n \)

For simplicity normalize:
\[ (z_0, w_0) = (0, 1) \]
\[ f = \frac{\beta}{\alpha}, \quad \alpha, \beta \in \mathcal{S}_n \quad \text{Schur polynomials of degree } n \]

\[ \mathcal{P}_n = \left\{ (\alpha, \beta) \in \mathcal{S}_n \times \mathcal{S}_n \mid f = \frac{\beta}{\alpha} \in \mathbb{C}_+ \right\} \]

\[ f \in \mathbb{C}_+ \quad \Rightarrow \quad \Re f = \frac{1}{2} \left( \frac{\beta}{\alpha} + \frac{\beta^*}{\alpha^*} \right) = \frac{1}{2} \frac{\alpha^* \beta + \alpha \beta^*}{\alpha \alpha^*} > 0 \]

\[ \frac{1}{2} (\alpha^* \beta + \alpha \beta^*) = \rho \rho^*, \quad \rho \in \mathcal{S}_n \quad \text{roots are spectral zeros} \]

\[ \Re(\alpha^* \beta) = |\rho|^2 \]
Complete parameterization

$$\mathcal{P}_n = \left\{ (\alpha, \beta) \in \mathcal{S}_n \times \mathcal{S}_n \mid f = \frac{\beta}{\alpha} \in \mathbb{C}_+ \right\}$$

**THEOREM.** To each monic $\rho \in \mathcal{S}_n$ there is a unique pair $(\alpha, \beta) \in \mathcal{P}_n$ such that $f = \beta/\alpha$ satisfies

(i) $f(z_k) = w_k, \quad k = 0, 1, \ldots, n$

(ii) $\text{Re}(\alpha^*/\beta) = |\rho|^2$

The correspondence $\rho \leftrightarrow \alpha$ is a diffeomorphism, which can be extended to the boundary as a homeomorphism.
Optimization approach

Given \((\rho, w)\), maximize

\[
\int_{-\pi}^{\pi} \left| \frac{\rho(e^{i\theta})}{\tau(e^{i\theta})} \right|^2 \log \left[ \text{Re} f(e^{i\theta}) \right] \, d\theta
\]

over all \(f \in \mathcal{C}_+\) subject to

\[
f(z_k) = w_k, \quad k = 0, 1, \ldots, n
\]

This optimization problem has a unique solution, which has the form

\[
f = \frac{\beta}{\alpha}, \quad \alpha, \beta \in \mathcal{S}_n \quad \text{where} \quad \text{Re}(\alpha^* \beta) = |\rho|^2
\]

determined via the dual problem:

\[
\tau(z) = \prod_{k=0}^{n} (1 - \bar{z}_k z)
\]
Given $(\rho, w)$, minimize the strictly convex functional

\[ J_\rho(q) = \text{Re} \sum_{k=0}^{n} w_k q_k - \int_{-\pi}^{\pi} \left| \frac{\rho(e^{i\theta})}{\tau(e^{i\theta})} \right|^2 \log Q(e^{i\theta}) \frac{d\theta}{2\pi} \]

over the convex set of all $(q_0, q_1, \ldots, q_n)$ such that

\[ Q(e^{i\theta}) := \text{Re} \left( \sum_{k=0}^{n} \frac{q_k}{1 - \bar{z}_k e^{i\theta}} \right) > 0, \quad \text{for all } \theta \in [-\pi, \pi] \]

THEOREM. There is a unique minimum.

Then \[ f = \frac{\beta}{\alpha}, \quad \alpha, \beta \in S_n \quad \text{where} \quad \left| \frac{\alpha(e^{i\theta})}{\tau(e^{i\theta})} \right|^2 = \hat{Q}(e^{i\theta}) \]
and \[ \text{Re}(\alpha^* \beta) = |\rho|^2 \]
Primal problem reformulated

\[ \mathbb{I}_P(\hat{\Phi}) := \int_{-\pi}^{\pi} P \log \frac{\hat{\Phi}}{2\pi} \, d\theta \longrightarrow \max \]

\[ \hat{\Phi}(z) := \frac{1}{2} \left[ \hat{G}(z) + \hat{G}(z^{-1})^T \right], \quad P(e^{i\theta}) := \left| \frac{\rho(e^{i\theta})}{\tau(e^{i\theta})} \right|^2 \]

Optimal solution: \[ \hat{\Phi}(z) = \frac{P(z)}{Q(z)} \quad \text{where } Q \text{ solution of the dual problem} \]

Maximum entropy solution for \[ P \equiv 1 \]
Dual problem reformulated

\[ \mathcal{J}_P(Q) := \int_{-\pi}^{\pi} \left[ \Phi Q - P \log Q \right] \frac{d\theta}{2\pi} \rightarrow \min \]

\[ \Phi(z) := \frac{1}{2} \left[ G(z) + G(z^{-1})^T \right], \quad P(e^{i\theta}) := \left| \frac{\rho(e^{i\theta})}{\tau(e^{i\theta})} \right|^2 \]

\[ Q(e^{i\theta}) = \text{Re} \sum_{k=0}^{r} q_k g_k(e^{i\theta}) \geq 0 \quad \text{where} \quad g_k(z) := \frac{1}{1 - \bar{z}_k z} \]

For \( P \equiv 1 \), the optimal solution: \( \Phi^{-1} \)

where \( \Phi \) optimal solution of primal problem
Kullback-Leibler divergence

\[ D(y\|\hat{y}) := \limsup_{N \to \infty} \frac{1}{N} \int_{\mathbb{R}^n} p^N_y(x) \log \frac{p^N_y(x)}{p^N_{\hat{y}}(x)} \, dx \]

If the stationary processes \( y \) and \( \hat{y} \) have spectral densities \( \Phi \) and \( \hat{\Phi} \), respectively, then

\[ D(y\|z) = \frac{1}{2} \int_{-\pi}^{\pi} \left[ (\Phi - \hat{\Phi})\hat{\Phi}^{-1} - \log(\Phi\hat{\Phi}^{-1}) \right] \frac{d\theta}{2\pi} \]

Prediction-error approximation

Anderson, Moore and Hawkes
Stoorvogel, van Schuppen
Consequently,

\[ D(y\|z) = \frac{1}{2} \mathbb{J}(\hat{\Phi}^{-1}) - \frac{1}{2} \left[ 1 + \int_{-\pi}^{\pi} \log \Phi \frac{d\theta}{2\pi} \right] \]

constant

The maximum entropy solution \( \hat{\Phi} \) is the minimum prediction-error approximant in the model class

\[ \hat{\Phi}^{-1}(z) = \text{Re} \sum_{k=0}^{r} q_k g_k(e^{i\theta}) \geq 0 \quad g_k(z) := \frac{1}{1 - \bar{z}_k z} \]

The Antoulas-Sorensen approach corresponds to the choice of basis functions in which \( z_1, z_2, \ldots, z_r \) are spectral zeros.
Conclusions

• The Antoulas-Sorensen solution is essentially
  - the maximum entropy solution
  - the minimum prediction-error solution in a
    model class with spectral zeros at spectral zeros of
    the function to be approximated

• It can in general be improved by smooth tuning of
  the spectral zeros and the interpolation points