Cut-elimination for a Logic with Generic Judgments

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Abstract

We present a logic in which signatures are explicit in the sequent, and of two different nature: one signature is associated to the whole sequent to account for eigenvariables of the sequent, the other is associated to each formula in the sequent, used to account for generic variables locally scoped over the formula. The logic is a version of intuitionistic logic extended with a new quantifier, ∇, to explicitly manipulate the local signature, and with a proof theoretic notion of definitions. We prove cut elimination for the logic and study the properties of the new quantifier in relation to other connectives.

1 Two-level sequents

In [3, 5], logic is used to reason about specifications of computational systems, that is, by encoding the specifications of the computational systems as Horn theories. One of the main issues is the treatment of object-level abstraction in the logic. In traditional sequent systems, there is essentially only one way to give a direct interpretation to the object-level abstractions, i.e., via the use of universal quantifier. At the proof-level, however, this means that different kinds of name-binding constructs in the object-systems are identified, i.e., as proof-level abstractions (eigenvariables). As a consequence of the properties of the logic, notions like *scoping* and *genericity* cannot be encoded faithfully. In the previous works [3, 5], these issues were dealt with by using term-level abstractions (λ-abstraction) in the logic, which results in a quite complicated syntax.

In this project, we are experimenting with a new form of sequent which would allow one to express directly a notion of locality of names, at the proof-level. We allow two level of signatures: one is global to the whole sequent, the other is local to a given formula. The local signature is intended to be a kind of variable binder within the formula, and hence α-equivalence is assumed. We introduce a new syntactical object built upon formulas:

$$\sigma \triangleright B.$$

Here σ denotes a list of variables, with types, and B is a formula. We call this syntactical object an *object-level judgment*, or judgment for short. The usual notion of α-conversion and capture-avoiding substitution apply to judgments. We denote formulas with letters A, B, C, D and judgments with $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$. The letter A and $\mathcal{A}$ are reserved for atomic formula and atomic judgment, respectively. We will often write B to denote the formula part of a judgment $\mathcal{B}$ if it is clear from context which local signature we are referring to. Sequents are expressions of the form

$$\Sigma : \Gamma \longrightarrow \mathcal{C}$$

where Σ is the sequent-level signature and Γ is a multiset of jugdments. Note that Σ is intended to be the set of eigenvariables. Constants can be made explicit in the sequent as well if needed, but we choose not to do so, just for the sake of simplicity of presentation. We assume a finite set of constants $\mathcal{K}$ is given when constructing derivations. We always assume that formulas are well-typed, given the type context Σ, the set of constants $\mathcal{K}$ and their respective formula-level signatures. Given α-conversion on judgments, we also assume that variables in formula-level signatures are different from the sequent-level signature and constants.

2 The logic $FO\lambda^{\Delta\nabla}$

The logic is a modification of $FO\lambda^{\Delta\mathbb{N}}$ [4]. We are not considering the induction rules, since it is not relevant to the issue of name binding which is the focus of this paper. The basic logic is an intuitionistic version of a subset of Church's Simple Theory of Types [2] in which meta-level formulas will be given the type o. The logical connectives are $\bot$, $\top$, $\wedge$, $\vee$, $\supset$, $\forall_\tau$, and $\exists_\tau$. The judgments $\sigma \triangleright \bot$ and $\sigma \triangleright \top$ will be abreviated as $\bot$ and $\top$, respectively. The quantification types τ (and thus the types of variables) are restricted to not contain o. The dynamics of formula-level signatures will be introduced by a new quantifier, ∇. In a sense, ∇ is actually a universal quantifier at the "object-logic". It is not subjected to the meta-level polarities, and therefore its left and right introduction rules are the same.

The rules for the core of logic $FO\lambda^{\Delta\nabla}$ is presented in Figure 2. The interaction between the global and local signatures takes place in the introduction rules for existential and universal quantifiers. In the rule for $\forall\mathcal{L}$ (and, dually, for $\exists\mathcal{R}$), the quantifier appears in the scope of the global signature Σ and the local signature σ. This quantifier can be instantiated (reading the rule bottom-up) with a term built from variables in both of these signatures. Similarly, in the rule for $\forall\mathcal{R}$ (and, dually, for $\exists\mathcal{L}$), the quantifier appears in the scope of the global signature Σ and the local signature σ. This quantifier can be instantiated (reading the rule bottom-up) with an eigenvariable whose intended range is over all terms built from variables in the local signature σ. Since, however, the eigenvariable (in this rule h) is stored in the global scope, its dependency on σ would be forgotten unless we employ some particular encoding technique. For this purpose, we use *raising* [6]: to denote a variable of type γ_0 that can range over some set of constants and over the variables in $\sigma = (x_1 : \gamma_1, \ldots, x_n : \gamma_n)$ $(n \geq 0)$, we can use instead the term $(hx_1 \ldots x_n)$ where the variable h ranges over the set of constants only: dependency on σ can be forgotten. Of course, the type of h will be $\gamma_1 \to \cdots \to \gamma_n \to \gamma_0$ instead of simply γ_0. In the inference rules of Figure 2, we write $(h\sigma)$ to denote $(hx_1 \ldots x_n)$.

The multicut (mc) rule is a generalization of cut due to Slaney [9], and is used to simplify the presentation of the cut-elimination proof.

2.1 Definitions and their introduction rules

We allow an atomic formula to be defined by other formulas. In effect, an atomic formula can be introduced in a derivation from its defining formulas.

Definition 1 A *definitional clause* is written $\forall\bar{x}[p\,\bar{t} \triangleq B]$, where p is a predicate constant, every free variable of the formula B is also free in at least one term in the list $\bar{t}$ of terms, and all variables free in $\bar{t}$ are contained in the list $\bar{x}$ of variables. The atomic formula $p\,\bar{t}$ is called the *head* of the clause, and the formula B is called the *body*. The symbol $\triangleq$ is used simply to indicate a definitional clause: it is not a logical connective. A *definition* is a (perhaps infinite) set of definitional clauses. The same predicate may occur in the head of multiple clauses of a definition: it is best to think of a definition as a mutually recursive definition of the predicates in the heads of the clauses.

$$\frac{}{\Sigma : \mathcal{A}, \Gamma \longrightarrow \mathcal{A}}\ init \qquad \frac{}{\Sigma : \bot, \Gamma \longrightarrow \mathcal{B}}\ \bot\mathcal{L} \qquad \frac{}{\Sigma : \Gamma \longrightarrow \top}\ \top\mathcal{R}$$

$$\frac{\Sigma : \sigma \triangleright B, \Gamma \longrightarrow \mathcal{D}}{\Sigma : \sigma \triangleright B \wedge C, \Gamma \longrightarrow \mathcal{D}}\ \wedge\mathcal{L} \qquad \frac{\Sigma : \sigma \triangleright C, \Gamma \longrightarrow \mathcal{D}}{\Sigma : \sigma \triangleright B \wedge C, \Gamma \longrightarrow \mathcal{D}}\ \wedge\mathcal{L}$$

$$\frac{\Sigma : \Gamma \longrightarrow \sigma \triangleright B \quad \Sigma : \Gamma \longrightarrow \sigma \triangleright C}{\Sigma : \Gamma \longrightarrow \sigma \triangleright B \wedge C}\ \wedge\mathcal{R}$$

$$\frac{\Sigma : \sigma \triangleright B, \Gamma \longrightarrow \mathcal{D} \quad \Sigma : \sigma \triangleright C, \Gamma \longrightarrow \mathcal{D}}{\Sigma : \sigma \triangleright B \vee C, \Gamma \longrightarrow \mathcal{D}}\ \vee\mathcal{L}$$

$$\frac{\Sigma : \Gamma \longrightarrow \sigma \triangleright B}{\Sigma : \Gamma \longrightarrow \sigma \triangleright B \vee C}\ \vee\mathcal{R} \qquad \frac{\Sigma : \Gamma \longrightarrow \sigma \triangleright C}{\Sigma : \Gamma \longrightarrow \sigma \triangleright B \vee C}\ \vee\mathcal{R}$$

$$\frac{\Sigma : \Gamma \longrightarrow \sigma \triangleright B \quad \Sigma : \sigma \triangleright C, \Gamma \longrightarrow \mathcal{D}}{\Sigma : \sigma \triangleright B \supset C, \Gamma \longrightarrow \mathcal{D}}\ \supset\mathcal{L} \qquad \frac{\Sigma : \sigma \triangleright B, \Gamma \longrightarrow \sigma \triangleright C}{\Sigma : \Gamma \longrightarrow \sigma \triangleright B \supset C}\ \supset\mathcal{R}$$

$$\frac{\Sigma, \sigma \vdash t : \tau \quad \Sigma : \sigma \triangleright B[t/x], \Gamma \longrightarrow \mathcal{C}}{\Sigma : \sigma \triangleright \forall_\tau x.B, \Gamma \longrightarrow \mathcal{C}}\ \forall\mathcal{L} \qquad \frac{\Sigma, h : \Gamma \longrightarrow \sigma \triangleright B[(h\ \sigma)/x]}{\Sigma : \Gamma \longrightarrow \sigma \triangleright \forall x.B}\ \forall\mathcal{R}$$

$$\frac{\Sigma, h : \sigma \triangleright B[(h\ \sigma)/x], \Gamma \longrightarrow \mathcal{C}}{\Sigma : \sigma \triangleright \exists x.B, \Gamma \longrightarrow \mathcal{C}}\ \exists\mathcal{L} \qquad \frac{\Sigma, \sigma \vdash t : \tau \quad \Sigma : \Gamma \longrightarrow \sigma \triangleright B[t/x]}{\Sigma : \Gamma \longrightarrow \sigma \triangleright \exists_\tau x.B}\ \exists\mathcal{R}$$

$$\frac{\Sigma : (\sigma, y) \triangleright B[y/x], \Gamma \longrightarrow \mathcal{C}}{\Sigma : \sigma \triangleright \nabla x.B, \Gamma \longrightarrow \mathcal{C}}\ \nabla\mathcal{L} \qquad \frac{\Sigma : \Gamma \longrightarrow (\sigma, y) \triangleright B[y/x]}{\Sigma : \Gamma \longrightarrow \sigma \triangleright \nabla x.B}\ \nabla\mathcal{R}$$

$$\frac{\Sigma : \mathcal{B}, \mathcal{B}, \Gamma \longrightarrow \mathcal{C}}{\Sigma : \mathcal{B}, \Gamma \longrightarrow \mathcal{C}}\ c\mathcal{L} \qquad \frac{\Sigma : \Gamma \longrightarrow \mathcal{C}}{\Sigma : \mathcal{B}, \Gamma \longrightarrow \mathcal{C}}\ \mathrm{w}\mathcal{L}$$

$$\frac{\Sigma : \Delta_1 \longrightarrow \mathcal{B}_1 \quad \cdots \quad \Sigma : \Delta_n \longrightarrow \mathcal{B}_n \quad \Sigma : \mathcal{B}_1, \ldots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}}{\Sigma : \Delta_1, \ldots, \Delta_n, \Gamma \longrightarrow \mathcal{C}}\ mc, \text{ where } n \geq 0$$

Figure 1: The core rules of $FO\lambda^{\Delta\nabla}$

We need to stratify the definition clauses, for the cut-elimination to hold [8]. For this purpose, we associate with each predicate p a natural number $\mathrm{lvl}(p)$, the level of p. The notion of level is generalized to formulas as follows.

Definition 2 Given a formula B, its *level* $\mathrm{lvl}(B)$ is defined as follows:

1. $\mathrm{lvl}(p\,\bar{t}) = \mathrm{lvl}(p)$
2. $\mathrm{lvl}(\bot) = \mathrm{lvl}(\top) = 0$
3. $\mathrm{lvl}(B \wedge C) = \mathrm{lvl}(B \vee C) = \max(\mathrm{lvl}(B), \mathrm{lvl}(C))$
4. $\mathrm{lvl}(B \supset C) = \max(\mathrm{lvl}(B) + 1, \mathrm{lvl}(C))$
5. $\mathrm{lvl}(\forall x.B) = \mathrm{lvl}(\nabla x.B) = \mathrm{lvl}(\exists x.B) = \mathrm{lvl}(B)$.

The level of a judgment is the level of its formula. We shall now require that for every definitional clause $\forall \bar{x}[p\,\bar{t} \triangleq B]$, $\mathrm{lvl}(B) \leq \mathrm{lvl}(p)$.

Definition rules make use of substitutions. We recall some basic definitions related to substitutions here. A substitution θ is a mapping from variables to terms, such that the *domain* of θ, $\mathrm{dom}(\theta) = \{x \mid \theta(x) \neq x\}$ is finite. The range of θ, denoted by $\mathrm{ran}(\theta)$, is defined as

$$\mathrm{ran}(\theta) = \{y \mid y \in \mathrm{fv}(\theta(x)), \text{ for all } x \in \mathrm{dom}(\theta)\},$$

where $\mathrm{fv}(t)$ is the set of free variables in t. Composition of substitutions is defined as $(\theta \circ \sigma)(x) = \sigma(\theta(x))$, for all variable x. Two substitutions θ and σ are considered equal if $\sigma(x) =_\eta \theta(x)$, for all variables x. The application of a substitution θ to a generic judgment $x_1, \ldots, x_n \triangleright B$, written as $(x_1, \ldots, x_n \triangleright B)\theta$, is $x_1, \ldots, x_n \triangleright B'$, if $(\lambda x_1 \ldots \lambda x_n.B)\theta$ is equal (modulo λ-conversion) to $\lambda x_1 \ldots \lambda x_n.B'$. If Γ is a multiset of generic judgments, then $\Gamma\theta$ is the multiset $\{J\theta \mid J \in \Gamma\}$. Finally, if Σ is a signature then $\Sigma\theta$ is the signature that results from removing from Σ the variables in the domain of θ and adding the variables that are free in the range of θ.

The following relation will be useful for describing the definition rules.

Definition 3 Let $\forall x_1, \ldots, x_n.[H' \triangleq B']$ be a definition clause and σ be a list of typed variables. The *raised definition clause* of $\forall \bar{x}.[H' \triangleq B']$ over σ is the definition clause $\forall y_1, \ldots, y_n.[H \triangleq B]$ where

$$B = B'[(y_1\ \sigma)/x_1, \ldots, (y_n\ \sigma)/x_n],$$
$$H = H'[(y_1\ \sigma)/x_1, \ldots, (y_n\ \sigma)/x_n].$$

The five-place relation $\mathrm{dfn}(\Sigma, \rho, \sigma \triangleright A, \theta, B)$ is defined to hold for the judgment $\sigma \triangleright A$, the formula B and the substitutions ρ and θ if

$$(\sigma \triangleright A)\rho =_{\beta\eta} (\sigma \triangleright H)\theta.$$

The variables $\bar{x}$ and $\bar{y}$ are chosen to be distinct from the variables in Σ and σ.

Remark: A more restricted form of the above definition would be to replace the substitutions ρ and θ with a unifier of the higher-order unification problem

$$\lambda\sigma.A = \lambda\sigma.H.$$

In fact, we can restrict it further by considering only the complete set of unifiers. We prefer the more general form since it simplifies the proof of cut-elimination. We will show later that they are all equivalent.

The right and left rules for atoms are

$$\frac{\Sigma : \Gamma \longrightarrow (\sigma \triangleright B)\theta}{\Sigma : \Gamma \longrightarrow \sigma \triangleright A}\ \mathit{def}\mathcal{R}, \text{ where } \mathrm{dfn}(\Sigma, \epsilon, \sigma \triangleright A, \theta, B)$$

$$\frac{\{\Sigma\rho : (\sigma \rhd B)\theta, \Gamma\rho \longrightarrow \mathcal{C}\rho \mid \mathrm{dfn}(\Sigma, \rho, \sigma \rhd A, \theta, B)\}}{\Sigma : \sigma \rhd A, \Gamma \longrightarrow \mathcal{C}}\ def\mathcal{L},$$

where ϵ is the empty substitution and $\Sigma\rho = (\Sigma \setminus \mathrm{dom}(\rho)) \cup \mathrm{ran}(\rho)$. Specifying a set of sequents as the premise should be understood to mean that each sequent in the set is a premise of the rule. Notice that in the $def\mathcal{L}$ rule, the free variables of the conclusion can be instantiated in the premises. In particular, a variable in Σ could possibly be replaced by several new variables.

As in $FO\lambda^{\Delta\mathbb{N}}$, the rules of $FO\lambda^{\Delta\nabla}$ may have variables that are free in the premise but not in the conclusion: this results from the eigenvariable y of $\forall\mathcal{R}$ and $\exists\mathcal{L}$, the term t of $\forall\mathcal{L}$ and $\exists\mathcal{R}$, and the substitutions ρ and θ of $def\mathcal{L}$. We view the choice of such variables as arbitrary and identify all derivations that differ only in the choice of variables that are not free in end-sequent.

We define an ordinal measure which corresponds to the height of a derivation. This measure will be used as the induction measure for many proofs of lemmas and theorems to follow.

Definition 4 Given a derivation Π with premise derivations $\{\Pi_i\}_i$, the measure $\mathrm{ht}(\Pi)$ is the least upper bound of $\{\mathrm{ht}(\Pi_i) + 1\}_i$.

Lemma 5 *Let* Π *be a derivation of* $\Sigma : \Gamma \longrightarrow \mathcal{C}$. *Then there is a derivation* Π' *of* $\Sigma, x : \Gamma \longrightarrow \mathcal{C}$, *where* $x \notin \Sigma$, *such that* $\mathrm{ht}(\Pi') \leq \mathrm{ht}(\Pi)$.

Proof By simple induction on $\mathrm{ht}(\Pi)$. For the case involving definition rules, it is enough to check that $\mathrm{dfn}(\{\Sigma, x\}, \rho, \mathcal{A}, \theta, B)$ implies $\mathrm{dfn}(\Sigma, \rho, \mathcal{A}, \theta, B)$. ∎

Lemma 6 *Let* Π *be a derivation of* $\Sigma : \Gamma \longrightarrow \mathcal{C}$ *and* θ *be a substitution. Then there is a derivation* $\Pi\theta$ *of* $\Sigma\theta : \Gamma\theta \longrightarrow \mathcal{C}\theta$ *such that* $\mathrm{ht}(\Pi\theta) \leq \mathrm{ht}(\Pi)$.

Proof By induction on $\mathrm{ht}(\Pi)$.

1. Suppose Π ends with a rule other than $def\mathcal{L}$, $def\mathcal{R}$, and rules that introduce quantifiers.

$$\frac{\begin{array}{c}\Pi_1\\ \Sigma : \Gamma_1 \longrightarrow \mathcal{C}_1\end{array} \quad \cdots \quad \begin{array}{c}\Pi_n\\ \Sigma : \Gamma_n \longrightarrow \mathcal{C}_n\end{array}}{\Sigma : \Gamma \longrightarrow \mathcal{C}}\ \bullet.$$

We take $\Pi\theta$ as the following derivation:

$$\frac{\begin{array}{c}\Pi_1\theta\\ \Sigma\theta : \Gamma_1\theta \longrightarrow \mathcal{C}_1\theta\end{array} \quad \cdots \quad \begin{array}{c}\Pi_n\theta\\ \Sigma\theta : \Gamma_n\theta \longrightarrow \mathcal{C}_n\theta\end{array}}{\Sigma\theta : \Gamma\theta \longrightarrow \mathcal{C}\theta}\ \bullet,$$

where $\Pi_i\theta$ is obtained by applying inductive hypotheses to Π_i.

2. Suppose Π ends with the $def\mathcal{L}$ rule

$$\frac{\left\{\begin{array}{c}\Pi^{(\rho,B)}\\ \Sigma\rho : (\sigma \rhd B)\theta, \Gamma'\rho \longrightarrow \mathcal{C}\rho\end{array}\right\}_{\mathrm{dfn}(\Sigma,\rho,\sigma\rhd A,\gamma,B)}}{\Sigma : \sigma \rhd A, \Gamma' \longrightarrow \mathcal{C}}\ def\mathcal{L}.$$

Suppose $\mathrm{dfn}(\Sigma\theta, \rho', (\sigma \rhd A)\theta, \gamma', B)$ holds. We have

$$((\sigma \rhd A)\theta)\rho' = (\sigma \rhd H)\gamma',$$

given a raised definition clause $\forall \bar{y}.[H \stackrel{\triangle}{=} B]$, where $\bar{y}$ are chosen to be different from Σ'. Then, obviously, $\mathrm{dfn}(\Sigma, \theta \circ \rho', \sigma \rhd A, \gamma', B)$ holds as well. Therefore we define $\Pi\theta$ as the derivation

$$\frac{\left\{\begin{array}{c}\Pi^{(\theta\circ\rho',B)}\\ \Sigma(\theta \circ \rho') : (\sigma \rhd B)\gamma', \Gamma'\theta\rho' \longrightarrow \mathcal{C}\theta\rho'\end{array}\right\}_{\mathrm{dfn}(\Sigma\theta,\rho',(\sigma\rhd A)\theta,\gamma',B)}}{\Sigma\theta : (\sigma \rhd A)\theta, \Gamma'\theta \longrightarrow \mathcal{C}\theta}\ def\mathcal{L}.$$

3. Suppose Π ends with the $def\mathcal{R}$ rule

$$\dfrac{\begin{array}{c}\Pi' \\ \Sigma : \Gamma \longrightarrow (\sigma \triangleright B)\rho\end{array}}{\Sigma : \Gamma \longrightarrow \sigma \triangleright A}\ def\mathcal{R}\ ,$$

where $\mathrm{dfn}(\Sigma, \epsilon, \sigma \triangleright A, \rho, B)$, given a raised definition clause $\forall \bar{y}.[H \triangleq B]$. By Definition 3, this means

$$\sigma \triangleright A = (\sigma \triangleright H)\rho.$$

Obviously, $(\sigma \triangleright A)\theta = (\sigma \triangleright H)\rho\theta$ and therefore $\mathrm{dfn}(\Sigma\theta, \epsilon, (\sigma \triangleright A)\theta, \rho \circ \theta, B)$ holds. We can then define $\Pi\theta$ as the derivation

$$\dfrac{\begin{array}{c}\Pi'\theta \\ \Sigma\theta : \Gamma\theta \longrightarrow (\sigma \triangleright B)\rho\theta\end{array}}{\Sigma\theta : \Gamma\theta \longrightarrow (\sigma \triangleright A)\theta}\ def\mathcal{R}\ ,$$

where $\Pi'\theta$ is obtained from Π' by inductive hypothesis.

4. Suppose Π ends with the $\forall\mathcal{L}$ rule

$$\dfrac{\Sigma, \sigma \vdash t : \tau \quad \begin{array}{c}\Pi' \\ \Sigma : \sigma \triangleright B\ t, \Gamma' \longrightarrow \mathcal{C}\end{array}}{\Sigma : \sigma \triangleright \forall_\tau x.Bx, \Gamma' \longrightarrow \mathcal{C}}\ \forall\mathcal{L}\ .$$

Then we define $\Pi\theta$ as

$$\dfrac{\Sigma\theta, \sigma \vdash t\theta : \tau \quad \begin{array}{c}\Pi'\theta \\ \Sigma\theta : \sigma \triangleright (B\ t)\theta, \Gamma'\theta \longrightarrow \mathcal{C}\theta\end{array}}{\Sigma\theta : \sigma \triangleright \forall_\tau x.(Bx)\theta, \Gamma'\theta \longrightarrow \mathcal{C}\theta}\ \forall\mathcal{L}\ ,$$

where $\Pi'\theta$ is obtained from Π' by inductive hypothesis. The case with $\exists\mathcal{R}$ is similar.

5. Suppose Π ends with the $\forall\mathcal{R}$ rule

$$\dfrac{\begin{array}{c}\Pi' \\ \Sigma, h : \Gamma \longrightarrow \sigma \triangleright B(h\ \sigma)\end{array}}{\Sigma : \Gamma \longrightarrow \sigma \triangleright \forall x.Bx}\ \forall\mathcal{R}\ .$$

We can assume without loss of generality that h and x are not free in θ. Then we define $\Pi\theta$ as

$$\dfrac{\begin{array}{c}\Pi'\theta \\ \Sigma\theta, h : \Gamma\theta \longrightarrow \sigma \triangleright (B\theta)(h\ \sigma)\end{array}}{\Sigma\theta : \Gamma\theta \longrightarrow \sigma \triangleright \forall x.(B\theta)x}\ \forall\mathcal{R}\ .$$

The case with $\exists\mathcal{L}$ is treated analogously.

6. Suppose Π ends with $\nabla\mathcal{R}$.

$$\dfrac{\begin{array}{c}\Pi' \\ \Sigma : \Gamma \longrightarrow (\sigma, y) \triangleright By\end{array}}{\Sigma : \Gamma \longrightarrow \sigma \triangleright \nabla x.Bx}\ \nabla\mathcal{R}$$

Then define $\Pi\theta$ as

$$\dfrac{\begin{array}{c}\Pi'\theta \\ \Sigma\theta : \Gamma\theta \longrightarrow (\sigma, y) \triangleright (B\theta)y\end{array}}{\Sigma\theta : \Gamma\theta \longrightarrow \sigma \triangleright \nabla x.(B\theta)x}\ \nabla\mathcal{R}$$

The case with $\nabla\mathcal{L}$ is similar.

Since each transformation step from Π to $\Pi\theta$ does not introduce extra applications of rules, $\mathrm{ht}(\Pi\theta) \leq \mathrm{ht}(\Pi)$. It can be smaller than $\mathrm{ht}(\Pi)$ because in the case of $def\mathcal{L}$, there could be fewer premises. ∎

3 Properties of the ∇ Quantifier

We use $\equiv$ to denote logical equivalence, i.e., $P \equiv Q$ means $P \supset Q$ and $Q \supset P$.

Proposition 7 *The following implications hold in $FO\lambda^{\Delta\nabla}$.*

1. $\nabla x.(Px \wedge Qx) \equiv (\nabla x.Px \wedge \nabla x.Qx)$.
2. $(\nabla x.Px \vee \nabla x.Qx) \equiv \nabla x.(Px \vee Qx)$.
3. $\nabla x.(Px \supset Qx) \equiv (\nabla x.Px \supset \nabla x.Qx)$.
4. $\nabla x.(Px \supset \bot) \equiv ((\nabla x.Px) \supset \bot)$.
5. $\nabla x \forall y.Pxy \equiv \forall h \nabla x.Px(hx)$.
6. $\nabla x \exists y.Pxy \equiv \exists h \nabla x.Px(hx)$.
7. $\nabla x \forall y.Pxy \supset \forall y \nabla x.Pxy$.

As a consequence, the ∇ quantifiers in a formula can be pushed to the atomic level, with the cost of raising the quantified variables in its scope.

A natural question to ask about ∇, in relation to its role as local binder, is whether the relative orders among consecutive ∇'s matters, or more precisely, whether the following is provable in $FO\lambda^{\Delta\nabla}$.

$$\nabla y \nabla x Bxy \supset \nabla x \nabla y Bxy$$

Of course, the statement is not true in the logic without definitions. With definitions, however, it holds with under certain restrictions. We consider a more general property below, under which the exchange of consecutive ∇'s is a simple corollary.

Definition 8 A definition $\mathbf{D}$ is *noetherian* if for every definition clause $\forall \bar{x}.[p\bar{t} \stackrel{\triangle}{=} B]$ in $\mathbf{D}$, it holds that $\mathrm{lvl}(p) > \mathrm{lvl}(B)$.

Proposition 9 *Given a noetherian definition, the sequent $\Sigma : \Gamma, \sigma \triangleright B \longrightarrow \sigma' \triangleright B$, where σ' is a permutation of σ, is provable in $FO\lambda^{\Delta\nabla}$.*

Proof We construct a derivation of $\Gamma, \sigma \triangleright B \longrightarrow \sigma' \triangleright B$ inductively. The induction is on the level of B with subordinate induction on the size of B. We can assume without loss of generality that all predicates in the definition are assigned levels greater than 0 and that all predicates in the sequent are defined (an undefined predicate p can be considered as defined by $p \stackrel{\triangle}{=} \bot$). For the base cases, we have either $B = \bot$ or $B = \top$, in which cases the respective derivations are

$$\frac{}{\Gamma, \sigma \triangleright \bot \longrightarrow \sigma' \triangleright \bot}\ \bot\mathcal{L} \qquad \text{and} \qquad \frac{}{\Gamma, \sigma \triangleright \top \longrightarrow \sigma' \triangleright \top}\ \top\mathcal{R}\ .$$

The inductive cases where B is not atomic formula are straightforward; we just apply the introduction rules for the outermost connective in B. For example, if $B = B_1 \wedge B_2$ then take the following derivation

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \Sigma : \Gamma, \sigma \triangleright B_1 \longrightarrow \sigma' \triangleright B_1\end{array}}{\Sigma : \Gamma, \sigma \triangleright B_1 \wedge B_2 \longrightarrow \sigma' \triangleright B_1}\ \wedge\mathcal{L} \qquad \dfrac{\begin{array}{c}\Pi_2\\ \Sigma : \Gamma, \sigma \triangleright B_2 \longrightarrow \sigma' \triangleright B_2\end{array}}{\Sigma : \Gamma, \sigma \triangleright B_1 \wedge B_2 \longrightarrow \sigma' \triangleright B_2}\ \wedge\mathcal{L}}{\Sigma : \Gamma, \sigma \triangleright B_1 \wedge B_2 \longrightarrow \sigma' \triangleright B_1 \wedge B_2}\ \wedge\mathcal{R}$$

where Π_1 and Π_2 are obtained from inductive hypothesis.

In the case where B is an atomic formula, suppose that $\mathrm{dfn}(\Sigma, \rho, \sigma \triangleright B, \theta, D)$ holds for a definition clause $\forall x_1, \ldots, x_n.[H' \stackrel{\triangle}{=} D']$ and its raised version $\forall h_1, \ldots, h_n.[H \stackrel{\triangle}{=} D]$. We can assume that all variables in σ and σ' are not free in θ or ρ (otherwise, apply renaming), so that $(\sigma \triangleright B)\theta$ can be written as $\sigma \triangleright B\theta$. We first apply $def\mathcal{L}$ to $\sigma \triangleright B$ to get the partial derivation

$$\frac{\{\Sigma : \Gamma\rho, \sigma \triangleright D\theta \longrightarrow \sigma' \triangleright B\rho\}_{\mathrm{dfn}(\Sigma,\rho,\sigma\triangleright B,\theta,D)}}{\Sigma : \Gamma, \sigma \triangleright B \longrightarrow \sigma' \triangleright B}\ def\mathcal{L}$$

For each premise of $def\mathcal{L}$ we must show that we can apply $def\mathcal{R}$ to $\sigma' \triangleright B\rho$ to get $\sigma' \triangleright D\theta$. By Definition 3,

$$\begin{array}{l} H = H'[(h_1\ \sigma)/x_1, \ldots, (h_n\ \sigma)/x_n] \\ D = D'[(h_1\ \sigma)/x_1, \ldots, (h_n\ \sigma)/x_n] \end{array}$$

and $\theta(\lambda\sigma.B) = \rho(\lambda\sigma.H)$. By the assumption on the variable-naming in σ, the latter implies that $B\rho = H\theta$. Let

$$\begin{array}{l} H'' = H'[(h'_1\ \sigma')/x_1, \ldots, (h'_n\ \sigma')/x_n] \\ D'' = D'[(h'_1\ \sigma')/x_1, \ldots, (h'_n\ \sigma')/x_n] \end{array}.$$

Here, we again assume that h'_i is different from any variable in σ, σ'. In order to apply $def\mathcal{R}$, we need to show that there exists a substitution γ such that

$$\lambda\sigma'.B\rho =_{\beta\eta} (\lambda\sigma'.H'')\gamma$$

and such that $D\theta = D''\gamma$. We define γ to be the following substitution.

$$\gamma(x) = \begin{cases} \lambda\sigma'.(\theta(h_i))\ \sigma, & \text{if } x = h'_i \text{ for some } i \in \{1, \ldots, n\}, \\ \theta(x), & \text{otherwise.} \end{cases}$$

It can be verified that

$$H''\gamma = H\theta = B\rho$$

and that

$$(\lambda\sigma'.D'')\gamma = \lambda\sigma'.D''\gamma = \lambda\sigma'.D\theta$$

We can now complete our previous partial derivation.

$$\frac{\left\{ \dfrac{\begin{array}{c}\Pi^{\rho,D} \\ \Sigma : \Gamma\rho, \sigma \triangleright D\theta \longrightarrow \sigma' \triangleright D\theta\end{array}}{\Sigma : \Gamma\rho, \sigma \triangleright D\theta \longrightarrow \sigma' \triangleright B\rho}\ def\mathcal{R} \right\}_{\mathrm{dfn}(\Sigma,\rho,\sigma\triangleright B,\theta,D)}}{\Sigma : \Gamma, \sigma \triangleright B \longrightarrow \sigma' \triangleright B}\ def\mathcal{L}$$

Here the derivation $\Pi^{\rho,D}$ is obtained by applying inductive hypothesis, which we can do since $\mathrm{lvl}(D) < \mathrm{lvl}(B)$. ∎

4 Cut-Elimination

We define a reduction relation between derivations in $FO\lambda^{\Delta\nabla}$, following [4]. The redex is always a derivation ending with a multicut, which is the only multicut in the derivation. We then define a measure on the redex and show that the reduction always produces redexes with smaller measure. Cut elimination for the general derivation then becomes a simple collorary.

Definition 10 Let Ξ be a derivation ending with a multicut rule, which is the only cut in the derivation:

$$\frac{\begin{array}{c}\Pi_1 \\ \Sigma : \Delta_1 \longrightarrow \mathcal{B}_1\end{array} \quad \cdots \quad \begin{array}{c}\Pi_n \\ \Sigma : \Delta_n \longrightarrow \mathcal{B}_n\end{array} \quad \begin{array}{c}\Pi \\ \Sigma : \mathcal{B}_1, \ldots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}\end{array}}{\Sigma : \Delta_1, \ldots, \Delta_n, \Gamma \longrightarrow \mathcal{C}}\ mc \quad .$$

We define a measure $\mu(\Xi)$ to be the tuple

$$\langle \max\{\mathrm{lvl}(\mathcal{B}_i)\}, \mathrm{def}(\Pi), \mathrm{contr}(\Pi), \sum \mid \mathcal{B}_i \mid, H \rangle \enspace .$$

The components of the tuple are defined as follows. Let $\{\Pi'_i\}_i$ be the premise derivations of Π. The measure $\mathrm{def}(\Pi)$ is defined as follows.

$$\mathrm{def}(\Pi) = \begin{cases} \mathrm{lub}(\{\mathrm{def}(\Pi'_i) + 1\}_i), & \text{if } \Pi \text{ ends with a } \mathit{def}\mathcal{L} \text{ rule} \\ \mathrm{lub}(\{\mathrm{def}(\Pi'_i)\}_i), & \text{otherwise.} \end{cases}$$

The measure $\mathrm{contr}(\Pi)$ is defined analogously, i.e., by counting the number of $c\mathcal{L}$ rules instead of $\mathit{def}\mathcal{L}$ rules. The measure $\mid \mathcal{B}_i \mid$ is the size of the formula in $\mathcal{B}_i$ and H is the sum of the height of the premise derivations, i.e. $H = \sum \mathrm{ht}(\Pi_i) + \mathrm{ht}(\Pi)$. The ordering on the measure μ is defined lexicographically on the ordering of its components.

We need to prove that these measures are preserved by substitution of derivations, in particular $\mathrm{def}(\Pi)$ and $\mathrm{contr}(\Pi)$. The proof of the following lemma follows straightforwardly from the construction of the derivation $\Pi\theta$ in the proof of Lemma 6.

Lemma 11 *Let Π be a derivation of $\Sigma : \Gamma \longrightarrow \mathcal{C}$ and let θ be a substitution. Then* $\mathrm{def}(\Pi\theta) \leq \mathrm{def}(\Pi)$ *and* $\mathrm{contr}(\Pi\theta) \leq \mathrm{contr}(\Pi)$.

Theorem 12 *Let Ξ be a derivation of $\Sigma : \Gamma \longrightarrow \mathcal{C}$ ending with a multicut, which is the only cut in the derivation. Then there exists a cut-free derivation of the same sequent.*

Proof Let Ξ be the derivation

$$\dfrac{\overset{\Pi_1}{\Sigma : \Delta_1 \longrightarrow \mathcal{B}_1} \quad \cdots \quad \overset{\Pi_n}{\Sigma : \Delta_n \longrightarrow \mathcal{B}_n} \quad \overset{\Pi}{\Sigma : \mathcal{B}_1, \ldots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}}}{\Sigma : \Delta_1, \ldots, \Delta_n, \Gamma \longrightarrow \mathcal{C}}\ mc \enspace .$$

If $n = 0$, Ξ reduces to the premise derivation Π.

For $n > 0$ we specify the reduction relation based on the last rule of the premise derivations. If the rightmost premise derivation Π ends with a left rule acting on a cut formula B_i, then the last rule of Π_i and the last rule of Π together determine the reduction rules that apply. We classify these rules according to the following criteria: we call the rule an *essential* case when Π_i ends with a right rule; if it ends with a left rule, it is a *right-commutative* case; if Π_i ends with the *init* rule, then we have an *axiom* case. When Π does not end with a left rule acting on a cut formula, then its last rule is alone sufficient to determine the reduction rules that apply. If Π ends in a rule acting on a formula other than a cut formula, then we call this a *left-commutative* case. A *structural* case results when Π ends with a contraction or weakening on a cut formula. If Π ends with the *init* rule, this is also an axiom case. For simplicity of presentation, we always show $i = 1$.

<u>*Essential cases:*</u>

$\wedge\mathcal{R}/\wedge\mathcal{L}$: If Π_1 and Π are

$$\dfrac{\overset{\Pi'_1}{\Sigma : \Delta_1 \longrightarrow \sigma \triangleright B'_1} \quad \overset{\Pi''_1}{\Sigma : \Delta_1 \longrightarrow \sigma \triangleright B''_1}}{\Sigma : \Delta_1 \longrightarrow \sigma \triangleright B'_1 \wedge B''_1}\ \wedge\mathcal{R} \qquad \dfrac{\overset{\Pi'}{\Sigma : \sigma \triangleright B'_1, \mathcal{B}_2, \ldots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}}}{\Sigma : \sigma \triangleright B'_1 \wedge B''_1, \mathcal{B}_2, \ldots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}}\ \wedge\mathcal{L} \enspace ,$$

then Ξ reduces to the derivation Ξ'

$$\dfrac{\overset{\Pi'_1}{\Sigma : \Delta_1 \longrightarrow \mathcal{B}'_1} \quad \overset{\Pi_2}{\Sigma : \Delta_2 \longrightarrow \mathcal{B}_2} \quad \cdots \quad \overset{\Pi_n}{\Sigma : \Delta_n \longrightarrow \mathcal{B}_n} \quad \overset{\Pi'}{\Sigma : \mathcal{B}'_1, \mathcal{B}_2, \ldots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}}}{\Sigma : \Delta_1, \ldots, \Delta_n, \Gamma \longrightarrow \mathcal{C}}\ mc \enspace .$$

The measure $\mu(\Xi)$ is smaller than $\mu(\Xi')$, since

$$\max\{\mathrm{lvl}(\mathcal{B}'_1), \mathrm{lvl}(\mathcal{B}_2), \ldots, \mathrm{lvl}(\mathcal{B}_n)\} \leq \max\{\mathrm{lvl}(\mathcal{B}_1), \ldots, \mathrm{lvl}(\mathcal{B}_n)\},$$

$$\mathrm{def}(\Pi') = \mathrm{def}(\Pi), \quad \mathrm{contr}(\Pi) = \mathrm{contr}(\Pi') \text{ and}$$

$$\mid \mathcal{B}'_1 \mid < \mid \mathcal{B}_1 \mid .$$

Therefore we can apply inductive hypothesis to Ξ' to obtain a cut free derivation. The case for the other $\wedge\mathcal{L}$ rule is symmetric.

$\vee\mathcal{R}/\vee\mathcal{L}$: If Π_1 and Π are

$$\dfrac{\begin{array}{c}\Pi'_1\\ \Sigma : \Delta_1 \longrightarrow \sigma \triangleright B'_1\end{array}}{\Sigma : \Delta_1 \longrightarrow \sigma \triangleright B'_1 \vee B''_1}\ \vee\mathcal{R}$$

$$\dfrac{\begin{array}{c}\Pi'\\ \Sigma : \sigma \triangleright B'_1, \mathcal{B}_2, \ldots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}\end{array} \quad \begin{array}{c}\Pi''\\ \Sigma : \sigma \triangleright B''_1, \mathcal{B}_2, \ldots, B_n, \Gamma \longrightarrow \mathcal{C}\end{array}}{\Sigma : \sigma \triangleright B'_1 \vee B''_1, \mathcal{B}_2, \ldots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}}\ \vee\mathcal{L} \quad ,$$

then Ξ reduces to a derivation Ξ'

$$\dfrac{\begin{array}{c}\Pi'_1\\ \Sigma : \Delta_1 \longrightarrow \mathcal{B}'_1\end{array} \quad \begin{array}{c}\Pi_2\\ \Sigma : \Delta_2 \longrightarrow \mathcal{B}_2\end{array} \quad \cdots \quad \begin{array}{c}\Pi_n\\ \Sigma : \Delta_n \longrightarrow \mathcal{B}_n\end{array} \quad \begin{array}{c}\Pi'\\ \Sigma : \mathcal{B}'_1, \mathcal{B}_2, \ldots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}\end{array}}{\Sigma : \Delta_1, \ldots, \Delta_n, \Gamma \longrightarrow \mathcal{C}}\ mc \quad .$$

As in previous case, the size of cut formulas decreases, and therefore inductive hypothesis applies to the reduct Ξ'. The case for the other $\vee\mathcal{R}$ rule is symmetric.

$\supset\mathcal{R}/\supset\mathcal{L}$: Suppose Π_1 and Π are

$$\dfrac{\begin{array}{c}\Pi'_1\\ \Sigma : \sigma \triangleright B'_1, \Delta_1 \longrightarrow \sigma \triangleright B''_1\end{array}}{\Sigma : \Delta_1 \longrightarrow \sigma \triangleright B'_1 \supset B''_1}\ \supset\mathcal{R}$$

$$\dfrac{\begin{array}{c}\Pi'\\ \Sigma : \mathcal{B}_2, \ldots, \mathcal{B}_n, \Gamma \longrightarrow \sigma \triangleright B'_1\end{array} \quad \begin{array}{c}\Pi''\\ \Sigma : \sigma \triangleright B''_1, \mathcal{B}_2, \ldots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}\end{array}}{\Sigma : \sigma \triangleright B'_1 \supset B''_1, \mathcal{B}_2, \ldots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}}\ \supset\mathcal{L} \quad .$$

Let Ξ_1 be

$$\dfrac{\left\{\begin{array}{c}\Pi_i\\ \Sigma : \Delta_i \longrightarrow \mathcal{B}_i\end{array}\right\}_{i\in\{2..n\}} \quad \begin{array}{c}\Pi'\\ \Sigma : \mathcal{B}_2, \ldots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{B}'_1\end{array}}{\Sigma : \Delta_2, \ldots, \Delta_n, \Gamma \longrightarrow \mathcal{B}'_1}\ mc$$

and let Ξ_2 be

$$\dfrac{\begin{array}{c}\Xi_1\\ \Sigma : \Delta_2, \ldots, \Delta_n, \Gamma \longrightarrow \mathcal{B}'_1\end{array} \quad \begin{array}{c}\Pi'_1\\ \Sigma : \mathcal{B}'_1, \Delta_1 \longrightarrow \mathcal{B}''_1\end{array}}{\Sigma : \Delta_1, \ldots, \Delta_n, \Gamma \longrightarrow \mathcal{B}''_1}\ mc \quad .$$

The derivation Ξ_1 has a smaller size of cut formula than Ξ, while other measures remain non-increasing. Therefore, inductive hypothesis can be applied to eliminate the multicut in Ξ_1. The measure $\mu(\Xi_2)$ is strictly smaller than $\mu(\Xi)$ because

$$\mathrm{lvl}(\mathcal{B}'_1) < \mathrm{lvl}(\mathcal{B}_1) \leq \max\{\mathrm{lvl}(\mathcal{B}_1), \ldots, \mathrm{lvl}(\mathcal{B}_n)\}.$$

Recall that $\mathrm{lvl}(B'_1 \supset B''_1) = \max\{\mathrm{lvl}(B'_1) + 1, \mathrm{lvl}(B''_1)\}$. Therefore, the multicut in Ξ_2 can be eliminated as well.

The derivation Ξ then reduces to the derivation Ξ'

$$\cfrac{\cfrac{\cfrac{\Xi_2'}{\Sigma : \ldots \longrightarrow \mathcal{B}_1''} \quad \left\{ \cfrac{\Pi_i}{\Sigma : \Delta_i \longrightarrow \mathcal{B}_i} \right\}_{i \in \{2..n\}} \quad \cfrac{\Pi''}{\Sigma : \mathcal{B}_1'', \{\mathcal{B}_i\}_{i\in\{2..n\}}, \Gamma \longrightarrow \mathcal{C}}}{\Sigma : \Delta_1, \ldots, \Delta_n, \Gamma, \Delta_2, \ldots, \Delta_n, \Gamma \longrightarrow \mathcal{C}}\ mc}{\Sigma : \Delta_1, \ldots, \Delta_n, \Gamma \longrightarrow \mathcal{C}}\ c\mathcal{L} \quad .$$

We use the double horizontal lines to indicate that the relevant inference rule (in this case, $c\mathcal{L}$) may need to be applied zero or more times. Again, since the cut formulas size decreases, we have $\mu(\Xi') < \mu(\Xi)$ and therefore inductive hypothesis can be applied to eliminate the multicut in Ξ'.

$\forall\mathcal{R}/\forall\mathcal{L}$: If Π_1 and Π are

$$\cfrac{\cfrac{\Pi_1'}{\Sigma, h : \Delta_1 \longrightarrow \sigma \triangleright B_1'[(h\ \sigma)/x]}}{\Sigma : \Delta_1 \longrightarrow \sigma \triangleright \forall_\tau x.B_1'}\ \forall\mathcal{R} \qquad \cfrac{\Sigma, \sigma \vdash t : \tau \quad \cfrac{\Pi'}{\Sigma : \sigma \triangleright B_1'[t/x], \mathcal{B}_2, \ldots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}}}{\Sigma : \sigma \triangleright \forall_\tau x.B_1', \mathcal{B}_2, \ldots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}}\ \forall\mathcal{L} \quad ,$$

then Ξ reduces to

$$\cfrac{\cfrac{\Pi_1'[\lambda\sigma.t/h]}{\Sigma : \Delta_1 \longrightarrow \sigma \triangleright B_1'[t/x]} \quad \left\{ \cfrac{\Pi_i}{\Sigma : \Delta_i \longrightarrow \mathcal{B}_i} \right\}_{i \in \{2..n\}} \quad \cfrac{\Pi'}{\Sigma : \ldots \longrightarrow \mathcal{C}}}{\Sigma : \Delta_1, \ldots, \Delta_n, \Gamma \longrightarrow \mathcal{C}}\ mc \quad .$$

The size of cut formulas decreases while other measures are non-increasing, therefore $\mu(\Xi') < \mu(\Xi)$.

$\exists\mathcal{R}/\exists\mathcal{L}$: If Π_1 and Π are

$$\cfrac{\Sigma, \sigma \vdash t : \tau \quad \cfrac{\Pi_1'}{\Sigma : \Delta_1 \longrightarrow \sigma \triangleright B_1'[t/x]}}{\Sigma : \Delta_1 \longrightarrow \sigma \triangleright \exists_\tau x.B_1'}\ \exists\mathcal{R} \qquad \cfrac{\cfrac{\Pi'}{\Sigma, h : \sigma \triangleright B_1'[(h\ \sigma)/x], \mathcal{B}_2, \ldots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}}}{\Sigma : \sigma \triangleright \exists x_\tau.B_1', \mathcal{B}_2, \ldots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}}\ \exists\mathcal{L} \quad ,$$

then Ξ reduces to

$$\cfrac{\cfrac{\Pi_1'}{\Sigma : \Delta_1 \longrightarrow \sigma \triangleright B_1'[t/x]} \quad \left\{ \cfrac{\Pi_i}{\Sigma : \Delta_i \longrightarrow \mathcal{B}_i} \right\}_{i \in \{2..n\}} \quad \cfrac{\Pi'[\lambda\sigma.t/h]}{\Sigma : \sigma \triangleright B_1'[t/x], \ldots \longrightarrow \mathcal{C}}}{\Sigma : \Delta_1, \ldots, \Delta_n, \Gamma \longrightarrow \mathcal{C}}\ mc \quad .$$

Size of cut formulas decreases.

$\nabla\mathcal{R}/\nabla\mathcal{L}$ Suppose Π_1 and Π are

$$\cfrac{\cfrac{\Pi_1'}{\Sigma : \Delta_1 \longrightarrow (\sigma, y) \triangleright B_1'[y/x]}}{\Sigma : \Delta_1 \longrightarrow \sigma \triangleright \nabla x.B_1'}\ \nabla\mathcal{R} \qquad \cfrac{\cfrac{\Pi'}{\Sigma : (\sigma, y) \triangleright B_1'[y/x], \mathcal{B}_2 \ldots, \mathcal{B}_n, \Gamma' \longrightarrow \mathcal{C}}}{\Sigma : \sigma \triangleright \nabla x.B_1', \mathcal{B}_2, \ldots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}}\ \nabla\mathcal{L} \quad .$$

Then Ξ reduces to the derivation Ξ'

$$\cfrac{\cfrac{\Pi_1'}{\Sigma : \Delta_1 \longrightarrow (\sigma, y) \triangleright B_1'[y/x]} \quad \left\{ \cfrac{\Pi_j'}{\Sigma : \Delta_j \longrightarrow \mathcal{B}_j} \right\}_{j \in [2\ldots n]} \quad \cfrac{\Pi'}{\Sigma : (\sigma, y) \triangleright B_1'[y/x], \ldots \longrightarrow \mathcal{C}}}{\Sigma : \Delta_1, \ldots, \Delta_n, \Gamma \longrightarrow \mathcal{C}}\ mc \quad .$$

Size of cut formula decreases while other measures remain non-increasing, therefore the multicut in Ξ' can be eliminated by applying inductive hypothesis.

$def\mathcal{R}/def\mathcal{L}$**:** Suppose Π_1 and Π are

$$\dfrac{\begin{matrix}\Pi_1' \\ \Sigma : \Delta_1 \longrightarrow \mathcal{B}_1'\theta\end{matrix}}{\Sigma : \Delta_1 \longrightarrow \mathcal{B}_1}\ def\mathcal{R} \qquad \dfrac{\left\{\begin{matrix}\Pi^{\rho,D} \\ \Sigma\rho : \mathcal{D}\gamma, \mathcal{B}_2\rho, \ldots, \mathcal{B}_n\rho, \Gamma\rho \longrightarrow \mathcal{C}\rho\end{matrix}\right\}}{\Sigma : \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}}\ def\mathcal{L} \quad .$$

By the $def\mathcal{R}$ rule in Π_1, $\mathrm{dfn}(\Sigma, \epsilon, \sigma \rhd B_1, \theta, B_1')$ holds. Then Ξ reduces to Ξ'

$$\dfrac{\begin{matrix}\Pi_1' \\ \Sigma : \Delta_1 \longrightarrow \mathcal{B}_1'\theta\end{matrix} \quad \left\{\begin{matrix}\Pi_i \\ \Sigma : \Delta_i \longrightarrow \mathcal{B}_i\end{matrix}\right\}_{i\in\{2..n\}} \quad \begin{matrix}\Pi^{\epsilon,B_1'} \\ \Sigma : \mathcal{B}_1'\theta, \mathcal{B}_2, \ldots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}\end{matrix}}{\Sigma : \Delta_1, \ldots, \Delta_n, \Gamma \longrightarrow \mathcal{C}}\ mc \quad .$$

By the definition of definition clause, we have $\mathrm{lvl}(\mathcal{B}_1') \leq \mathrm{lvl}(\mathcal{B}_1)$, and therefore the maximum level of cut formulas is non-increasing. However, $\mathrm{def}(\Pi^{\epsilon,B_1'}) < \mathrm{def}(\Pi)$, therefore $\mu(\Xi') < \mu(\Xi)$ and inductive hypothesis can be applied to remove the multicut in Ξ'.

Left-commutative cases:

$\bullet\mathcal{L}/\circ\mathcal{L}$**:** Suppose Π ends with a left rule other than $c\mathcal{L}$ and $w\mathcal{L}$ acting on B_1, and Π_1 is

$$\dfrac{\left\{\begin{matrix}\Pi_1^i \\ \Sigma' : \Delta_1^i \longrightarrow B_1\end{matrix}\right\}}{\Sigma : \Delta_1 \longrightarrow B_1}\ \bullet\mathcal{L} \quad ,$$

where $\bullet\mathcal{L}$ is any left rule except $\supset\mathcal{L}$, $def\mathcal{L}$, and Σ is a subset of Σ'. Then Ξ reduces to

$$\dfrac{\left\{\dfrac{\begin{matrix}\Pi_1^i \\ \Sigma' : \Delta_1^i \longrightarrow \mathcal{B}_1\end{matrix} \quad \left\{\begin{matrix}\Pi_j' \\ \Sigma' : \Delta_j \longrightarrow \mathcal{B}_j\end{matrix}\right\}_{j\in\{2..n\}} \quad \begin{matrix}\Pi' \\ \Sigma' : \mathcal{B}_1, \ldots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}\end{matrix}}{\Sigma' : \Delta_1^i, \Delta_2, \ldots, \Delta_n, \Gamma \longrightarrow \mathcal{C}}\ mc\right\}}{\Sigma : \Delta_1, \Delta_2, \ldots, \Delta_n, \Gamma \longrightarrow \mathcal{C}}\ \bullet\mathcal{L} \quad ,$$

where Π_j' and Π' are obtained from Π_j and Π by applying Lemma 5. Since $\mathrm{ht}(\Pi_j') \leq \mathrm{ht}(\Pi_j)$ and $\mathrm{ht}(\Pi') \leq \mathrm{ht}(\Pi)$, the measure H in the multicuts in Ξ' is strictly smaller than the same measure in Ξ. Since other measures remain unchanged, $\mu(\Xi') < \mu(\Xi)$, and therefore inductive hypothesis can be applied to eliminate the multicuts in Ξ'.

$\supset\mathcal{L}/\circ\mathcal{L}$**:** Suppose Π ends with a left rule other than $c\mathcal{L}$ and $w\mathcal{L}$ acting on B_1 and Π_1 is

$$\dfrac{\begin{matrix}\Pi_1' \\ \Sigma : \Delta_1' \longrightarrow \sigma \rhd D_1'\end{matrix} \quad \begin{matrix}\Pi_1'' \\ \Sigma : \sigma \rhd D_1'', \Delta_1' \longrightarrow \mathcal{B}_1\end{matrix}}{\Sigma : \sigma \rhd D_1' \supset D_1'', \Delta_1' \longrightarrow \mathcal{B}_1}\ \supset\mathcal{L} \quad .$$

Let Ξ_1 be

$$\dfrac{\begin{matrix}\Pi_1'' \\ \Sigma : \sigma \rhd D_1'', \Delta_1' \longrightarrow \mathcal{B}_1\end{matrix} \quad \begin{matrix}\Pi_2 \\ \Sigma : \Delta_2 \longrightarrow \mathcal{B}_2\end{matrix} \quad \cdots \quad \begin{matrix}\Pi_n \\ \Sigma : \Delta_n \longrightarrow \mathcal{B}_n\end{matrix} \quad \begin{matrix}\Pi \\ \Sigma : \mathcal{B}_1, \ldots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}\end{matrix}}{\Sigma : \sigma \rhd D_1'', \Delta_1', \Delta_2, \ldots, \Delta_n, \Gamma \longrightarrow \mathcal{C}}\ mc \quad .$$

Then Ξ reduces to

$$\dfrac{\dfrac{\begin{matrix}\Pi_1' \\ \Sigma : \Delta_1' \longrightarrow \sigma \rhd D_1'\end{matrix}}{\Sigma : \Delta_1', \Delta_2, \ldots, \Delta_n, \Gamma \longrightarrow \sigma \rhd D_1'}\ w\mathcal{L} \quad \begin{matrix}\Xi_1 \\ \Sigma : \sigma \rhd D_1'', \Delta_1', \Delta_2, \ldots, \Delta_n, \Gamma \longrightarrow \mathcal{C}\end{matrix}}{\Sigma : \sigma \rhd D_1' \supset D_1'', \Delta_1', \Delta_2, \ldots, \Delta_n, \Gamma \longrightarrow \mathcal{C}}\ \supset\mathcal{L} \quad .$$

The multicut in Ξ_1 can be eliminated by inductive hypothesis since the total height of its premises are smaller than the total height of the premises of the multicut in Ξ, while other measures are equal.

$def\mathcal{L}/\circ\mathcal{L}$**:** If Π ends with a left rule other than $c\mathcal{L}$ and $w\mathcal{L}$ acting on B_1 and Π_1 is

$$\dfrac{\left\{\begin{array}{c}\Pi_1^{\rho,D}\\ \Sigma\rho : (\sigma \triangleright D)\theta, \Delta_1'\rho \longrightarrow \mathcal{B}_1\rho\end{array}\right\}}{\Sigma : \sigma \triangleright A, \Delta_1' \longrightarrow \mathcal{B}_1}\ def\mathcal{L} \quad .$$

By the definition of $def\mathcal{L}$ rule, the relation $\mathrm{dfn}(\Sigma, \rho, \sigma \triangleright A, \theta, D)$ holds for a given raised definition clause $\forall \bar{x}.[H \triangleq D]$ where $\bar{x}$ are chosen to be different from the variables in Σ. Then Ξ reduces to

$$\dfrac{\left\{\dfrac{\begin{array}{c}\Pi_1^{\rho,D}\\ \Sigma\rho : (\sigma \triangleright D)\rho, \Delta_1'\rho \longrightarrow \mathcal{B}_1\rho\end{array} \quad \left\{\begin{array}{c}\Pi_i\rho\\ \Sigma\rho : \Delta_i\rho \longrightarrow \mathcal{B}_i\rho\end{array}\right\}_{i\in\{2..n\}} \quad \begin{array}{c}\Pi\rho\\ \Sigma\rho : \ldots \longrightarrow \mathcal{C}\rho\end{array}}{\Sigma\rho : (\sigma \triangleright D)\theta, \Delta_1'\rho, \Delta_2\rho, \ldots, \Delta_n\rho, \Gamma\rho \longrightarrow \mathcal{C}\rho}\ mc\right\}}{\Sigma : \sigma \triangleright A, \Delta_1', \Delta_2, \ldots, \Delta_n, \Gamma \longrightarrow \mathcal{C}}\ def\mathcal{L} \quad .$$

Applying a substitution to a derivation does not increase the height of the derivation, therefore, for each multicut in the reduct, the total height of the premises is strictly smaller than the total height of premise derivations in Ξ, and hence inductive hypothesis can be applied to eliminate those multicuts.

Right-commutative cases:

$-/\circ\mathcal{L}$**:** Suppose Π is

$$\dfrac{\left\{\begin{array}{c}\Pi^i\\ \Sigma' : \mathcal{B}_1, \ldots, \mathcal{B}_n, \Gamma^i \longrightarrow \mathcal{C}\end{array}\right\}}{\Sigma : \mathcal{B}_1, \ldots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}}\ \circ\mathcal{L} \quad ,$$

where $\Sigma' \supseteq \Sigma$ and $\circ\mathcal{L}$ is any left rule other than $\supset\mathcal{L}$, $def\mathcal{L}$, acting on a judgment other than $\mathcal{B}_1, \ldots, \mathcal{B}_n$. Then Ξ reduces to

$$\dfrac{\left\{\dfrac{\begin{array}{c}\Pi_1'\\ \Sigma' : \Delta_1 \longrightarrow \mathcal{B}_1\end{array} \quad \cdots \quad \begin{array}{c}\Pi_n'\\ \Sigma' : \Delta_n \longrightarrow \mathcal{B}_n\end{array} \quad \begin{array}{c}\Pi^i\\ \Sigma' : \mathcal{B}_1, \ldots, \mathcal{B}_n, \Gamma^i \longrightarrow \mathcal{C}\end{array}}{\Sigma' : \Delta_1, \ldots, \Delta_n, \Gamma^i \longrightarrow C}\ mc\right\}}{\Sigma : \Delta_1, \ldots, \Delta_n, \Gamma \longrightarrow C}\ \circ\mathcal{L} \quad .$$

The total height of the premises of each multicut in the reduct decreases, therefore inductive hypothesis can be applied to eliminate the multicuts.

$-/\supset\mathcal{L}$**:** Suppose Π is

$$\dfrac{\begin{array}{c}\Pi'\\ \Sigma : \mathcal{B}_1, \ldots, \mathcal{B}_n, \Gamma' \longrightarrow \sigma \triangleright D'\end{array} \quad \begin{array}{c}\Pi''\\ \Sigma : \mathcal{B}_1, \ldots, \mathcal{B}_n, \sigma \triangleright D'', \Gamma' \longrightarrow \mathcal{C}\end{array}}{\Sigma : \mathcal{B}_1, \ldots, \mathcal{B}_n, \sigma \triangleright D' \supset D'', \Gamma' \longrightarrow \mathcal{C}}\ \supset\mathcal{L} \quad .$$

Let Ξ_1 be

$$\dfrac{\begin{array}{c}\Pi_1\\ \Sigma : \Delta_1 \longrightarrow \mathcal{B}_1\end{array} \quad \cdots \quad \begin{array}{c}\Pi_n\\ \Sigma : \Delta_n \longrightarrow \mathcal{B}_n\end{array} \quad \begin{array}{c}\Pi'\\ \Sigma : \mathcal{B}_1, \ldots, \mathcal{B}_n, \Gamma' \longrightarrow \sigma \triangleright D'\end{array}}{\Sigma : \Delta_1, \ldots, \Delta_n, \Gamma' \longrightarrow \sigma \triangleright D'}\ mc$$

and Ξ_2 be

$$\dfrac{\begin{array}{c}\Pi_1\\ \Sigma : \Delta_1 \longrightarrow \mathcal{B}_1\end{array} \quad \cdots \quad \begin{array}{c}\Pi_n\\ \Sigma : \Delta_n \longrightarrow \mathcal{B}_n\end{array} \quad \begin{array}{c}\Pi''\\ \Sigma : \mathcal{B}_1, \ldots, \mathcal{B}_n, \sigma \triangleright D'', \Gamma' \longrightarrow \mathcal{C}\end{array}}{\Sigma : \Delta_1, \ldots, \Delta_n, \sigma \triangleright D'', \Gamma' \longrightarrow \mathcal{C}}\ mc \quad .$$

Then Ξ reduces to

$$\frac{\begin{array}{c}\Xi_1\\ \Sigma : \Delta_1, \dots, \Delta_n, \Gamma' \longrightarrow \sigma \triangleright D'\end{array} \quad \begin{array}{c}\Xi_2\\ \Sigma : \Delta_1, \dots, \Delta_n, \sigma \triangleright D'', \Gamma' \longrightarrow \mathcal{C}\end{array}}{\Sigma : \Delta_1, \dots, \Delta_n, \sigma \triangleright D' \supset D'', \Gamma' \longrightarrow \mathcal{C}} \supset \mathcal{L} \quad .$$

The multicuts in Ξ_1 and Ξ_2 have lower total height of premise derivations than the multicut in Ξ, and therefore can be eliminated by applying inductive hypothesis.

$-/def\mathcal{L}$**:** If Π is

$$\frac{\left\{\begin{array}{c}\Pi^{\rho,D}\\ \Sigma\rho : \mathcal{B}_1\rho, \dots, \mathcal{B}_n\rho, (\sigma \triangleright D)\theta, \Gamma'\rho \longrightarrow C\rho\end{array}\right\}}{\Sigma : \mathcal{B}_1, \dots, \mathcal{B}_n, \sigma \triangleright A, \Gamma' \longrightarrow \mathcal{C}} \; def\mathcal{L} \quad .$$

Then Ξ reduces to

$$\frac{\left\{\dfrac{\left\{\begin{array}{c}\Pi_i\rho\\ \Sigma\rho : \Delta_i\rho \longrightarrow \mathcal{B}_i\rho\end{array}\right\}_{i\in\{1..n\}} \quad \begin{array}{c}\Pi^{\rho,D}\\ \Sigma\rho : \{\mathcal{B}_i\rho\}_{i\in\{1..n\}}, D\sigma, \Gamma'\rho \longrightarrow C\rho\end{array}}{\Sigma\rho : \Delta_1\rho, \dots, \Delta_n\rho, (\sigma \triangleright D)\theta, \Gamma'\rho \longrightarrow \mathcal{C}\rho} \; mc\right\}}{\Sigma : \Delta_1, \dots, \Delta_n, \sigma \triangleright A, \Gamma' \longrightarrow \mathcal{C}} \; def\mathcal{L} \quad .$$

Since $\mathrm{def}(\Pi^{\rho,D}) < \mathrm{def}(\Pi)$, we can apply inductive hyptothesis to remove the multicuts.

$-/\circ\mathcal{R}$**:** If Π is

$$\frac{\left\{\begin{array}{c}\Pi^i\\ \Sigma' : \mathcal{B}_1, \dots, \mathcal{B}_n, \Gamma^i \longrightarrow \mathcal{C}^i\end{array}\right\}}{\Sigma : \mathcal{B}_1, \dots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}} \; \circ\mathcal{R} \quad ,$$

where $\circ\mathcal{R}$ is any right rule, then Ξ reduces to

$$\frac{\left\{\dfrac{\begin{array}{c}\Pi'_1\\ \Sigma' : \Delta_1 \longrightarrow \mathcal{B}_1\end{array} \quad \cdots \quad \begin{array}{c}\Pi'_n\\ \Sigma' : \Delta_n \longrightarrow \mathcal{B}_n\end{array} \quad \begin{array}{c}\Pi^i\\ \Sigma' : \mathcal{B}_1, \dots, \mathcal{B}_n, \Gamma^i \longrightarrow \mathcal{C}^i\end{array}}{\Sigma' : \Delta_1, \dots, \Delta_n, \Gamma^i \longrightarrow \mathcal{C}^i} \; mc\right\}}{\Sigma : \Delta_1, \dots, \Delta_n, \Gamma \longrightarrow \mathcal{C}} \; \circ\mathcal{R} \quad .$$

Here the derivation Π'_i is obtained from Π_i by applying Lemma 5. The multicuts in the reduct can be then eliminated by applying inductive hypothesis.

<u>*Structural cases:*</u>

$-/c\mathcal{L}$**:** If Π is

$$\frac{\begin{array}{c}\Pi'\\ \Sigma : \mathcal{B}_1, \mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}\end{array}}{\Sigma : \mathcal{B}_1, \dots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}} \; c\mathcal{L} \quad ,$$

then Ξ reduces to

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \Sigma : \Delta_1 \longrightarrow \mathcal{B}_1\end{array} \quad \left\{\begin{array}{c}\Pi_i\\ \Sigma : \Delta_i \longrightarrow \mathcal{B}_i\end{array}\right\}_{i\in\{1..n\}} \quad \begin{array}{c}\Pi'\\ \Sigma : \mathcal{B}_1, \mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}\end{array}}{\Sigma : \Delta_1, \Delta_1, \Delta_2, \dots, \Delta_n, \Gamma \longrightarrow \mathcal{C}} \; mc}{\Sigma : \Delta_1, \dots, \Delta_n, \Gamma \longrightarrow \mathcal{C}} \; c\mathcal{L} \quad .$$

The measure $\mathrm{contr}(\Pi') < \mathrm{contr}(\Pi)$, while the maximum level of cut formulas does not change and $\mathrm{def}(\Pi) = \mathrm{def}(\Pi')$. Therefore $\mu(\Xi') < \mu(\Xi)$ and we can apply inductive hypothesis to remove the multicut in the reduct.

$-/w\mathcal{L}$: If Π is

$$\dfrac{\begin{array}{c}\Pi' \\ \Sigma : \mathcal{B}_2, \ldots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}\end{array}}{\Sigma : \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}}\ c\mathcal{L} \quad ,$$

then Ξ reduces to

$$\dfrac{\dfrac{\left\{\begin{array}{c}\Pi_i \\ \Sigma : \Delta_i \longrightarrow \mathcal{B}_i\end{array}\right\}_{i\in\{2..n\}} \quad \begin{array}{c}\Pi' \\ \Sigma : \mathcal{B}_2, \ldots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}\end{array}}{\Sigma : \Delta_2, \ldots, \Delta_n, \Gamma \longrightarrow \mathcal{C}}\ mc}{\Sigma : \Delta_1, \Delta_2, \ldots, \Delta_n, \Gamma \longrightarrow \mathcal{C}}\ w\mathcal{L} \quad .$$

The total size of cut formulas decreases in the reduct, therefore we can apply inductive hypothesis to remove the multicut.

<u>*Axiom cases:*</u>

init$/-$: If Π_1 ends with the *init* rule, that is, $\mathcal{B}_1 \in \Delta_1$, then Ξ reduces to

$$\dfrac{\dfrac{\begin{array}{c}\Pi_2 \\ \Sigma : \Delta_2 \longrightarrow \mathcal{B}_2\end{array} \quad \cdots \quad \begin{array}{c}\Pi_n \\ \Sigma : \Delta_n \longrightarrow \mathcal{B}_n\end{array} \quad \begin{array}{c}\Pi \\ \Sigma : \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n, \Gamma \longrightarrow \mathcal{C}\end{array}}{\Sigma : \mathcal{B}_1, \Delta_2, \ldots, \Delta_n, \Gamma \longrightarrow \mathcal{C}}\ mc}{\Sigma : \Delta_1, \Delta_2, \ldots, \Delta_n, \Gamma \longrightarrow \mathcal{C}}\ w\mathcal{L} \quad .$$

The size of cut formulas and the total height of premise derivations of multicut decreases, while other measures are non-increasing, therefore inductive hypothesis can be applied to remove the multicut.

$-/$*init*: If Π ends with the *init* rule and $\mathcal{C}$ is a judgment in Γ, then Ξ reduces to

$$\dfrac{}{\Sigma : \Delta_1, \ldots, \Delta_n, \Gamma \longrightarrow \mathcal{C}}\ init \quad .$$

If Π ends with the *init* rule, but $\mathcal{C}$ is not a judgment in Γ, then $\mathcal{C}$ must be one of the cut judgments, say $\mathcal{B}_1$. In this case Ξ reduces to

$$\dfrac{\begin{array}{c}\Pi_1 \\ \Sigma : \Delta_1 \longrightarrow \mathcal{B}_1\end{array}}{\Sigma : \Delta_1, \ldots, \Delta_n, \Gamma \longrightarrow \mathcal{B}_1}\ w\mathcal{L}$$

∎

Corollary 13 *For every derivation Ψ of $\Sigma : \Gamma \longrightarrow C$ there exists a cut-free derivation Ψ' of the same sequent.*

5 Conclusions and Future Works

The design of this logic was motivated by the use of logic in reasoning about specification of computations. As noted in [7], there are two different roles of eigenvariables in such specifications: eigenvariables as fresh, scoped constants and eigenvariables as site for substitutions (instantiations). These two behaviors are not fully compatible, especially when we need to reason about left behavior of the judgments in the sequent. A new layer of scoping (abstraction) seems therefore quite natural to add to the sequent. In [7], this logic is used to encode and reason about examples in π-calculus and object-logic provability in which the separation of the quantifiers $\forall$ and ∇ are clearly shown.

The cut-elimination proof we have seen is a modular extension of McDowell's proof [3, 4]. The cases involving the new quantifier ∇ are surprisingly easy to deal with, due to its isolated nature, that is, its introduction rules are totally confined within the local judgments. This suggests that extending further the logic with induction rules should not be difficult. A more challenging thing to do is to add coinduction, which is an entirely different issue. Some more examples, like in reasoning about security protocols [1], are interesting to explore.

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