# Space-Time-Frequency Degrees of Freedom: Fundamental Limits for Spatial Information

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*Abstract*— **We bound the number of electromagnetic signals which may be observed over a frequency range $[F - W, F + W]$ a time interval $[0, T]$ within a sphere of radius $R$. We show that the such constrained signals may be represented by a series expansion whose terms are bounded exponentially to zero beyond a threshold. Our result implies there is a finite amount of information which may be extracted from a region of space via electromagnetic radiation.**

## I. INTRODUCTION

Wireless communication is fundamentally limited by the physics of the medium. Electromagnetic wave propagation has been given [1, 2] as a motivation for developing such limits: information is ultimately carried on electromagnetic waves. Narrowband degrees of freedom (dimensionality) results have been given for dense multipath [3–5] and subsequently extended to sparse systems. Here, the signal bandwidth is negligible: dimensionality results are defined in wavelengths.

Narrow-band wavefields were shown to have limited concentrations [6]. The limit was based upon the free-space Helmholtz (wave) equation – a time independent variation of the electromagnetic wave [7]. Such waves may be represented by a functional series, whose terms are bounded exponentially toward zero beyond some limit. This limit was used to describe a random MIMO channel in dense [8] multipath and provide capacity results.

More recent work – including wide-band MIMO motivates analysis of the capability of spatially diverse signals to support multiplexing over significant bandwidths.

> Given a region, bounded by radius $R$, centre-frequency $F$, bandwidth $2W$ and observation time $T$, what is the number $\mathcal{D}$ of wireless (electromagnetic) signals which may be observed?

In [9] an approximate dimensionality result was given. This bound was excessively complex — resulting in a loose over-bound. In this work we provide exponential error bounds — reflecting [3], and provide a tighter bound on the dimensionality of 3D-spatial broadband signals.

The remainder of this paper is arranged as follows: We provide a truncation point and bound the error for electromagnetic signals in space in Section II. This gives our main result in Theorem 1. Section III gives plots of the degrees of freedom,

while Section IV provides an to MIMO mutual information. We draw conclusions in Section V. Proofs are in the Appendix.

## II. DIMENSIONALITY

Existing dimensionality results for signals 3D space, with non-trivial bandwidth are limited by

$$\mathcal{D}_{WT} = 2WT + 1 \tag{1}$$

$$\mathcal{D}_{space} = \left( \left\lceil \frac{e\pi RF}{c} \right\rceil + 1 \right)^2 \tag{2}$$

where (1) is from [10], formalised in [11] and (2) is from [12]. Fundamentally, we seek to develop a result which combines both (1) and (2): broadband, spatially diverse signals.

Source-free (propagating) electric fields $\Psi(\mathbf{r}, t)$, are solutions of the free-space Maxwell wave equation [7]:

$$\left( \triangle - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \cdot \Psi(\mathbf{r}, t) = 0 \tag{3}$$

where $\triangle = (\partial^2/\partial x^2, \partial^2/\partial y^2, \partial^2/\partial z^2)$ and $c = 3 \times 10^8 \text{ms}^{-1}$ is the speed of light. The vector $\mathbf{r} = (r, \theta, \phi)$ with $0 \leq \theta < \pi$, $0 \leq \phi < 2\pi$ denotes position. We now formally pose:

*Problem 1: Given a function in space-time $x(\mathbf{r}, t)$ which is non-zero for $|\mathbf{r}| \leq R$ and $t \in [0, T]$ and has a frequency component in $[F - W, F + W]$ and satisfies (3); what number $\mathcal{D}$ of signals $\varphi(\mathbf{r}, t)$ are required to parameterize $x(\mathbf{r}, t)$?*

Observe that we may represent any signal which is constrained to $[0, T] \times [0, R]$ and satisfying (3) by [13]:

$$\Psi_{\mathbf{k}}(\mathbf{r}, t) = \exp\left(-\iota|\mathbf{k}|ct - \iota\mathbf{k} \cdot \mathbf{r}\right) \tag{4}$$

where $\mathbf{k}$ is the vector wave-number and $\iota = \sqrt{-1}$. The magnitude $|\mathbf{k}| = 2\pi f/c$ is scalar wave-number.

We may then express any radio signal as (an infinite) series expansion (24). The series itself is not important: simply that it exists, and we may truncate the series at some point.

This series may be truncated at a point $\mathcal{N}(\mathbf{r}, t; f)$ which is an increasing function of frequency, time and space. Our first step is to define the truncation point at which *the majority* of the signal energy is captured – this will then provide us with our degrees-of-freedom result.

*Lemma 1 (Truncation Point): The critical threshold (a function of $|\mathbf{r}|$, $t$) at frequency $f$ is*

$$\mathcal{N}(\mathbf{r}, t; f) = \mathcal{N}_T(t; f) + \mathcal{N}_{\mathbb{S}}(\mathbf{r}; f) \tag{5}$$

$$\mathcal{N}_T(t; f) = \lceil e\pi\Delta_f t \rceil \tag{6}$$

$$\mathcal{N}_{\mathbb{S}}(\mathbf{r}; f) = \left\lceil e\pi f \frac{|\mathbf{r}|}{c} \right\rceil \tag{7}$$

Once we have chosen an appropriate threshold, increasing $\mathcal{N}$ reduces the truncation error exponentially.

*Lemma 2 (Truncation Error): The truncation error is*

$$\epsilon_{\mathcal{N}} \leq 2/e \approx 0.74 \tag{8}$$

*and for any $\alpha, \delta \geq 1$*

$$\epsilon_{\mathcal{N}+\delta+\alpha} < \epsilon_{\mathcal{N}} \cdot e^{2-\delta-\alpha} \tag{9}$$

Lemma 1 allows us to approximate any broadband radio signal with a finite number of terms. Since this bound is exponentially tight (from Lemma 2) we may apply Lemma 1 to give a dimensionality result:

*Theorem 1 (3D dimensionality $\mathcal{D}_{3D}$): The number of orthogonal electromagnetic waves which may be observed in a three-dimensional spatial region bounded by radius $R$, over frequency range $F \pm W$ and time interval $[0, T]$ is*

$$\mathcal{D}_{3D} \approx (e\pi 2WT + 1)\left(\frac{e\pi R(F-W)}{c} + 1\right)^2 \tag{10a}$$

$$+ e\pi 2WT\left(\frac{e\pi R}{c}\right)^2\left[2FW - \frac{2}{3}W^2\right] \tag{10b}$$

$$+ \left(\frac{e\pi R}{c}\right)^2 4FW + e\pi 2WT\left[\left(\frac{e\pi R}{c}\right)F + \frac{1}{6}\right] \tag{10c}$$

Since $F \geq W$ (by definition), so all terms in (10) are non-negative.

The term (10b) gives an estimate of the effect of signal bandwidth on the degrees of freedom in space. For practical applications we expect $R \ll c$ and $TW$ to be small ($TW < 10$). Thus, for (10b) to be non-negligible $\sqrt{FW} \approx c$. In general (10c) will be dominated by (10a) and (10b)
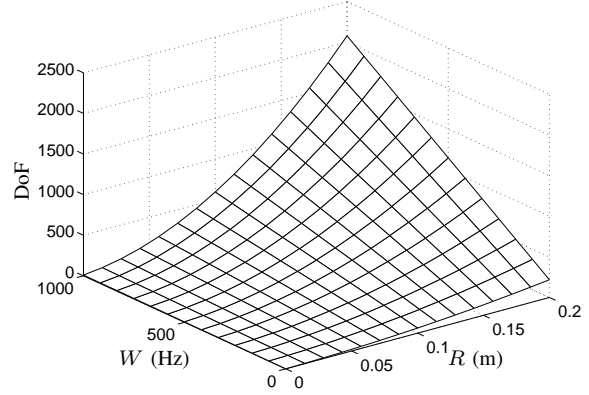
### A. Asymptotic Results

For $R \to 0$ (10) reduces to $7e\pi TW/3 + 1$ which overbounds (1). For $T, W \to 0$, (10) reduces to (2) while for $W \neq 0, T \to 0$ (10) reduces to (2) with centre frequency $F + W$. For extreme broadband signals $F = W$, and

$$\mathcal{D}_{3D} \propto 2e\pi TW\left(\frac{e\pi R}{c}\right)^2\frac{4}{3}W^2 \tag{11}$$

and when all parameters are non-trivial

$$\mathcal{D}_{3D} \propto 2e\pi TW\left(\frac{e\pi R}{c}\right)^2\left(F^2 + \frac{W^2}{3}\right) \tag{12}$$



(a) DoF vs bandwidth $W$ and region radius $R$
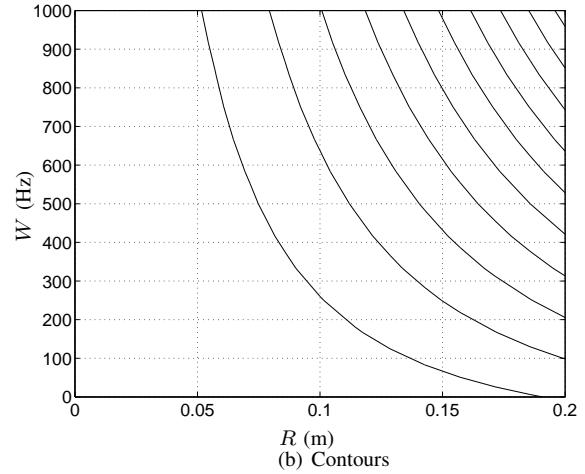


(b) Contours

Fig. 1. Number of degrees of freedom for moderate $W$ and $T = 0.5ms$, $F = 2.4GHz$, $\lambda = 0.125m$: Curvature toward bottom of Fig. 1(b) denotes saturation wrt. radius.

### III. PLOTS

We have considered two common spatially diverse scenarios. In Figure 1 we have used a centre frequency $F = 2.4$GHz, 1kHz bandwidth and $R < 2\lambda$. Figure 1(a) shows a mesh of the degrees of freedom, while Figure 1(b) gives a contour plot. The super-linear growth in DoF can be seen as both $R$ and $W$ increase.

Figure 2 shows the DoF for a broadband signal $W \leq F$ with centre frequency $F = 2.4$MHz. In this case the centre wave-length is 125m (and at $F + W$, $\lambda_{\min} = 62.5m$) so $R \ll \lambda/2$. In this case at $R \to 0$ we see the usual $2WT$ linear growth in DoF, at $R \approx \lambda_{\min}/4$ we see a knee-point in the DoF surface, corresponding to spatial degrees of freedom becoming effective. This can be seen by the curvature of the contours near $R = 5m$ in Figure 2(b). This suggests that standard MIMO designs may be worthwhile for much denser array spacings: at 2.4GHz some MIMO-like capacity improvement (over a 2WT result) may be seen at 3cm antenna separation for sufficient
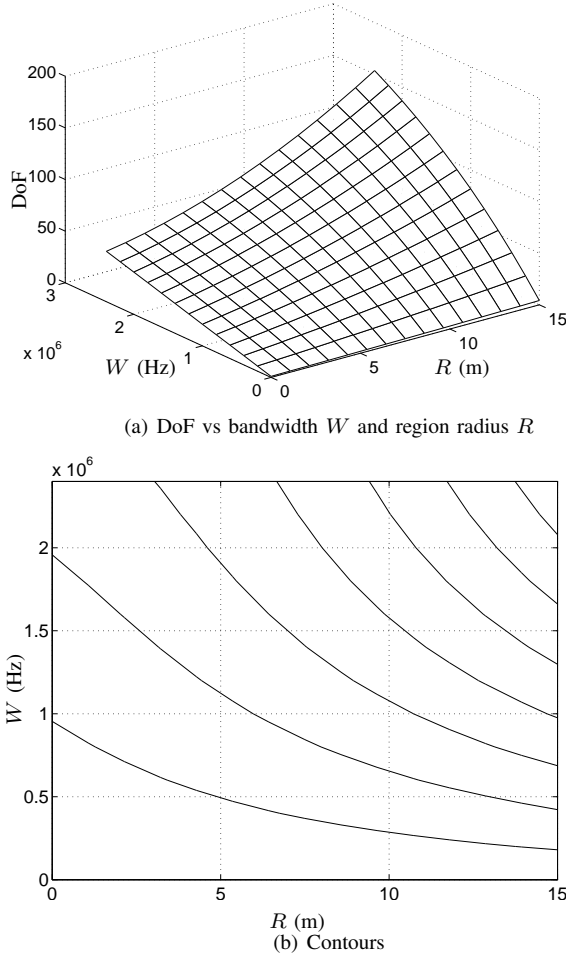
(a) DoF vs bandwidth $W$ and region radius $R$



(b) Contours

Fig. 2. Number of degrees of freedom for large $W$, small $R$ and $T = 1\mu s$, F=2.4MHz, $\lambda = 125m$

## IV. EXAMPLE: MUTUAL INFORMATION $\mathcal{I}$

We assume all channel eigenvalues are equal magnitude in space and frequency, up to $\mathcal{N}_T$ and $\mathcal{N}_S$ respectively. The transmitter uses (4) as matched filters for the channel and sends uniform power $\rho$ on the subset of modes with non-neglible magnitude. This is a reasonable capacity approximation [14].

A naïve application of (10) to mutual information would be $\mathcal{I} = \mathcal{D}\log(1 + \rho/\mathcal{D}) \leq \rho$. This ignores the random nature of the MIMO channel: spatial signals are mixed through a random scattering channel, while frequency signals are not. Assume both transmitter and receiver have identical geometries and are situated in dense scatter.

Consider Figure 3, every horizontal dashed line may be considered as an input element with each element operating at $2W$ different frequency taps (eg. through OFDM). At frequency $F - W \leq f \leq F + W$ there are $N_t = (e\pi Rf/c + 1)^2$, independent input signals from (2). Then there are

$$N = \sum_f N_t(f) = \sum_{f=F-W}^{F+W} (e\pi Rf/c + 1)^2 \qquad (13)$$

parallel input channels and mutual information at $f$ is $\mathcal{I}_f$

$$\mathcal{I}_f = \log\det\left(I_{N_t} + \frac{\rho}{N}XX^*\right) \to N_t\log\left(1 + P\frac{N_t}{N}\right)$$

Where $X$ is a Gaussian random matrix of dimension $N_t(f)$, see for example [6]. Since each frequency channel is independent, the total mutual information is found by combining (13) and

$$\mathcal{I} = \sum_{f=F-W}^{F+W} N_t(f)\log\left(1 + \rho\frac{N_t(f)}{N}\right) \qquad (14)$$

which may be calculated numerically. Note, in the case of $N_t(f) = \text{const}$ we return to the classic parallel channel result $\mathcal{I} = \rho N_t$. For $N_t(f)$ an increasing function of $f$, the sum (14) is increased, thus

$$\mathcal{I} \geq \rho\left(\frac{e\pi R(F - W)}{c} + 1\right)^2 \qquad (15)$$

Due to random scattering, spatial modes provide a linear increase in capacity, while frequency modes provide parallel channels.

## V. CONCLUSION

We have shown that the degrees of freedom for a spherically restricted broadband wireless signal is proportional to the surface area of the spatial region, the square of the frequency and bandwidth and the DoF of the broadband signal itself. We have shown that the error associated with truncating such a signal at $N$ terms decreases exponentially as $N$ increases.

As an example we have shown that broadband spatial communication systems may have a capacity beyond that expected by combining MIMO capacity results by with parallel frequency channels.

## APPENDIX I
## OPERATOR MATERIAL

*Definition 1 (Truncation Projection D): Given the spatial interval $\mathbb{S}$ and time interval $[0, T]$, the truncation operator $D$ sets a field $f(\mathbf{r}, t)$ to zero outside the time- and space-intervals.*

$$Df(\mathbf{r}, t) = \begin{cases} f(\mathbf{r}, t) & \mathbf{r} \in \mathbb{S} \text{ and } t \in [0, T] \\ 0 & else \end{cases} \qquad (16)$$

*Definition 2 (Wavefield Projection W): The wavefield projection $W$ projects a field $f(\mathbf{r}, t)$ onto solutions $\Psi(\mathbf{r}, t)$ of (3).*

$$Wf(\mathbf{r}, t) = \sum_{l,m,n} f_{l,m,n}\Psi_{l,m,n}(\mathbf{r}, t) \qquad (17)$$

$$f_{l,m,n} = \iint f(\mathbf{r}, t)\overline{\Psi_{l,m,n}(\mathbf{r}, t)}\, dt\, d\mathbf{r}$$

$\mathbf{k}$ *is the wave vector in three dimensions* $\mathbf{k} = (k_x, k_y, k_z)$ *with scalar wave number* $k = |\mathbf{k}| = (k_x^2 + k_y^2 + k_z^2)^{1/2}$ *[13, eqn.6.94 p.759]. $\Psi_{l,m,n}$ is given by (4) with the values of $k$ chosen discretely:*

$$k = \frac{2\pi}{cT}p \quad k_x = \frac{2\pi}{L}l \quad k_y = \frac{2\pi}{L}m \quad k_z = \frac{2\pi}{L}m \qquad (18)$$

Using (16) and (17) we may write the signal observed in $\mathbb{S} \times T$ as:

$$g(\mathbf{r}, t) = WDf(\mathbf{r}, t) \tag{19}$$

*Lemma 3:* $WD$ is a compact operator.

Lemma 3 emphasizes that although there are infinitely many independent electromagnetic waves, only a finite number of electromagnetic signals may be resolved within the region $\mathbb{S} \times T$. The implication of this is that any approximation for a given signal has a bounded error.

## APPENDIX II
## PROOFS

*Proof:* [Lemma 1]

From (4)

$$\Psi_{\mathbf{k}}(\mathbf{r}, t) = \exp(-\iota k_{\min} ct) \exp\left(-\iota \acute{k} ct - \iota \mathbf{k} \cdot \mathbf{r}\right) \tag{20}$$

$$= \exp(-\iota k_{\min} ct) \hat{\Psi}_{\mathbf{k}}(\mathbf{r}, t) \tag{21}$$

where $\mathbf{k} \cdot \mathbf{r}$ denotes the vector dot product and $0 \le \acute{k} \le 4\pi W/c$. We wish to bound the number of terms required to approximate this function. Note $\exp(-\iota k_{\min} ct)$ has exactly one degree of freedom, so we may equivalently calculate the DoF for $\hat{\Psi}$. Using the Jacobi-Anger expansion [15, eqn2.45, pp.32] and summation theorem [15, eqn2.29, pp.27]

$$e^{-\iota \mathbf{k} \cdot \mathbf{r}} = 4\pi \sum_{n=0}^{\infty} \iota^n j_n(k\|\mathbf{r}\|) \sum_{m=-n}^{n} Y_n^m(\hat{\mathbf{r}}) \overline{Y_n^m(\hat{\mathbf{k}})} \tag{22}$$

where $j_n(z) = \sqrt{\frac{\pi}{2z}} J_{n+\frac{1}{2}}(z)$ is a spherical Bessel function.

From [13, 8.534.1]

$$e^{-\iota c \acute{k} t} = \sum_{p=0}^{\infty} \iota^p (2p+1) j_p(c\acute{k}t) \tag{23}$$

Combining (22) and (23)

$$\Psi^{(P,N)}(\mathbf{r}, t) = 4\pi \sum_{p}^{P} \iota^p (2p+1) j_p(c\acute{k}t)$$
$$\times \sum_{n,m}^{N} \iota^n j_n(k\|\mathbf{r}\|) Y_n^m(\hat{\mathbf{r}}) \overline{Y_n^m(\hat{\mathbf{k}})} \tag{24}$$

We may truncate $\Psi(\mathbf{r}, t)$ and bound the error as in [3]:

$$\epsilon_{(P,N)} = \left| \Psi(\mathbf{r}, t) - \Psi^{(P,N)}(\mathbf{r}, t) \right|$$
$$= 4\pi \left| \sum_{p>N}^{\infty} \iota^p (2p+1) j_p(c\acute{k}t) \right.$$
$$\left. \times \sum_{n>N}^{\infty} \iota^n j_n(k\|r\|) \sum_{m=-n}^{n} Y_n^m(\hat{\mathbf{r}}) \overline{Y_n^m(\hat{\mathbf{k}})} \right|$$

Taking the absolute values inside the summation and using $\left| \sum_{m=-n}^{n} Y_n^m(\hat{\mathbf{r}}) \overline{Y_n^m(\hat{\mathbf{k}})} \right| \le (2n+1)/(2\pi)$ [15, pp.27] gives

$$\epsilon_{P,N} \le 2 \sum_{p>N}^{\infty} (2p+1) \left| j_p(c\acute{k}t) \right| \sum_{n>N}^{\infty} (2n+1) |j_n(k|r|)| \tag{25}$$

From [3, 12]

$$|j_n(x)| \le \frac{\sqrt{\pi}}{2} \frac{1}{\Gamma(n+3/2)} \left(\frac{x}{2}\right)^n \tag{26}$$

Using the identity $(2n+1)/\Gamma(n+3/2) = 1/\Gamma(n+1/2)$ and [16, pp.257] $\Gamma(n+1/2) > e^{-n-1/2}(n+1/2)^n (2\pi)^{1/2}$

$$\epsilon_{P,N} < \frac{e}{2} \sum_{p>P} \left[ \frac{(c\acute{k}t)e}{2(p+1/2)} \right]^p \sum_{n>N} \left[ \frac{(k|\mathbf{r}|)e}{2(n+1/2)} \right]^n \tag{27}$$

$$< \frac{e}{2} \sum_{p>P} \left[ \frac{(c\acute{k}t)e}{2(P+1)} \right]^p \sum_{n>N} \left[ \frac{(k|\mathbf{r}|)e}{2(N+1)} \right]^n \tag{28}$$

Both sums (28) converge if $P+1 > ec\acute{k}t/2$ and $N+1 > ek|\mathbf{r}|/2$.

$$\epsilon_{P,N} < \frac{e(N+1)(P+1) \left(\frac{ec\acute{k}t}{P+1}\right)^P \left(\frac{ek|\mathbf{r}|/c}{N+1}\right)^N}{(2N+2-x)(2P+2-y)2^{N+P-1}} \tag{29}$$

$$< 2e \left(\frac{e\pi \acute{f} t}{P+1}\right)^P \left(\frac{e\pi f|\mathbf{r}|/c}{N+1}\right)^N \tag{30}$$

$0 \le \acute{f} \le 2W$ and $F - W \le f \le F + W$ $\qquad\blacksquare$

*Proof:* [Lemma 2] Define $x = eck t$ and $y = ekR$ then using (30)

$$\frac{\epsilon_{N+\alpha, P+\delta}}{\epsilon_{N,P}} = \left(\frac{P+1}{P+1+\delta}\right)^{P+\delta} \left(\frac{N+1}{N+1+\delta}\right)^{N+\alpha}$$
$$\times \left(\frac{x}{P+1}\right)^{\delta} \left(\frac{y}{N+1}\right)^{\alpha}$$

Now $\alpha, \delta \ge 1$, $P+1 > x$ and $N+1 > y$ by definition

$$\frac{\epsilon_{N+\alpha, P+\delta}}{\epsilon_{N,P}} \le \left(\frac{P+1}{P+\delta}\right)^{P+\delta} \left(\frac{N+1}{N+\delta}\right)^{N+\alpha} \le e^{1-\delta} e^{1-\alpha}$$

Using the identity $(1 + a/N)^N < e^a$ $\qquad\blacksquare$

*Proof:* [Theorem 1] From (30) $P$ may found directly by using the maximum values of $t$ and $\acute{k}$:

$$P = e\pi 2WT \tag{31}$$

Truncating (30) at $P$ gives $\mathcal{D}_T = 2e\pi WT + 1$ terms. The same technique cannot be used for $N$, since the bound becomes excessively loose. Instead we use a geometric argument:

Consider Figure 3. The dotted vertical lines give the frequency constraints, while the top line $n = e\pi Rf/c$ gives the spatial constraint of (7). The constraint set defines a trapezium in the space-frequency plane (or trapezoid when time is included). The heights of the trapezium are found from (7) with $f = F \pm W$. Note that the figure is not drawn to scale: for any reasonable value of $R$, the slope of the line $e\pi R/c$ is almost zero.

Each horizontal dashed line represents a collection of time-frequency signals which may be observed at a point in space. Spatial diversity allows observation of multiple collections. Each collection has an intrinsic dimensionality of $e\pi 2\Delta_W T$ where $\Delta_W$ is the effective (frequency) bandwidth of the spatial
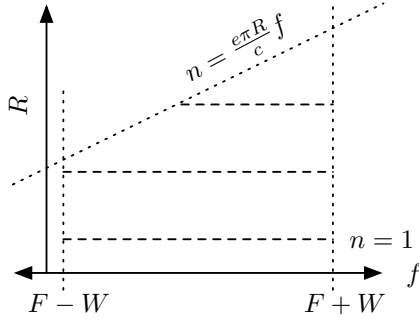
Fig. 3. Geometry for spatial functions. $F - W \leq f \leq F + W$ and $0 \leq |\mathbf{r}| \leq R$. Time forms a third dimension (into the page).

observation. The dimensionality result is obtained by counting each collection, with appropriate dimensionality for each.

The collections are enumerated by the spatial degrees of freedom $n$. Each collection is scaled by the spherical Bessel function $j_n(k|\mathbf{r}|)$ from (22). These functions have a natural high-pass characteristic: for $k|\mathbf{r}| < n$ then $j_n(k|\mathbf{r}|) \to 0$. At low values of $n$,

$$0 \leq n \leq e\pi R(F - W)/c = N_0 \qquad (32)$$

each Bessel function is already active and thus each collection has the full $e\pi 2\Delta_W T = e\pi 2WT$ degrees of freedom. For higher values of $n$,

$$N_0 = e\pi R(F - W)/c < n \leq e\pi R(F + W)/c = N_1 \quad (33)$$

each collection has a reduced DoF, since the Bessel functions are only activated part-way through the frequency band.

$$N_{TW(n)} = \begin{cases} 2e\pi WT + 1 & 0 \leq n < N_0 \\ e\pi \left( F + W - \frac{c}{e\pi R}n \right)^+ T + 1 & N_0 \leq n \leq N_1 \end{cases}$$

For each value of $n$ in Figure 3, there are $2n+1$ independent spatial modes. Thus the total degrees of freedom is given by

$$\mathcal{D} = \sum_{n=0}^{N_1} N_{TW(n)}(2n + 1) \leq \mathcal{D}_1 + \mathcal{D}_2 \qquad (34)$$

$$\mathcal{D}_1 = (2e\pi WT + 1) \sum_{n=0}^{N_0} (2n + 1) \qquad (35)$$

$$\mathcal{D}_2 = \sum_{n=N_0}^{N_1} \left[ e\pi(F + W)T - \frac{cT}{R}n + 1 \right](2n + 1) \qquad (36)$$

We may solve (35):

$$\mathcal{D}_1 = (2e\pi WT + 1) \left( \frac{e\pi R}{c}(F - W) + 1 \right)^2, \qquad (37)$$

Evaluating (36) and combining with (37) gives the result. ∎

*Proof:* [Lemma 3] We are considering bandlimited electromagnetic signals, in this case $W$ may be decomposed into a band-limiting projection $B$ [11] and a spatial wavefield (or Helmholtz) projection $G$ [8].

$$Bf(\mathbf{r}, t) = \frac{1}{2\pi} \int_{F-W}^{F+W} d\omega \, e^{\iota\omega t} \int_{-\infty}^{\infty} dt \, e^{-\iota\omega t} f(\mathbf{r}, t) \qquad (38)$$

$$Gf(\mathbf{r}, t) = \sum_{\mathbf{k}} f_{\mathbf{k}} \exp\left( \iota\mathbf{k} \cdot \mathbf{r} \right) \qquad (39)$$

Write (19) as:

$$g(\mathbf{r}, t) = GBDf(\mathbf{r}, t) \qquad (40)$$

We know [11] the operator $BD$ is compact, since it maps a unit ball in $\mathcal{L}_2$ (finite energy signals) to an essentially finite dimensional ball (of approximate dimension $2WT + 1$). From [4] $G$ is a projection and thus bounded so [17, Lem.8.3-2 p.422] the product $G \cdot BD$ is compact. ∎

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