INTRINSIC FINITE DIMENSIONALITY OF RANDOM MULTIPATH FIELDS

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ABSTRACT

We study the dimensions or degrees of freedom of random multipath fields in wireless communications. Random multipath fields are presented as solutions to the wave equation in an infinite-dimensional vector space. We prove a universal bound for the dimension of random multipath field in the mean square error sense. The derived maximum dimension is directly proportional to the radius of the two-dimensional spatial region where the field is coupled to. Using the Karhunen-Loeve expansion of multipath fields, we prove that, among all random multipath fields, isotropic random multipath achieves the maximum dimension bound. These results mathematically quantify the imprecise notion of rich scattering that is often used in multiple-antenna communication theory and show that even the richest scatterer (isotropic) has a finite intrinsic dimension.

1. INTRODUCTION

1.1. Motivation and Problem Statement

Wireless communications use *space* as the physical medium for information transfer. Recently, multiple-sensor wireless communications, such as multiple-input multiple-output (MIMO) systems have been proposed to use space as a promising resource of diversity and capacity enhancement [1, 2]. In the capacity analysis of MIMO systems, the fundamentally limiting factor of spatial diversity is attributed to the minimum number of sensors (antennas) and their spacing [1, 3]. However, array sensors are artifacts of implementation, rather than intrinsic limiting factors [4]. It is gradually becoming clear that the extent to which reliable communication is feasible in space is primarily constrained by the dynamics of wave propagation [4–6], and the sensor setup is a secondary issue.

In wireless communication systems, the unknown multipath field is modelled as a random process. The predicted capacity enhancement of multiple-sensor wireless communications usually hinges on the imprecise assumption of *rich* multipath scattering and the associated independence of elements in the MIMO channel matrix. However, the term *rich scattering* needs to be refined and quantified based on the dynamics of multipath field random process and wave equation. For example, [7] studied the degrees of freedom in MIMO channels and their effects on the capacity of MIMO systems, based on a virtual MIMO channel representation.

In this paper, we aim to find the intrinsic limits on the dimensions or degrees of freedom for random multipath fields when they are observed in, or coupled to a region of space. This region of space is where multiple sensors may be potentially located. This analysis provides a mathematical measure for random scattering richness. The practical significance of studying the dimensionality of a random multipath field is the fundamental limit that it poses on the capacity of MIMO systems even with a large number of antennas.

1.2. Approach and Contributions

To find the dimensions of random multipath fields, we detach ourselves from the conventional matrix representation of multiple-antenna channels [1, 2]. Instead, we study multipath fields as the *functional* solutions to the wave equation [8]. As a result, the multipath field is represented in a countable, infinite-dimensional, and linear vector space, where vectors consist of functions.

The advantage of the functional wave representation is that 1) it is general enough to be applied for *any* narrowband multipath environment, regardless of the number or nature of multipath sources, 2) it accommodates representation of random multipath fields with a general spatial correlation function [9], and 3) it allows us to determine the effective number of dimensions (in the infinite-dimensional functional space) that essentially contribute to the coupling of multipath fields to a spatial region.

In this paper, we extend the results in [4–6] to explicitly consider the random nature of multipath fields. The main contributions of the paper are summarized as:

1. We prove that a random multipath field has an intrinsic maximum dimension in the mean square error (MSE) sense. This dimension is directly related to the radius of the two-dimensional (2-D) spatial region where the field is coupled to. Therefore, the dimension of the spatial region is a key factor in defining the degrees of freedom or richness of multipath.
2. Using the Karhunen-Loeve expansion of multipath fields, we prove that only isotropic uniform scattering achieves the upper bound on multipath dimension and hence, is the richest scatterer in the minimum mean square error (MMSE) sense. We show that random multipath fields with limited angle of arrival have smaller dimensions than the isotropic case.
3. As a by-product of our analysis, we provide a systematic numerical algorithm to find the eigenvalues of the multipath spatial correlation function. Unlike the Fredholm equation method [10, 11], the proposed numerical technique does not require selecting quadrature points.

In light of intrinsic finite dimensionality of random multipath fields (even isotropic multipath), the asymptotic analysis and predictions on MIMO capacity enhancements should be cautiously interpreted. In particular, increasing the number of antennas beyond the dimension of multipath field would be ineffectual.

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2. PRESENTATION OF RANDOM MULTIPATH FIELDS

In order not to obscure the approach, we present the main results for a narrowband, 2-D, random multipath farfield. Our approach is generalizable to three-dimensional (3-D) random multipath fields in a straightforward manner.

Let $\mathbf{x} \in \mathbb{R}^2$ represent a vector in 2-D space, with $\|\mathbf{x}\|$ denoting its Euclidean distance from the origin and $\widehat{\mathbf{x}} = \mathbf{x}/\|\mathbf{x}\|$ denoting the unit vector in the direction of $\mathbf{x}$. In polar format, $\mathbf{x} = (\|\mathbf{x}\|, \phi(\mathbf{x}))$.

Let $F(\mathbf{x})$ denote a finite, complex-valued, and narrowband multipath field in a region of interest $\mathbf{x} \leqslant R$, for a finite range R. It is assumed that the sources that are generating the field are external to the region. The multipath field $F(\mathbf{x})$ satisfies the linear partial differential equation, also known as Helmholtz wave equation [8]

$$\triangle F(\mathbf{x}) + k^2 F(\mathbf{x}) = 0, \tag{1}$$

where $\triangle$ denotes Laplacian, $k \triangleq 2\pi/\lambda$ is the wave number and λ is the wavelength. The solution to (1) may be written in two equivalent formats. First, multipath field $F(\mathbf{x})$ is written as a superposition of plane waves from all azimuth directions ϕ. That is

$$F(\mathbf{x}) = \int_0^{2\pi} a(\phi) e^{ik\mathbf{x}.\widehat{\mathbf{y}}} d\phi, \tag{2}$$

where $a(\phi)$ is the complex-valued and *random* gain of scatterers as a function of the direction of arrival ϕ and $\widehat{\mathbf{y}} \triangleq (1, \phi)$. The alternative presentation of multipath field $F(\mathbf{x})$ is given as [8]

$$F(\mathbf{x}) = \sum_{m=-\infty}^{+\infty} i^m \alpha_m J_m(k\|\mathbf{x}\|) \frac{e^{im\phi(\mathbf{x})}}{\sqrt{2\pi}}, \tag{3}$$

where $J_m(\cdot)$ is the m^{th} order Bessel function of the first kind and α_m is the m^{th} order Fourier series coefficient of the gain $a(\phi)$ in (2). The advantage of (3) over (2) is that the multipath field $F(\mathbf{x})$ in (3) is represented as a countable sum of complete orthonormal basis functions $i^m e^{im\phi(\mathbf{x})}/\sqrt{2\pi}$. This allows analyzing the effects of wave field truncation and its essential dimensionality. The basis set in (3) is synthesizing the field $F(\mathbf{x})$ over the circle with radius $\|\mathbf{x}\|$. In Section 3, we show that this presentation is enough for bounding the field truncation MSE over a 2-D disc with radius R.

The actual setting of scatterers that generate the multipath field $F(\mathbf{x})$ is usually unknown. Therefore, it is reasonable to represent multipath field $F(\mathbf{x})$ as a random process. Referring to (2), the scattering gain $a(\phi)$ is random and so is α_m in (3). We assume uncorrelated scattering, which means that the random gains $a(\phi)$ and $a(\phi')$ at two distinct incident angles are uncorrelated from each other and the angular power spectrum is defined as

$$\mathcal{E}\left\{a(\phi)a^*(\phi')\right\} \triangleq \left\{ \begin{array}{cc} P(\phi) & \phi = \phi' \\ 0 & \text{otherwise} \end{array} \right., \tag{4}$$

where $\mathcal{E}\{\cdot\}$ and $*$ denote expectation and complex conjugate, respectively. Using (4) and following a few intermediate steps, the cross-correlation of α_m and α_n in (3) is derived as

$$\mathcal{E}\left\{\alpha_m \alpha_n^*\right\} = \gamma_{m-n}, \tag{5}$$

where γ_{m-n} is the $(m-n)^{th}$ Fourier series coefficient of the angular power spectrum $P(\phi)$. In [9], the spatial correlation function of multipath fields $F(\mathbf{x}_1)$ and $F(\mathbf{x}_2)$ at points $\mathbf{x}_1$ and $\mathbf{x}_2$ was given as

$$\begin{aligned} \rho(\mathbf{x}_2 - \mathbf{x}_1) &= \int_0^{2\pi} P(\phi) e^{ik(\mathbf{x}_2-\mathbf{x}_1).\widehat{\mathbf{y}}} d\phi \\ &= \sum_{m=-\infty}^{+\infty} i^m \gamma_m J_m(k\|\mathbf{x}_2 - \mathbf{x}_1\|) \frac{e^{im\phi_{21}}}{\sqrt{2\pi}}, \end{aligned} \tag{6}$$

where ϕ_{21} is the direction of the vector that connects $\mathbf{x}_1$ to $\mathbf{x}_2$.

3. DIMENSIONALITY OF RANDOM MULTIPATH FIELDS

In this section, we prove a universal bound on the dimensionality of random multipath fields in the MSE sense. Jones *et. al.* in [4] proved that the truncation of multipath field $F(\mathbf{x})$ in (3) to $L = N + \Delta$ terms $\widehat{F}_L(\mathbf{x})$ $(\Delta \geqslant 0)$

$$\widehat{F}_L(\mathbf{x}) = \sum_{m=-L}^{L} i^m \alpha_m J_m(k\|\mathbf{x}\|) \frac{e^{im\phi(\mathbf{x})}}{\sqrt{2\pi}}, \tag{7}$$

results in an exponentially decaying normalized error

$$\varepsilon_L(\mathbf{x}) \triangleq \frac{\left|F(\mathbf{x}) - \widehat{F}_L(\mathbf{x})\right|}{\|a\|} \leqslant 0.1613 e^{-\Delta}, \tag{8}$$

provided that N is chosen as $N = \lceil e\pi\|\mathbf{x}\|/\lambda\rceil$, $\lceil\cdot\rceil$ being the integer ceiling function. Therefore, it was concluded that the essential dimensionality of $F(\mathbf{x})$ is $2N + 1 = 2\lceil e\pi\|\mathbf{x}\|/\lambda\rceil + 1$ and is directly related to the radius of the region where the field is coupled to.

Since the multipath field is a random process, we wish to compute the normalized mean square truncation error over a 2-D disc with radius R, when the field in (3) is truncated to N terms

$$\bar{\varepsilon}_N(R) = \frac{\int_0^R \int_0^{2\pi} \mathcal{E}\left\{\left|F(\mathbf{x}) - \widehat{F}_N(\mathbf{x})\right|^2\right\} d\phi r dr,}{\int_0^R \int_0^{2\pi} \mathcal{E}\left\{|F(\mathbf{x})|^2\right\} d\phi r dr}, \tag{9}$$

where $r = \|\mathbf{x}\|$. After a few steps and using the orthogonality of basis functions in (3), the MSE in (9) is simplified to

$$\bar{\varepsilon}_N(R) = \frac{\int_0^R 2\sum_{m=N+1}^{+\infty} J_m^2(kr) r dr}{1/2R^2}. \tag{10}$$

The following theorem provides a universal upper bound for the MSE in (10).

Theorem 1 - Universal MSE Upper Bound for the Truncation of 2-D Random Multipath Field: *The 2-D random multipath field in* (3) *can be truncated to $L = N + \Delta$ terms with a normalized truncation MSE in* (9) *bounded by*

$$\bar{\varepsilon}_L(R) \leqslant 0.0093 e^{-2\Delta}, \tag{11}$$

provided that N is chosen as $N = \lceil e\pi R/\lambda\rceil$ and $\Delta \geqslant 0$.□

Before we proceed to the proof, we note that in the derivation of multipath truncation MSE in (9)-(11), we did not assume anything about the multipath spatial correlation function $\rho(\mathbf{x}_2 - \mathbf{x}_1)$ or equivalently the angular power spectrum $P(\phi)$ in (6). Therefore, (11) is a universal bound on multipath truncation MSE, regardless of multipath scattering spatial correlation. Therefore, no matter how *rich* the scatter is, the field can be truncated to $2N + 1$ terms in (7) with a very small normalized MSE. This is summarized as the following upper bound on the essential dimension of random multipath fields.

Corollary 1.1 - Universal Bound on the Dimension of 2-D Random Multipath Fields: *The dimension of a 2-D random multipath field (in the MSE sense) coupled to a disc of radius R is bounded by*

$$2N + 1 = 2\lceil e\pi R/\lambda\rceil + 1. \tag{12}$$

Proof of Theorem 1: First, we prove that, in general, the following summation is bounded by

$$S_M(r) \triangleq 2 \sum_{m=M+1}^{+\infty} J_m^2(kr) \leqslant 0.0093, \tag{13}$$

if M is chosen as $M = \lceil e\pi r/\lambda \rceil$. To this end, we upper bound each term in (13) by using an upper bound for Bessel functions [12]. Defining $z \triangleq kr/2$, (13) is bounded as

$$S_M(r) = 2 \sum_{m=M+1}^{+\infty} J_m^2(2z) \leqslant 2 \sum_{m=M+1}^{+\infty} \frac{z^{2m}}{(m!)^2}. \tag{14}$$

By changing the summation variable $n = m - (M+1)$, we obtain

$$S_M(r) \leqslant 2 \frac{z^{2(M+1)}}{((M+1)!)^2} \sum_{n=0}^{+\infty} \frac{((M+1)!)^2}{((n+M+1)!)^2} z^{2n}. \tag{15}$$

We upper bound the coefficients in the summation to obtain

$$\sum_{n=0}^{+\infty} \frac{((M+1)!)^2}{((n+M+1)!)^2} z^{2n} \leqslant \sum_{n=0}^{+\infty} \frac{z^{2n}}{(M+2)^{2n}} = \frac{1}{1 - \frac{z^2}{(M+2)^2}}, \tag{16}$$

where the equality is written by assuming $M+2 > z$ for the sum of geometric series. Now, we use the Stirling bound, $((M+1)!)^2 > 2\pi(M+1)(M+1)^{2(M+1)}e^{-2(M+1)}$, to write

$$S_M(r) \leqslant \frac{1}{\pi(M+1)} \left(\frac{ez}{M+1}\right)^{2(M+1)} \frac{1}{1 - \frac{z^2}{(M+2)^2}}. \tag{17}$$

Now, using the exponential inequality $\left(1 + \frac{ez}{M+1}\right)^{2(M+1)} < e^{2ez}$, (17) may be bounded as

$$S_M(r) \leqslant \frac{1}{\pi(M+1)} e^{2(ez-M-1)} \frac{1}{1 - \frac{z^2}{(M+2)^2}}. \tag{18}$$

If we choose $ez \leqslant M = \lceil ez \rceil = \lceil e\pi r/\lambda \rceil < ez + 1$, we obtain

$$S_M(r) \leqslant \frac{0.0498}{(M+1)}. \tag{19}$$

We can further examine $S_M(r)$ at critical points of the ceiling function ($z = k/e$, $k \in \mathbb{N}$) to obtain a universal upper bound

$$S_M(r) \leqslant 0.0093. \tag{20}$$

Based on (20), it is also concluded that $\bar{\varepsilon}_N(R) \leqslant 0.0093$, if for each summation in (10) we choose $N = \lceil e\pi R/\lambda \rceil \geqslant \lceil e\pi r/\lambda \rceil$.

It is also possible to show that for $\Delta' \geqslant 0$, $M = \lceil ez \rceil$, and $M+2 > z$, the following ratio holds

$$\frac{S_{M+\Delta'}(r)}{S_M(r)} \leqslant e^{-2\Delta'}. \tag{21}$$

This inequality carries over to the MSE in (10) by choosing $L = N + \Delta$, where $N = \lceil e\pi R/\lambda \rceil$ corresponds to the radius of the 2-D disc R. This completes the proof. □

4. KARHUNEN-LOEVE EXPANSION OF RANDOM MULTIPATH FIELDS

In Section 3, we found an upper bound on the MSE of multipath truncation when the field is represented by the natural choice of exponential orthogonal basis in (3). In this section, we aim to find the *optimal* representation of the random field $F(\mathbf{x})$ and bring the field's spatial correlation into the picture. The optimal field representation results in the MMSE for the field truncation and reveals its true dimensionality. It is known that the Karhunen-Loeve (KL) expansion is the unique, MMSE optimal expansion of a random process [13]. Based on the KL theory, the optimal expansion of a random multipath field $F(\mathbf{x})$ may be written as

$$F(\mathbf{x}) = \sum_{m=0}^{+\infty} \sqrt{\mu_m} \beta_m g_m(\mathbf{x}), \tag{22}$$

where β_m is a sequence of uncorrelated (white) random variables with unit variance, μ_m and $g_m(\mathbf{x})$ are the m^{th} eigenvalue and eigenfunction of the spatial correlation function $\rho(\mathbf{x}_2 - \mathbf{x}_1)$, respectively.

The major difference between the orthogonal representations of the random multipath field in (3) and (22) is that the random variables α_m in (3) may be correlated with covariance coefficients given in (5), whereas all correlation is taken away from β_m in (22).

The truncation of multipath field $F(\mathbf{x})$ to $2N+1$ terms in (22) results in the MMSE that is already bounded by the derived MSE in Theorem 1 for $N \geqslant \lceil e\pi R/\lambda \rceil$.

Now, we consider two special cases of isotropic uniform scattering and a single plane wave (unidirectional power spectrum).

Special Case 1 - Isotropic Uniform Scattering: In isotropic scattering, multipath is uniformly coming from all directions. In this case, angular power spectrum is $P(\phi) = 1/2\pi$ for $\phi \in [0, 2\pi)$. Using (5), we conclude that α_m in (3) is already an uncorrelated random sequence. Since the KL expansion is unique [13] and by comparing (3) and (22), we conclude that

$$g_m(\mathbf{x}) = e^{im\phi(\mathbf{x})}/\sqrt{2\pi}$$

is the orthonormal eigenfunction of $\rho(\mathbf{x}_2 - \mathbf{x}_1)$ and

$$\mu_m = 2\pi J_m^2(k\|\mathbf{x}\|)$$

is its eigenvalue. Therefore, the bound in (10) is, in fact, the MMSE bound for the truncation of isotropic multipath field. It is concluded that isotropic scattering is the *richest* random multipath and its dimension achieves the upper bound in Corollary 1.1.

Special Case 2 - Unidirectional Power Spectrum: A single path may be considered as the least random type of scattering. In this case, the angular power spectrum is $P(\phi) = \delta(\phi - \phi_0)$ for some $\phi_0 \in [0, 2\pi)$. It is easier to determine the eigenfunction of $\rho(\mathbf{x}_2 - \mathbf{x}_1)$ directly from its integral definition in (6). To see this, we write

$$\begin{aligned} \rho(\mathbf{x}_2 - \mathbf{x}_1) &= \int_0^{2\pi} \delta(\phi - \phi_0) e^{ik(\mathbf{x}_2 - \mathbf{x}_1)\cdot\hat{\mathbf{y}}} d\phi \\ &= e^{ik(\mathbf{x}_2 - \mathbf{x}_1)\cdot\hat{\mathbf{y}}_0} = e^{ik\mathbf{x}_2\cdot\hat{\mathbf{y}}_0} e^{-ik\mathbf{x}_1\cdot\hat{\mathbf{y}}_0}. \end{aligned} \tag{23}$$

Therefore, $e^{ik\mathbf{x}\cdot\hat{\mathbf{y}}_0}/\sqrt{2\pi}$ is the single eigenfunction and the associated eigenvalue is 2π. Hence, the dimension of a single plane wave is 1.

Based on the above discussions, we obtain the following conclusion:

Dimensionality of Random Multipath Fields with Restricted Angle of Arrival: *The dimension of a 2-D random multipath field (in the MMSE sense), which is coupled to a disc of radius R is upper bounded by* $2\lceil e\pi R/\lambda \rceil + 1$ *and lower bounded by 1.*

5. A SYSTEMATIC NUMERICAL METHOD FOR EIGENVALUE CALCULATION

In Section 2 and Section 4, it was observed that the multipath field $F(\mathbf{x})$ may be represented in two different orthonormal bases in (3) and (22). These representations are equivalent:

$$\begin{aligned} F(\mathbf{x}) &= \sum_{m=-\infty}^{+\infty} i^m \alpha_m J_m(k\|\mathbf{x}\|) \frac{e^{im\phi(\mathbf{x})}}{\sqrt{2\pi}} \\ &= \sum_{m=0}^{+\infty} \sqrt{\mu_m}\beta_m g_m(\mathbf{x}). \end{aligned} \tag{24}$$

It is evident from (24) that the change of orthonormal bases from $i^m e^{im\phi(\mathbf{x})}/\sqrt{2\pi}$ to $g_m(\mathbf{x})$ *whitens* the random sequence $\alpha_m J_m(k\|\mathbf{x}\|)$ to obtain the random sequence $\sqrt{\mu_m}\beta_m$ with variance μ_m. Here, we assume that $r = \|\mathbf{x}\|$ is fixed and only the eigenvalues of $\rho(\mathbf{x}_2 - \mathbf{x}_1)$ over the circle with radius $r = \|\mathbf{x}\|$ are to be evaluated. Therefore, we propose the following algorithm to obtain systematic numerical estimates of the eigenvalues μ_m.

Algorithm 1 - Systematic Numerical Calculation of μ_m:

1. *Truncate the random sequence $\alpha_m J_m(kr)$ in (24). Based on [4] and our results, choose the truncation length $N \geqslant \lceil e\pi r\lambda \rceil$. Therefore, the random vector*

$$\mathbf{v} = \begin{bmatrix} \alpha_{-N}J_{-N}(kr) & \cdots & \alpha_N J_N(kr) \end{bmatrix}^T$$

   is obtained (T denotes matrix transpose).

2. *Using (5), form the covariance matrix for the random vector $\mathbf{v}$ defined as $\mathbf{\Gamma} = \mathcal{E}\{\mathbf{v}\mathbf{v}^*\}$.*
3. *The eigenvalues of the covariance matrix $\mathbf{\Gamma}$ are estimates of the first $2N+1$ eigenvalues of the covariance function $\rho(\mathbf{x}_2 - \mathbf{x}_1)$.*

Unlike the quadrature-based numerical technique that solves the Fredholm eigenvalue equation [10,11], Algorithm 1 does not require selecting quadrature points. In Fig. 1 we have computed the eigenvalues of spatial correlation function $\rho(\mathbf{x}_2 - \mathbf{x}_1)$ using Algorithm 1. The angular power spectrum in (6) is $P(\phi) = 1/2\Delta$, which is nonzero for $\phi \in [-\Delta, \Delta]$. Δ varies from 0.1π (highly directional scattering) to π (isotropic scattering). The radius is $r = \|\mathbf{x}\| = 0.5\lambda$. It is verified that the number of significant eigenvalues decreases with decreasing the angular range of the angular power spectrum $P(\phi)$.

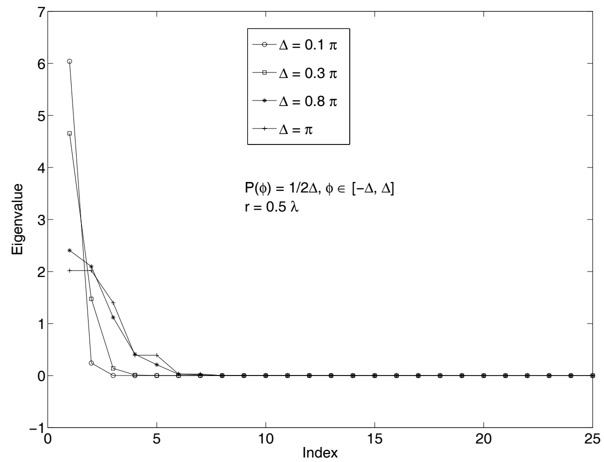

Fig. 1. The eigenvalues of spatial correlation function $\rho(\mathbf{x}_2 - \mathbf{x}_1)$ are computed using Algorithm 1. The number of significant eigenvalues and hence, multipath richness decreases with decreasing the angular range of the angular power spectrum $P(\phi)$.

6. CONCLUSIONS

We proved that all random multipath fields have an intrinsic universal finite dimension in the MSE sense, which is imposed by the dynamics of wave propagation in space. This dimension is directly related to the radius of the spatial region where the field is coupled to. The derived maximum dimension of multipath fields is achieved by isotropic scattering. All other random multipath fields have smaller degrees of freedom. We also presented a systematic numerical algorithm to calculate the eigenvalues of the spatial correlation function.

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