On non-coherent Rician fading channels with average and peak power limited input

*Rasika R. Perera, Tony S. Pollock, and Thushara D. Abhayapala**
Department of Information Engineering
Research School of Information Sciences and Engineering
The Australian National University, ACT 0200, AUSTRALIA
rasika.perera@anu.edu.au

***Abstract*— Communication over a discrete time-varying Rician fading channel is considered, where neither the transmitter nor the receiver has the knowledge of the channel state information (CSI) except the fading statistics. We provide closed form expressions to the mutual information under both average and peak input power constraint when its distribution is complex Gaussian. The results are compared with the existing capacity results showing the importance of the perfect CSI at the receiver at moderate and high signal to noise ratio (SNR). Furthermore, we show that the peak input power constraint gives better mutual information compared to the average power constraint input in Rician fading. Also, we show the information loss with a Gaussian distributed input compared to channel capacity is negligible at low SNR. Hence, there is no penalty for not knowing the channel perfectly at the receiver in the low SNR regime.**

***Index Terms*— Channel capacity, mutual information, Rician fading, Gaussian-quadrature, differential entropy**

I. INTRODUCTION

Optimal information transfer over fading channels is a prime challenge in wireless communications. The knowledge of channel capacity and the optimal input distribution in fading channels motivates to develop powerful codes such as Turbo codes which enable us to operate near Shannon limits. Unlike the non-fading channel [1], and coherent Rayleigh fading channels [2]–[5], finding the capacity and optimal input distribution of non-coherent fading channels is considered as a difficult problem in the literature.

The independent and identically distributed (i.i.d.) Gaussian is the capacity achieving input distribution in non-fading channel [1], Rayleigh fading channel with CSI [6]. Furthermore, white complex Gaussian input achieves the capacity in coherent Rician fading channel [7]. However, the optimal input distribution of the non-coherent fading channel is not Gaussian. In this paper, we consider information transfer over a non-coherent Rician fading channel when the input distribution is complex Gaussian. We show the mutual information under both average and peak power limited Gaussian inputs in closed form and compare with channel capacity. Our results prove that the channel knowledge at low SNR does not provide a significant capacity improvement.

Communication over a non-coherent Rayleigh fading channel with average input power constraint is initially considered by Richters [8] in which he conjectured that the capacity achieving input distribution is discrete with a finite number of mass points. Richters's work is extended by Abou-Faycal [9] with a meticulous proof for the optimal input from which the capacity is computed numerically using convex optimisation. The channel capacity is found numerically identifying the optimal number of discrete inputs, their probabilities, and locations which satisfy the Kuhn Tucker condition for optimality for a given input power. Extending this work, Gursoy, Poor and Verdu [10] considered the communication over non-coherent Rician fading channels under both average and peak power limited inputs. It is shown that in both cases, the input distribution is discrete with a finite number of mass points. Similar to [9], numerical capacity results for Rician channel is found in [10].

We deploy the discrete input shown in [10] for capacity analysis to compare with mutual information obtained when the input distribution is complex Gaussian. The integrals are presented in closed form using the Gaussian quadrature where abscissas are precisely the roots of corresponding orthogonal polynomial and weighting functions which normalise orthogonal functions [11].

II. RICIAN MODEL

We consider the following time-varying memoryless Rician fading channel model

$$y = mx + ax + n. \tag{1}$$

Random variables a, n which represent the channel fading and noise respectively are i.i.d. *circular symmetric*[1] zero mean complex Gaussian. It is assumed that both a, n are independent of each other and of the input with a unit variance (i.e. the variance in each dimension is 1/2). m is the deterministic complex constant. x and y denote the channel input and output. In (1), the time index is omitted for simplicity.

The Rician model (1) is appropriate when there is a line of sight (LOS) component present in addition to the multipath signals. In such a situation, random multipath components arriving at different angles are superimposed on a stationary dominant signal, having the effect of adding a dc component to the random multipath. Further, the Rician model includes both unfaded Gaussian channel and the Rayleigh fading channel as two special cases.

T. S. Pollock and T. D. Abhayapala have appointments with National ICT Australia (NICTA).

[1] The distribution of a complex variable ω is said to be circularly symmetric if for any deterministic $-\pi \leq \theta \leq \pi$, the distribution of random variable $e^{j\theta}\omega$ is identical to the distribution of ω.

In (1), the fading is assumed to be flat. This assumption is valid when the channel has constant gain and linear phase response over a bandwidth which is greater than the bandwidth of the transmitted signal. Also it is assumed that the fading is independent at every symbol period, hence the channel coherence time is one. In fast fading systems or with a mobile receiver, the channel estimation may become difficult through the training sequence. Hence, we consider the non-coherent case where both the transmitter and receiver has no CSI except the fading statistics.

We use random variables $x \in X$ and $y \in Y$ to represents specific values of the channel input and output respectively. Also we use $h(\cdot)$ for differential entropy, $I(\cdot;\cdot)$ for mutual information and $\mathrm{Ei}(\cdot)$ for the exponential integral.

III. Peak power limited Rician channel

A. Mutual Information

Consider the non-coherent Rician fading channel with the peak power constraint input $|x|^2 \leq P_p$. The mutual information of channel model (1) is given by

$$I(X;Y) = \int_{\mathbb{C}}\int_{\mathbb{C}} f_{Y|X}(y|x)f_X(x) \times \log\left[\frac{f_{Y|X}(y|x)}{\int_{\mathbb{C}} f_{Y|V}(y|v)f_V(v)dv}\right] dxdy, \tag{2}$$

where the conditional probability density function (pdf) of the output given input [12]

$$f_{Y|X}(y|x) = \frac{1}{\pi(|x|^2+1)}\exp\left(\frac{-|y-mx|^2}{|x|^2+1}\right). \tag{3}$$

In (2), $\mathbb{C}$ denotes the complex domain. Since the channel output y has a finite second moment, we can find the differential entropy $h(y|x)$.

***Lemma** 1: Differential entropy of the output given input in the channel model* (1) *is given by*

$$h(Y|X) = \mathrm{E}_x\{\log[\pi e(1+|x|^2)]\} \ \geq \ \log(\pi e). \tag{4}$$

Proof: See Appendix VII-A. □

Jensen's inequality [13] provides the upper bound $h(Y|X) \leq \log[\pi e(1+P_a)]$ where P_a is the average power constraint at the input. Unlike Rayleigh fading [14], Rician fading carry the phase information when the channel is unknown. Therefore, phase carry information for $|m| > 0$. We use the polar coordinates

$$x = ue^{j\phi}, \quad y = ve^{j\psi} \quad u,v \geq 0 \quad \phi,\varphi \in [-\pi,\pi) \tag{5}$$

where u, ϕ are the input magnitude and phase respectively and v, φ are the magnitude and phase of the received signal. The following result is shown in [10].

***Theorem** 1: Mutual information* (2) *of the non-coherent Rician fading channel is bounded by*

$$I(X;Y) \leq h(v) + \log(2\pi) + \int_0^\infty f_V(v;F_U)\log v dv - \int_0^\infty \log[\pi e(1+u^2)]dF_U(u) \tag{6}$$

with equality for independent v and ϕ with uniformly distributed ϕ. Further, the upper bound in (6) *is achieved by independent u and ϕ with ϕ uniformly distributed.*

$F_U(\cdot)$ is the cumulative distribution function of input magnitude of the channel. Define

$$r = v^2, \quad s = u \quad r,s \geq 0, \tag{7}$$

where $h(S) = h(U)$, $s \in S$ and $u \in U$. The output differential entropy is given by

$$h(R) = h(V) + \int_0^\infty f_V(v)\log(2v)dv, \tag{8}$$

where $r \in R$ and $v \in V$. We use the upper bound in (6) with ϕ uniformly distributed as the mutual information. Substituting $h(V)$ from (8) in (6) we get

$$I(X;Y) = h(Y) - h(Y|X) = h(R) - h(R|S) \tag{9a}$$

$$= h(R) - \int_0^\infty \log(1+s^2)dF_S(s) - 1. \tag{9b}$$

The new output differential entropy

$$h(R) = -\int_0^\infty \left\{\int_0^\infty f_{R|S}(r|s)dF_S(s)\right\} \times \log\left\{\int_0^\infty f_{R|S}(r|s)dF_S(s)\right\} dr \tag{10}$$

can be found with the knowledge of $f_{R|S}(r|s)$. This conditional pdf of the Rician channel output given the input is given by [12]

$$f_{R|S}(r|s) = \frac{1}{(1+s^2)}\exp\left(-\frac{R+Ks^2}{1+s^2}\right) I_0\left(\frac{2\sqrt{K}s\sqrt{R}}{1+s^2}\right) \tag{11}$$

where $K = |m|^2/\mathrm{E}\{|a|^2\} = |m|^2$ is the Rician factor.

B. Non-Coherent Rician Capacity

Using mutual information derived in (9a), we can pose the channel capacity

$$C_{\text{peak}} = \sup_{\substack{F_S(\cdot) \\ |s|^2 \leq P_p}} I(S;R) \tag{12}$$

which depends on the Rician factor K and the input power constraint P_p. The following theorem is given [12] for the capacity.

***Theorem** 2: For the non-coherent Rician fading channel* (1) *where the input is subject to a peak power constraint $|x|^2 \leq P_p$, uniformly distributed input phase that is independent of the amplitude is optimal and the capacity achieving amplitude distribution is discrete with a finite number of mass points.*

Using theorem 2, the mutual information for the discrete input $f_S(s) = \sum_{i=1}^N p_i\delta(s-s_i)$ with probabilities p_i,

$$I_{\text{peak}}(S;R) = -\sum_{i=1}^N \int_0^\infty p_i f_{R|S}(r|s_i)\log\left[\sum_{i=1}^N p_i f_{R|S}(r|s_i)\right] dr - \sum_{i=1}^N p_i\log(1+{s_i}^2) - 1. \tag{13}$$

The channel capacity

$$C_{\text{peak}} = \sup_{|s|^2 \leq P_p} I_{\text{peak}}(S;R) \tag{14}$$

can be computed numerically using the Kuhn Tucker [15] condition for optimality.

C. Mutual Information with Gaussian Input

The capacity achieving input distribution is Gaussian in channel (1) when the CSI is perfectly known at the receiver only or at both the transmitter and receiver [7]. Since the optimal input distribution is discrete, we look at the mutual information achieved when the input distribution is complex Gaussian.

For a Gaussian distributed input x, with an average power constraint $\mathrm{E}\{|x|^2\} \leq P_a$, the distribution of s

$$f_S(s) = \frac{2s}{P_a}\exp\left(-\frac{s^2}{P_a}\right), \quad 0 \leq s \leq \infty \tag{15}$$

is Rayleigh. Since the input is peak power limited, we define the truncated and normalised input as the distribution of s. The new pdf is given by

$$f_{\text{TS}}(s) = \frac{2s}{P_a(1-e^{-\nu})}\exp\left(-\frac{s^2}{P_a}\right), \quad 0 \leq s \leq \sqrt{P_p} \tag{16}$$

where ν is the peak to average power ratio.

***Lemma** 2: Mutual information of the non-coherent Rician fading channel when the input distribution is complex Gaussian with peak power constraint is given by* (9a) *where*

$$\begin{aligned}
h(R) = &-\sum_{\ell=1}^{n}\sum_{j=1}^{m} \frac{2W_j A_\ell \nu e^{-\nu{\tau_\ell}^2}\exp\left(-\frac{KP_p{\tau_\ell}^2}{1+P_p{\tau_\ell}^2}\right)}{(1-e^{-\nu})\log(\frac{1}{\tau_\ell})} \\
&\times I_o\left(\frac{2\sqrt{KP_p\kappa_j\tau_\ell}}{\sqrt{1+P_p{\tau_\ell}^2}}\right)\log\left\{\sum_{i=1}^{n}\frac{2W_iA_i\nu e^{-\nu{\tau_i}^2}}{(1+P_p{\tau_i}^2)(1-e^{-\nu})}\right. \\
&\times\frac{\exp\left(-\frac{KP_p{\tau_i}^2}{1+P_p{\tau_i}^2}\right)}{\log(\frac{1}{\tau_i})}\exp\left(-\frac{\kappa_j(1+P_p{\tau_\ell}^2)}{1+P_p{\tau_i}^2}\right) \\
&\left.\times I_0\left(\frac{2\tau_i\sqrt{KP_p\kappa_j}\sqrt{1+P_p{\tau_\ell}^2}}{1+P_p{\tau_i}^2}\right)\right\}
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
h(R|S) = &\frac{1}{(1-e^{-\nu})}\left\{\exp\left(\frac{\nu}{P_p}\right)\left\{\mathrm{Ei}\left[-\left(\frac{\nu(1+P_p)}{P_p}\right)\right]\right.\right. \\
&\left.\left.-\mathrm{Ei}\left(\frac{-\nu}{P_p}\right)\right\} - e^{-\nu}\log(1+P_p)\right\} + 1.
\end{aligned} \tag{18}$$

Proof: See Appendix VII-B. □

Both (17) and (18) are shown in closed form using Gauss-Legendre and Gauss-Laguree quadrature. A, W are the weights and τ, κ are the roots of Legendre and Laguree polynomials respectively. Using lemma 2, the mutual information when the input distribution is complex Gaussian with peak constraint input power can be computed numerically. The roots and the weights are in tabulated form [11] where the computations error is zero upon proper selection of the polynomial order.

IV. Average power limited Rician channel

A. Mutual Information

We consider the Rician channel model (1) with the phase noise in the line of sight (LOS) component. The new channel model is given by

$$y = me^{j\vartheta}x + ax + n \tag{19}$$

where ϑ is assumed to be an i.i.d. uniform random variable on $[-\pi, \pi)$. It is assumed that both a and ϑ are known by neither the receiver nor the transmitter. Here we consider the input is average power constraint, $\mathrm{E}\{|x|^2\} \leq P_a$. To obtain the output conditional pdf, we use the following theorem [10].

***Theorem** 3: In a non-coherent Rician fading channel* (19) *with a phase noise in the LOS component, the channel output, y, is conditionally Gaussian given x and ϑ. The conditional probability distribution of the channel output given input,*

$$\begin{aligned}
f_{Y|X}(y|x) = &\frac{1}{\pi(1+|x|^2)}\exp\left(-\frac{|y|^2+|m|^2|x|^2}{1+|x|^2}\right) \\
&\times I_0\left(\frac{2|m||y||x|}{1+|x|^2}\right).
\end{aligned} \tag{20}$$

Introducing new random variables $z = |y|^2$, $g = |x|$, the conditional distribution of z given g can be obtained:

$$f_{Z|G}(z|g) = \frac{1}{1+g^2}\exp\left(-\frac{z+Kg^2}{1+g^2}\right)I_0\left(\frac{2\sqrt{Kzg}}{1+g^2}\right) \tag{21}$$

where the Rician factor $K = |m|^2$ and $g \in G$, $z \in Z$. Since the above transformations are one to one, we have

$$I(X;Y) = I(|X|;|Y|) = I(G;Z). \tag{22}$$

The mutual information of Rician channel (19) with phase noise is given by

$$I(Z;G) = h(Z) - h(Z|G) \tag{23a}$$

$$\begin{aligned}
&= -\int_0^\infty f_Z(z)\log[f_Z(z)]dz \\
&\quad - \mathrm{E}_g\left\{\int_0^\infty f_{Z|G}(z|g)\log[f_{Z|G}(z|g)]dz\right\}
\end{aligned} \tag{23b}$$

where $f_Z(z) = \int_0^\infty f_{Z|G}(z|g)f_G(g)dg$.

B. Capacity

The channel capacity

$$C_{\text{ave}} = \sup_{\substack{F_S(\cdot) \\ \mathrm{E}|s|^2 \leq P_a}} I(G;Z) \tag{24}$$

depends on the Rician factor K and the constraint P_a. The following theorem [10] characterises the capacity achieving input in channel model (19).

***Theorem** 4: For the non-coherent Rician fading channel with phase noise* (19) *where the input is subject to an average power constraint $\mathrm{E}\{|x|^2\} \leq P_a$, the capacity achieving input amplitude distribution is discrete with a finite number of mass points.*

Hence, the mutual information for the discrete input $F_G(g) = \sum_{i=1}^{N} p_i \delta(g - g_i)$:

$$I_{\text{ave}}(G;Z) = \sum_{i=1}^{N} \int_0^{\infty} p_i f_{Z|G}(z|g_i) \times \log\left[\frac{f_{Z|G}(z|g_i)}{\sum_{j=1}^{N} p_j f_{Z|G}(z|g_j)}\right] dz \tag{25}$$

can be used to compute the capacity

$$C_{\text{ave}} = \sup_{\mathrm{E}\{|s|^2\} \leq P_a} I_{\text{ave}}(G;Z). \tag{26}$$

The optimal mass point probabilities and their locations are to be found verifying the Kuhn Tucker condition [15].

C. Mutual Information with Gaussian Input

For the average power constraint input $\mathrm{E}\{|x|^2\} \leq P_a$, we use the distribution (15) when its distribution is complex Gaussian.

***Lemma** 3: Mutual information of non-coherent Rician fading channel with phase noise in the LOS component, and the input is average power constraint complex Gaussian is given by* (23a) *where*

$$\begin{aligned} h(Z) = &-\sum_{\ell=1}^{n}\sum_{j=1}^{m} 2W_j \omega_\ell v_\ell \exp\left(-\frac{KP_a {v_\ell}^2}{1+P_a {v_\ell}^2}\right) \\ &\times I_0\left(\frac{2v_\ell\sqrt{K\kappa_j P_a}}{1+P_a {v_\ell}^2}\right) \log\Bigg\{\sum_{i=1}^{n} \frac{2\omega_i v_i}{1+P_a {v_i}^2} \\ &\times \exp\left(-\frac{\kappa_j(1+P_a {v_\ell}^2)}{1+P_a {v_i}^2}\right) \exp\left(-\frac{KP_a {v_i}^2}{1+P_a {v_i}^2}\right) \\ &\times I_0\left(\frac{2v_i\sqrt{K\kappa_j P_a(1+P_a {v_\ell}^2)}}{1+P_a {v_i}^2}\right)\Bigg\} \end{aligned} \tag{27}$$

and

$$\begin{aligned} h(Z|G) = &-\sum_{i=1}^{n}\sum_{j=1}^{m} 2W_i \omega_j v_j \exp\left(-\frac{KP_a {v_j}^2}{1+P_a {v_j}^2}\right) \\ &\times I_0\left(\frac{2v_j\sqrt{K\kappa_i P_a}}{\sqrt{1+P_a {v_j}^2}}\right) \log\left\{\frac{e^{-\kappa_i}\exp\left(\frac{KP_a {v_j}^2}{1+P_a {v_j}^2}\right)}{1+P_a {v_j}^2}\right. \\ &\left.\times I_0\left(\frac{2v_j\sqrt{K\kappa_i P_a}}{\sqrt{1+P_a {v_j}^2}}\right)\right\}. \end{aligned} \tag{28}$$

Proof: See Appendix VII-C. □

Both (27) and (28) are shown in closed form using Gauss-Laguree and Gauss-Hermite quadrature. W, ω are the weights and κ, v are the roots of Laguree and Hermite polynomials respectively [16]. From the expressions (27) and (28), the mutual information of Rician channel (19) can be computed numerically at any SNR.

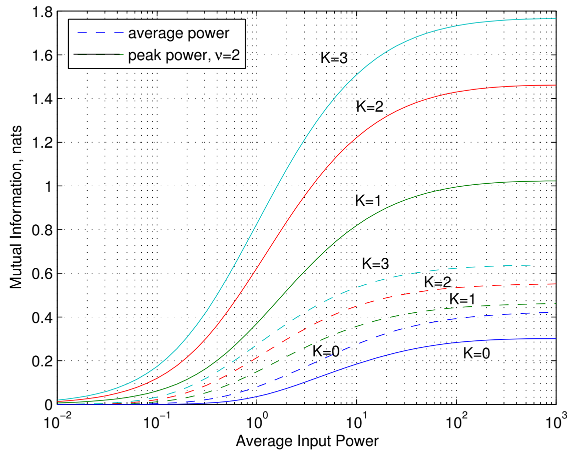

Fig. 1. Mutual information of a non-coherent Rician fading channel when the input distribution is complex Gaussian for both average and peak power constraint inputs.

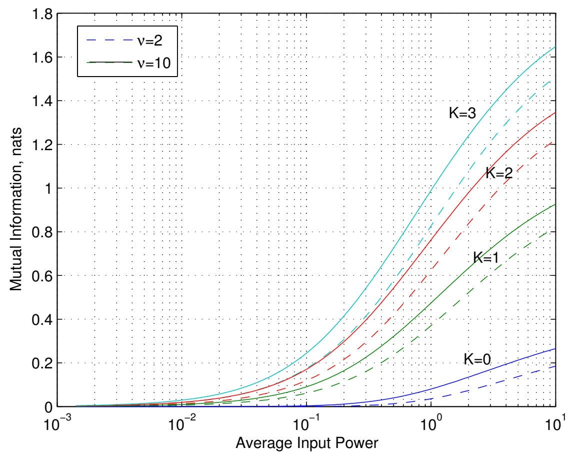

Fig. 2. Mutual information of non-coherent Rician fading channel for the Gaussian distributed input with peak power constraint. The figure shows the mutual information for peak to average power ratio, $\nu \in \{2, 10\}$.

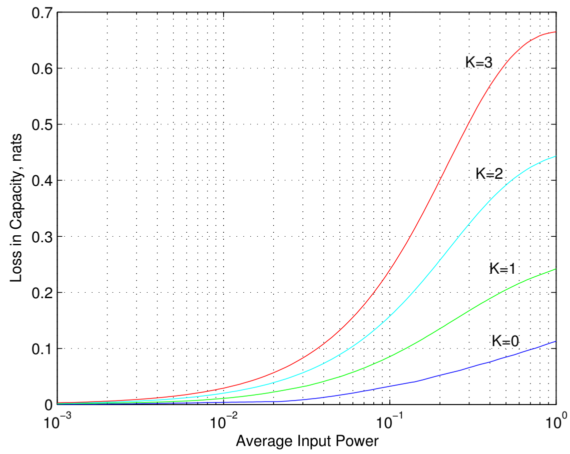

Fig. 3. Capacity loss with mutual information obtained with the Gaussian input under average power constraint. At very low SNR, Gaussian distribution gives mutual information close to the channel capacity.

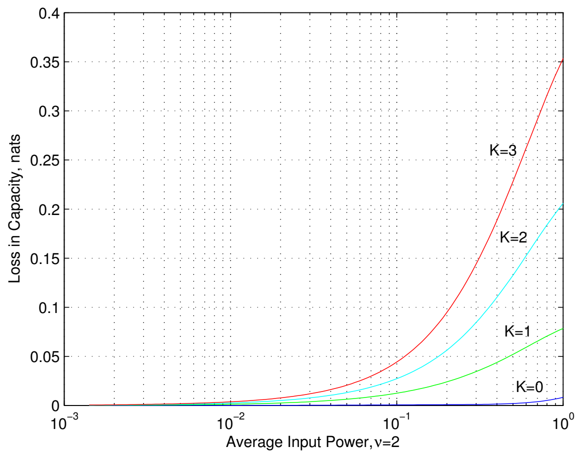

Fig. 4. Capacity loss with mutual information obtained with the Gaussian input under peak power constraint, $\nu = 2$. At very low SNR, Gaussian distribution gives mutual information close to the channel capacity.

V. Numerical Results

Using the closed form expressions derived in Lemma 2, and 3, the mutual information can be numerically computed precisely. The roots and weighting factors of the quadrature polynomials are available in tabulated form [11], [16]. Fig. 1 depicts the mutual information when the input distribution is complex Gaussian under both average and peak power constraints. Mutual information is bounded by SNR in both cases for each Rician factor, $K \in \{0, 1, 2, 3\}$. Furthermore, the mutual information with only the peak power constraint is dominant except for $K = 0$, in case of Rayleigh fading where no LOS component present. Our results show that in Rayleigh fading, the mutual information is optimal when the input is average power constraint than peak.

The channel response for various peak to average power ratio ν is observed under the peak power constraint. Fig. 2 shows the mutual information for $\nu \in \{2, 10\}$ with the average input power for each Rician factor K. The increase in mutual information is noted as ν increases.

Channel capacity under the peak and average power constraint inputs can be numerically computed using Theorems 2 and 4 respectively. Fig. 3 and 4 depict the loss in capacity in each case when the input is Gaussian distributed instead of the optimal. At low SNR, it is clear that loss is insignificant. Hence it is hard to distinguish the difference in mutual information and capacity. The i.i.d. Gaussian is the optimal input in coherent Rician channel [7]. At very low SNR the same distribution gives mutual information close to the capacity when neither the receiver nor the transmitter has perfect CSI. Therefore, the knowledge of CSI at very low SNR is not important.

The loss increases at high SNR since the mutual information in both cases is bounded by the SNR. Therefore, the $\%$ loss in capacity at particular SNR compared to mutual information also increases.

Consider two SNR's $SNR_2 > SNR_1 > 20dB$ where the capacities are C_1, and C_2. Since the channel capacity is monotonically increasing with SNR, $C_1 < C_2$. Define the corresponding mutual information M_1 and M_2 where $M_1 \simeq M_2 = M$. This assumption is valid since the mutual information when the input distribution is Gaussian is bounded by the SNR. For $SNR > 20dB$, the rate of increase is very slow. The fractional loss in capacity in both cases $L_1 = (C_1 - M)/C_1 < 1$, and $L_2 = (C_2 - M)/C_2 < 1$ are related by

$$L_2 - L_1 = (1 - L_1)\left(1 - \frac{C_1}{C_2}\right) > 0, \qquad (29)$$

indicating the significance of optimal input at high SNR.

VI. Conclusions

In this paper, we investigated the mutual information of the Rician fading channel with both average and peak power constraint input when its distribution is complex Gaussian. The purpose of this exercise is to identify the performance of Gaussian signalling in the presence of LOS component compared to traditional Rayleigh fading.

The performance of Gaussian signalling is proven poor at high SNR. However, in the low SNR regime the mutual information obtained under both average and peak input power constraint approximately match the channel capacity. Since the i.i.d. Gaussian distribution is optimal in coherent Rician channels, in the low SNR regime, it provides the mutual information close to non-coherent capacity. The specular (LOS) component with $K > 0$ can transfer most of energy without significant reduction in mutual information in which the peak power constraint provides higher mutual information.

VII. Appendix

A. *Proof of Lemma 1*

The output conditional entropy is given by

$$h(Y|X) = -\int_{\mathbb{C}}\int_{\mathbb{C}} f_{Y|X}(y|x)\log[f_{Y|X}(y|x)]dy f_X(x)dx. \qquad (30)$$

For the output conditional pdf (3), we get

$$h(Y|X) = \mathrm{E}_x\left\{\frac{\log[\sqrt{\pi(1+|x|^2)}]}{\sqrt{\pi(1+|x|^2)}}\int_{\mathbb{C}}\exp\left(-\frac{|y-mx|^2}{1+|x|^2}\right)dy \right.$$
$$\left. - \int_{\mathbb{C}}\frac{|y-mx|^2}{\sqrt{\pi}(1+|x|^2)^{\frac{2}{3}}}\exp\left(-\frac{|y-mx|^2}{1+|x|^2}\right)dy\right\}. \qquad (31)$$

Using the theorem [17, Page 105], which is an extension of the fundamental theorem of calculus, we can evaluate the complex integrals involved in (31). The following integral solutions [18]

$$\int \exp\left(-\frac{|x-n|^2}{k}\right)dx = \frac{\sqrt{\pi k}}{2}\mathrm{erf}\left(\frac{x-n}{\sqrt{k}}\right),$$

and

$$\int |x-n|^2\exp\left(-\frac{|x-n|^2}{k}\right)dx = \frac{(n-kx)}{2}$$
$$\times\exp\left(-\frac{|x-n|^2}{k}\right) + \frac{k^{\frac{3}{2}}}{4}\sqrt{\pi}\mathrm{erf}\left(\frac{x-n}{\sqrt{k}}\right),$$

gives

$$h(Y|X) = \mathrm{E}_x\left\{\log\pi(1+|x|^2) + 1\right\}, \qquad (32)$$

the result used in (4).

B. Proof of Lemma 2

The inner integral in (10)

$$f_R(r) = \int_0^\infty f_{R|S}(r|s) dF_S(s) \qquad (33)$$

can be solved using Gauss-Legendre quadrature by substituting $t = s/\sqrt{P_p}$. Using the integral solution [11]

$$\int_0^1 \log\left(\frac{1}{\tau}\right) f(\tau) d\tau \simeq \sum_{i=1}^N A_i f(\tau_i), \qquad (34)$$

we get

$$f_R(r) = \sum_{i=1}^N \frac{2A_i \nu e^{-\nu \tau_i{}^2} \exp\left(\frac{r + K P_p \tau_i{}^2}{1 + P_p \tau_i{}^2}\right)}{(1 + P_p \tau_i{}^2)(1 - e^{-\nu}) \log\left(\frac{1}{\tau_i}\right)} I_0\left(\frac{2\sqrt{K P_p r} \tau_i}{1 + P_p \tau_i{}^2}\right). \qquad (35)$$

The output entropy $h(R) = -\int_0^\infty f_R(r) \log[f_R(r)] dr$ can be simplified to (17) by substituting $\kappa = r/(1 + P_p \tau_\ell{}^2)$, $\ell = \{1, 2, 3, ...N\}$ for the $\ell^t h$ term of $h(R)$ with the Gauss-Laguree approximation on the integral $\int_0^\infty e^{-\kappa} f(\kappa) d\kappa \simeq \sum_{j=1}^M W_i f(\kappa_i)$. With $f_{\mathrm{TS}}(s)$ in (16), we get the output conditional entropy

$$h(R|S) = \int_0^{\sqrt{P_p}} \frac{2s\log(1+s^2)\exp\left(-\frac{s^2}{P_a}\right)}{P_a(1 - e^{-\nu})} ds + 1. \qquad (36)$$

Using the integral solution [18]

$$\int s \exp\left(-\frac{s^2}{k}\right) \log(1+s^2) ds = \frac{k}{2}\left\{\exp\left(\frac{1}{k}\right) \mathrm{Ei}\left(-\frac{1+s^2}{k}\right) - \exp\left(-\frac{s^2}{k}\right)\log(1+s^2)\right\} \qquad (37)$$

and applying the limits we obtain (18). Note that

$$\begin{aligned} \begin{matrix} h(R/S) \\ P_p \to \infty \end{matrix} &= -\exp\left(\frac{1}{P_a}\right)\mathrm{Ei}\left(-\frac{1}{P_a}\right) + 1 \\ &= C_{\mathrm{rcsi}} + 1, \end{aligned} \qquad (38)$$

the result derived in [19] for Rayleigh fading with average power constraint where C_{rcsi} is the channel capacity of coherent single antenna Rayleigh fading channel.

C. Proof of Lemma 3

We substitute $v^2 = g/P_a$ to

$$f_Z(z) = \int_0^\infty f_{Z|G}(z|g) f_G(g) dg$$

where $f_{Z|G}(z|g)$ is given in (21). Using the Gauss-Hermite quadrature integral solution [16]

$$\int_0^\infty e^{-v^2} f(v) dv \simeq \sum_{i=1}^N \omega_i f(v_i)$$

we get

$$f_Z(z) = \sum_{i=1}^N \frac{2\omega_i v_i \exp\left(-\frac{z + K P_a v_i{}^2}{1 + P_a v_i{}^2}\right)}{1 + P_a v_i{}^2} I_0\left(\frac{2\sqrt{K P_a z} v_i}{1 + P_a v_i{}^2}\right). \qquad (39)$$

Again we substitute $\kappa = z/(1 + P_a v_\ell{}^2)$, $\ell = \{1, 2, 3, ...N\}$ for $h(Z)$ in (26) and finally obtain (27). The proof for $h(Z|G)$ in (28) is similar using Gauss-Hermite and Gauss-Laguree quadrature polynomials.

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