Achieving Maximum Capacity from a Fixed Region of Space

Tharaka A. Lamahewa[†], *Tony S. Pollock*[⋆], *Thushara D. Abhayapala*[†⋆]

[†]Department of Information Engineering, The Australian National University, Canberra ACT 0200, Australia
and [⋆]National ICT Australia, Locked Bag 8001, Canberra, ACT 2601, Australia.
{tharaka.lamahewa, thushara.abhayapala}@anu.edu.au, tony.pollock@nicta.com.au

***Abstract*— Previous results have shown channel capacity of multiple-antenna array communication systems linearly scales with the number of antennas. In reality, by increasing the number of antennas within a fixed region of space the antenna array become dense and spatial correlation (non-ideal antenna placement) significantly limits the capacity. In this paper, we derive a spatial precoder which eliminates the effects of non-ideal antenna placement on the capacity performance of spatially constrained dense MIMO systems. The precoder is derived based on fixed and known parameters of MIMO channels, namely the antenna spacing and antenna placement which are known at the transmitter. Therefore, with this design, the precoder is fixed for fixed antenna placement and the transmitter does not require any feedback of channel state information (partial or full) from the receiver. Closed form solutions for the spatial precoder is derived and numerical results are presented to show the capacity improvements obtained for two types of spatially constrained antenna arrays.**

I. Introduction

Multiple-input multiple-output (MIMO) communications systems using multi-antenna arrays simultaneously during transmission and reception have generated significant interest in recent years. Theoretical work of [1] and [2] showed the potential for significant capacity increases in wireless channels via spatial multiplexing with sparse antenna arrays. However, in reality by increasing the number of antennas within a fixed region of space the antenna array become dense and spatial correlation significantly limits the channel capacity [3]. The achievable capacities of MIMO channels and power allocation schemes to achieve these capacities under various assumptions of channel state information (CSI) has been the subject of recent research work in information theory.

Previous studies [3–9] have given insights and bounds into the effects of correlated channels and [7–9] have specifically studied the capacity of spatially constrained dense antenna arrays. Above studies have assumed that the perfect CSI is known only to the receiver. In [1, 10, 11] various power allocation schemes (or water filling strategies) have been derived assuming perfect CSI or partial CSI (e.g. channel covariance) is available at the transmitter through feedback. However, performance of these schemes heavily depends on the accuracy of the feedback information.

In [7] it was shown that there exists a theoretical antenna saturation point at which the maximum achievable capacity for a fixed region occurs, and further increases in the number of antennas in the region will not give further capacity gains. However, it was also shown that due to non-ideal antenna placement, capacity achieved from a fixed region of space is always lower than the theoretical maximum capacity, and in this case the capacity achieved corresponds to a smaller region with optimally placed antennas within.

In contrast, in this paper we show that the theoretical maximum capacity for a fixed region of space can be achieved via linear spatial precoding, which basically eliminates the detrimental effects of non-ideal antenna placement. This linear spatial precoder is designed based on previously unutilized fixed and known parameters of a MIMO channel, the antenna spacing and antenna placement.

The spatial channel model proposed in [12] provides us a way to incorporate antenna spacing and antenna placement details into the precoder design. In this model, MIMO channel is decomposed into deterministic and random parts, where deterministic parts are related to the transmitter and receiver antenna configurations (antenna spacing and antenna placement) and the random part is related to the scattering environment surrounding the transmitter and receiver antenna arrays.

Unlike the power allocation schemes found in the literature [1, 10, 11] our new scheme does not require any feedback information from the receiver. Our novel scheme utilizes channel state information contained in the antenna locations, which has previously been ignored. Furthermore, this new scheme can be used in stationary channels as well as non-stationary channels.

Notations: Throughout the paper, the following notations will be used: The matrix $\boldsymbol{I}_n$ is the $n \times n$ identity matrix and bold lower (upper) letters denote vectors (matrices). $[\cdot]^\dagger$ denotes the conjugate transpose operation. The notation $E\{\cdot\}$ denotes the mathematical expectation, $|\cdot|$ denotes the matrix determinant, tr$\{\cdot\}$ denotes the matrix trace, $\overline{f(\cdot)}$ denotes the complex conjugate of function $f(\cdot)$, $\lceil . \rceil$ denotes the ceiling operator and $\|\cdot\|$ denotes the Euclidean length of a vector.

II. System Model

Consider a MIMO system consisting of n_T transmit antennas and n_R receive antennas. The original $n_T \times 1$ data vector sent from the transmitter is denoted by $\boldsymbol{s}$ with $E\{\boldsymbol{s}\boldsymbol{s}^\dagger\} =$

$P_T/n_T \boldsymbol{I}_{n_T}$, where P_T is the total transmit power. Before each data vector is transmitted, it is multiplied by a fixed linear spatial precoder matrix $\boldsymbol{F}$ of size $n_T \times n_T$, so the $n_R \times 1$ received signal becomes

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{w},$$

where $\boldsymbol{x} = \boldsymbol{F}\boldsymbol{s}$ is the $n_T \times 1$ baseband transmitted signal vector from n_T antennas with input signal covariance matrix

$$\boldsymbol{Q} = E\left\{\boldsymbol{x}\boldsymbol{x}^\dagger\right\} = \frac{P_T}{n_T}\boldsymbol{F}\boldsymbol{F}^\dagger, \tag{1}$$

$\boldsymbol{w}$ is the $n_R \times 1$ white Gaussian noise matrix in which elements are zero-mean independent Gaussian distributed random variables with variance $1/2$ per dimension and $\boldsymbol{H}$ is the $n_R \times n_T$ random flat fading channel matrix. Note that P_T is also the average signal-to-noise (SNR) at each receiver antenna. In this work we adapt the spatial channel model derived in [12] to represent $\boldsymbol{H}$ and in the next section we briefly review this channel model.

A. Channel Model

Suppose n_T transmit antennas located at positions $\boldsymbol{u}_t$, $t = 1, 2, \cdots, n_T$ relative to the transmitter array origin, and n_R receive antennas located at positions $\boldsymbol{v}_r$, $r = 1, 2, \cdots, n_R$ relative to the receiver array origin. $r_T \geq \max \|\boldsymbol{u}_t\|$ and $r_R \geq \max \|\boldsymbol{v}_r\|$ denote the radius of spheres that contain all the transmitter and receiver antennas, respectively. We assume that scatterers are distributed in the far field from the transmitter and receiver antennas and regions containing the transmit and receive antennas are distinct.

Here we consider the situation where the multipath is restricted to the azimuth plane only (2-D scattering environment[1]), having no field components arriving at significant elevations. In this case, the channel matrix $\boldsymbol{H}$ can be decomposed as [12]

$$\boldsymbol{H} = \boldsymbol{J}_R \boldsymbol{H}_S \boldsymbol{J}_T^\dagger, \tag{2}$$

where $\boldsymbol{J}_R$ is the $n_R \times (2N_R+1)$ receiver configuration matrix,

$$\boldsymbol{J}_R = \begin{bmatrix} \mathcal{J}_{-N_R}(\boldsymbol{v}_1) & \cdots & \mathcal{J}_{N_R}(\boldsymbol{v}_1) \\ \mathcal{J}_{-N_R}(\boldsymbol{v}_2) & \cdots & \mathcal{J}_{N_R}(\boldsymbol{v}_2) \\ \vdots & \ddots & \vdots \\ \mathcal{J}_{-N_R}(\boldsymbol{v}_{n_R}) & \cdots & \mathcal{J}_{N_R}(\boldsymbol{v}_{n_R}) \end{bmatrix},$$

$\boldsymbol{J}_T$ is the $n_T \times (2N_T+1)$ transmitter configuration matrix,

$$\boldsymbol{J}_T = \begin{bmatrix} \mathcal{J}_{-N_T}(\boldsymbol{u}_1) & \cdots & \mathcal{J}_{N_T}(\boldsymbol{u}_1) \\ \mathcal{J}_{-N_T}(\boldsymbol{u}_2) & \cdots & \mathcal{J}_{N_T}(\boldsymbol{u}_2) \\ \vdots & \ddots & \vdots \\ \mathcal{J}_{-N_T}(\boldsymbol{u}_{n_T}) & \cdots & \mathcal{J}_{N_T}(\boldsymbol{u}_{n_T}) \end{bmatrix},$$

with

$$\mathcal{J}_n(\boldsymbol{x}) \triangleq J_n(k\|\boldsymbol{x}\|)e^{in(\phi_x - \pi/2)} \tag{3}$$

as the *spatial-to-mode* function which maps the antenna location $\boldsymbol{x} \equiv (\|\boldsymbol{x}\|, \phi_x)$ in the polar coordinate system to the n-th communication mode[2] of the region [14], where $J_n(\cdot)$ are the Bessel functions of the first kind of order n and $k = 2\pi/\lambda$ is the wave number with λ the wave length. $2N_T + 1$ and $2N_R + 1$ are the number of effective[3] communication modes at the transmit and receive regions, respectively. Note, N_T and N_R are defined by the size of the regions containing all the transmit and receive antennas, respectively [15]. In our case,

$$N_T = \left\lceil \frac{ker_T}{\lambda} \right\rceil \quad \text{and} \tag{4}$$

$$N_R = \left\lceil \frac{ker_R}{\lambda} \right\rceil, \tag{5}$$

where $e \approx 2.7183$.

Finally, $\boldsymbol{H}_S$ is the $(2N_R+1) \times (2N_T+1)$ random complex scattering channel matrix with (p,q)-th element given by

$$\{\boldsymbol{H}_S\}_{p,q} = \int_0^{2\pi}\int_0^{2\pi} g(\phi,\psi)e^{i(q-N_T-1)\phi}e^{-i(p-N_R-1)\psi}\mathrm{d}\phi\mathrm{d}\psi$$

representing the complex scattering gain between the $(q - N_T - 1)$-th mode of the scatter-free transmit region and $(p - N_R - 1)$-th mode of the scatter-free receiver region, where $g(\phi,\psi)$ is the effective random complex scattering gain function for signals with angle-of-departure ϕ from the scatter-free transmitter region and angle-of-arrival ψ at the scatter-free receiver region.

The channel matrix decomposition (2) separates the channel into three distinct regions of interest: the scatter-free region around the transmitter antenna array, the scatter-free region around the receiver antenna array and the complex random scattering environment which is the complement of the union of two antenna array regions. In other words, MIMO channel is decomposed into deterministic and random matrices, where deterministic portions $\boldsymbol{J}_T$ and $\boldsymbol{J}_R$ represent the physical configuration of the transmitter and the receiver antenna arrays, respectively, and the random portion represents the complex scattering environment between the transmitter and the receiver antenna regions.

The rank of the channel matrix $\boldsymbol{H}$ gives the effective number of independent parallel channels between the transmit and receive antenna arrays, and thus determines the capacity of the communications system. From the decomposition (2), $\mathrm{rank}\{\boldsymbol{H}\} = \min\{\mathrm{rank}(\boldsymbol{J}_T), \mathrm{rank}(\boldsymbol{J}_R), \mathrm{rank}(\boldsymbol{H}_s)\}$. For a large number of antennas in a rich scattering environment (this is the scenario we consider in this paper), rank of the channel matrix $\boldsymbol{H}$ becomes $\min\{2N_T+1, 2N_R+1\}$. Therefore the number of available communication modes for the transmit and receive regions limits the capacity of the system.

[1] Similar results can be obtained using the 3-D spatial model derived in [13]

[2] The set of modes form a basis of functions for representing a multipath wave field.

[3] Although there are infinite number of modes excited by an antenna array, there are only finite number of modes (2N+1) which have sufficient power to carry information.

III. Capacity of spatially constrained antennas

The ergodic capacity of n_T transmit and n_R receive antennas is given by [1],

$$\widetilde{C} = E\left\{\log\left|\boldsymbol{I}_{n_R} + \boldsymbol{H}\boldsymbol{Q}\boldsymbol{H}^\dagger\right|\right\},$$

where $\boldsymbol{Q} = E\left\{\boldsymbol{x}\boldsymbol{x}^\dagger\right\}$ is the input signal covariance matrix. In the following we will assume that the channel matrix $\boldsymbol{H}$ is fully known at the receiver and it is also partially known at the transmitter, where deterministic parts of the channel such as antenna spacing and antenna geometry are considered as partial channel information.

In this paper, we consider the case where the receiver array consists of large number of receive antennas. It was shown in [16] that the total received power at the receiver array should remain a constant for a given region, regardless of number of antennas in it. In this situation, the normalized ergodic capacity is given by

$$\widetilde{C} = E\left\{\log\left|\boldsymbol{I}_{n_R} + \frac{1}{n_R}\boldsymbol{H}\boldsymbol{Q}\boldsymbol{H}^\dagger\right|\right\}, \tag{6}$$

where the scaling factor $1/n_R$ scales the channel variances to $E\left\{|h_{r,t}|^2\right\}/n_R$, which assures the total received power remains a constant as the number of antennas is increased.

Substitution of (2) into (6) gives the ergodic capacity

$$\begin{aligned}\widetilde{C} &= E\left\{\log\left|\boldsymbol{I}_{n_R} + \frac{1}{n_R}\boldsymbol{J}_R\boldsymbol{H}_s\boldsymbol{J}_T^\dagger\boldsymbol{Q}\boldsymbol{J}_T\boldsymbol{H}_s^\dagger\boldsymbol{J}_R^\dagger\right|\right\},\\ &= E\left\{\log\left|\boldsymbol{I}_{n_T} + \frac{1}{n_R}\boldsymbol{Q}\boldsymbol{J}_T\boldsymbol{H}_s^\dagger\boldsymbol{J}_R^\dagger\boldsymbol{J}_R\boldsymbol{H}_s\boldsymbol{J}_T^\dagger\right|\right\}, \end{aligned}\tag{7}$$

where the second equality follows from the determinant identity $|\boldsymbol{I}+\boldsymbol{A}\boldsymbol{B}| = |\boldsymbol{I}+\boldsymbol{B}\boldsymbol{A}|$.

Let $\widetilde{\boldsymbol{H}} = \boldsymbol{J}_R\boldsymbol{H}_s = [\widetilde{\boldsymbol{h}}_1^\dagger, \widetilde{\boldsymbol{h}}_2^\dagger, \cdots, \widetilde{\boldsymbol{h}}_{n_R}^\dagger]^\dagger$, where $\widetilde{\boldsymbol{h}}_r$ is a $1\times(2N_T+1)$ row-vector of $\widetilde{\boldsymbol{H}}$, which corresponds to the complex channel gains from $(2N_T+1)$ transmit modes to the r-th receiver antenna, then $(2N_T+1)\times(2N_T+1)$ transmitter modal correlation matrix can be defined as

$$\boldsymbol{R}_{\widetilde{\boldsymbol{H}}} \triangleq E\left\{\widetilde{\boldsymbol{h}}_r^\dagger\widetilde{\boldsymbol{h}}_r\right\}, \forall r$$

where (n,n')-th element of $\boldsymbol{R}_{\widetilde{\boldsymbol{H}}}$ gives the modal correlation between n-th and n'-th modes in the transmit region.

Similar to [7], we consider the situation where the receiver aperture of radius r_R has optimally placed (uncorrelated) $n_R = 2N_R+1$ antennas, which corresponds to independent $\widetilde{\boldsymbol{h}}_r$ vectors, then the sample transmitter modal correlation matrix is given by

$$\widehat{\boldsymbol{R}}_{\widetilde{\boldsymbol{H}}} = \frac{1}{n_R}\sum_{r=1}^{n_R}\widetilde{\boldsymbol{h}}_r^\dagger\widetilde{\boldsymbol{h}}_r.$$

For a large number of receive antennas, the sample transmitter modal correlation matrix $\widehat{\boldsymbol{R}}_{\widetilde{\boldsymbol{H}}}$ converges to $\boldsymbol{R}_{\widetilde{\boldsymbol{H}}}$ as $r_R \to \infty$. Since $\widetilde{\boldsymbol{H}}^\dagger\widetilde{\boldsymbol{H}} = \sum_{r=1}^{n_R}\widetilde{\boldsymbol{h}}_r^\dagger\widetilde{\boldsymbol{h}}_r$, then for a large number of uncorrelated receive antennas, the ergodic capacity (7) converges to the deterministic quantity C,

$$\lim_{r_R\to\infty}\widetilde{C} = C \triangleq \log\left|\boldsymbol{I}_{n_T} + \boldsymbol{Q}\boldsymbol{J}_T\boldsymbol{R}_{\widetilde{\boldsymbol{H}}}\boldsymbol{J}_T^\dagger\right|. \tag{8}$$

This analytical capacity expression allows us to investigate the effects of transmit antenna configuration, scattering environment and the input signal covariance matrix $\boldsymbol{Q}$ on ergodic capacity. However, in this paper, our main objective is to *find the optimum transmit power allocation scheme which reduces the effects of non-ideal antenna placement on the capacity performance of a communication system*.

IV. Optimization Problem Setup

Assume that the scatterers generate an isotropic diffuse field at the transmitter, which corresponds to independent elements of scattering channel matrix $\boldsymbol{H}_S$. With this assumption we have $\boldsymbol{R}_{\widetilde{\boldsymbol{H}}} = \boldsymbol{I}_{2N_T+1}$ and (8) reduces to

$$C = \log\left|\boldsymbol{I}_{n_T} + \boldsymbol{Q}\boldsymbol{J}_T\boldsymbol{J}_T^\dagger\right|. \tag{9}$$

In this case, we see that the capacity obtained from a fixed region of space is dependent on the transmit antenna configuration and also on the input signal covariance matrix.

In (9), (q,r)-th element of scatter-free transmit matrix product $\boldsymbol{J}_T\boldsymbol{J}_T^\dagger$ is given by,

$$\begin{aligned}\left\{\boldsymbol{J}_T\boldsymbol{J}_T^\dagger\right\}_{q,r} &= \sum_{n=-N_T}^{N_T}\mathcal{J}_n(\boldsymbol{u}_q)\overline{\mathcal{J}_n(\boldsymbol{u}_r)},\\ &= J_0(k \parallel \boldsymbol{u}_q - \boldsymbol{u}_r \parallel)\end{aligned}$$

which follows from a special case of Gegenbauer's Addition Theorem [17, page 363]. For a rich scattering environment, $J_0(k \parallel \boldsymbol{u}_q - \boldsymbol{u}_r \parallel)$ gives the spatial correlation between the complex envelopes of the transmitted signals from antennas q and r [18]. It is well known that the presence of spatial correlation between antenna elements limits the capacity of MIMO systems. So the main objective is to reduce the effects of spatial correlation (non-ideal antenna placement in our case) on MIMO capacity by designing $\boldsymbol{Q}$ (and hence the linear precoder $\boldsymbol{F}$) to maximize the deterministic capacity (9) for a given antenna placement.

If the channel matrix $\boldsymbol{H}$ is known only to the receiver, then as shown in [1], transmission of statistically independent equal power signals each with a Gaussian distribution will be optimal. In this case $\boldsymbol{Q} = (P_T/n_T)\boldsymbol{I}_{n_T}$. In what follows we will refer to this scheme as equal power loading.

A. Optimum input signal covariance

Writing $\boldsymbol{J}_T$ as the singular value decomposition (svd) $\boldsymbol{J}_T = \boldsymbol{U}_T\boldsymbol{\Lambda}_T\boldsymbol{V}_T^\dagger$, then (9) becomes

$$C = \log\left|\boldsymbol{I}_{n_T} + \boldsymbol{U}_T^\dagger\boldsymbol{Q}\boldsymbol{U}_T\boldsymbol{T}\right|,$$

where $\boldsymbol{T} = \boldsymbol{\Lambda}_T\boldsymbol{\Lambda}_T^\dagger$ is a diagonal matrix with squared singular values of $\boldsymbol{J}_T$ (or the eigen-values of spatial correlation matrix $\boldsymbol{J}_T\boldsymbol{J}_T^\dagger$) on the diagonal.

The optimum input signal covariance $\boldsymbol{Q}$ is obtained by solving the optimization problem:

$$\begin{aligned} \max \quad & \log\left|\boldsymbol{I}_{n_T} + \boldsymbol{U}_T^\dagger \boldsymbol{Q}\boldsymbol{U}_T\boldsymbol{T}\right| \\ \text{subject to} \quad & \boldsymbol{Q} \succeq 0,\ \mathrm{tr}\{\boldsymbol{Q}\} = P_T, \\ & \mathrm{tr}\{\boldsymbol{U}_T^\dagger \boldsymbol{Q}\boldsymbol{U}_T\boldsymbol{T}\} = P_T, \end{aligned} \tag{10}$$

where we assumed $\boldsymbol{Q}$ is non-negative definite ($\boldsymbol{Q} \succeq \boldsymbol{0}$). The power constraint $\mathrm{tr}\{\boldsymbol{Q}\} = P_T$ ensures the total power transmitted from n_T transmit antennas is P_T and the second power constraint $\mathrm{tr}\{\boldsymbol{U}_T^\dagger \boldsymbol{Q}\boldsymbol{U}_T\boldsymbol{T}\} = P_T$ ensures the total power assigned to effective modes at the scatter-free transmit region is also P_T.

Let $\widetilde{\boldsymbol{Q}} = \boldsymbol{U}_T^\dagger \boldsymbol{Q}\boldsymbol{U}_T$. Since $\boldsymbol{U}_T$ is unitary, maximization/minimization over $\boldsymbol{Q}$ can be carried equally well over $\widetilde{\boldsymbol{Q}}$. Furthermore, $\widetilde{\boldsymbol{Q}}$ is non-negative definite since $\boldsymbol{Q}$ is non-negative definite. Therefore, the optimization problem (10) becomes[4]

$$\begin{aligned} \min \quad & -\log\left|\boldsymbol{I}_{n_T} + \widetilde{\boldsymbol{Q}}\boldsymbol{T}\right| \\ \text{subject to} \quad & \widetilde{\boldsymbol{Q}} \succeq 0,\ \mathrm{tr}\{\widetilde{\boldsymbol{Q}}\} = P_T,\ \mathrm{tr}\{\widetilde{\boldsymbol{Q}}\boldsymbol{T}\} = P_T. \end{aligned} \tag{11}$$

By applying Hadamard's inequality on $\left|\boldsymbol{I}_{n_T} + \widetilde{\boldsymbol{Q}}\boldsymbol{T}\right|$ gives that this determinant is maximized when $\widetilde{\boldsymbol{Q}}\boldsymbol{T}$ is diagonal [1]. Therefore $\widetilde{\boldsymbol{Q}}$ must be diagonal as $\boldsymbol{T}$ is diagonal. Since $\widetilde{\boldsymbol{Q}}\boldsymbol{T}$ is a non-negative definite diagonal matrix with non-negative entries on its diagonal, $\boldsymbol{I} + \widetilde{\boldsymbol{Q}}\boldsymbol{T}$ forms a positive definite matrix. As a result, the objective function of our optimization problem is convex [19, page 73]. Therefore the optimization problem (11) above is a convex minimization problem because the objective function and the inequality constraint are convex and equality constraints are affine.

Let $\widetilde{q}_i = [\widetilde{\boldsymbol{Q}}]_{i,i}$ and $t_i = [\boldsymbol{T}]_{i,i}$. Optimization problem (11) then reduces to finding $\widetilde{q}_i > 0$ such that

$$\begin{aligned} \min \quad & -\sum_{i=1}^{n_T} \log(1 + t_i\widetilde{q}_i) \\ \text{subject to} \quad & \widetilde{\boldsymbol{q}} \succeq 0, \quad \boldsymbol{1}^T\widetilde{\boldsymbol{q}} = P_T,\ \boldsymbol{t}^T\widetilde{\boldsymbol{q}} = P_T, \end{aligned}$$

where $\widetilde{\boldsymbol{q}} = [\widetilde{q}_1, \widetilde{q}_2, \cdots, \widetilde{q}_{n_T}]^T$, $\boldsymbol{t} = [t_1, t_2, \cdots, t_{n_T}]^T$ and $\boldsymbol{1}$ denotes the vector of all ones. Introducing Lagrange multipliers $\boldsymbol{\lambda} \in \mathbb{R}^{n_T}$ for the inequality constraint $-\boldsymbol{q} \preceq 0$ and $\upsilon, \mu \in \mathbb{R}$ for equality constraints $\boldsymbol{1}^T\widetilde{\boldsymbol{q}} = P_T$ and $\boldsymbol{t}^T\widetilde{\boldsymbol{q}} = P_T$, respectively, we obtain the Karush-Kuhn-Tucker (K.K.T) conditions

$$\begin{aligned} \widetilde{\boldsymbol{q}} \succeq 0, \quad \boldsymbol{\lambda} \succeq 0, \quad \boldsymbol{1}^T\widetilde{\boldsymbol{q}} = P_T, \quad \boldsymbol{t}^T\widetilde{\boldsymbol{q}} = P_T \\ \lambda_i\widetilde{q}_i = 0, \quad i = 1, 2, \cdots, n_T \\ -\frac{t_i}{1 + t_i\widetilde{q}_i} - \lambda_i + \upsilon + \mu t_i = 0, \quad i = 1, 2, \cdots, n_T. \end{aligned} \tag{12}$$

[4]Maximization of $f(x)$ is equivalent to minimization of $-f(x)$.

Note that λ_i in (12) can be eliminated since it acts as a slack variable[5], giving new K.K.T conditions

$$\widetilde{\boldsymbol{q}} \succeq 0, \quad \boldsymbol{1}^T\widetilde{\boldsymbol{q}} = P_T, \quad \boldsymbol{t}^T\widetilde{\boldsymbol{q}} = P_T$$

$$q_i\left(\upsilon + \mu t_i - \frac{t_i}{1 + t_i\widetilde{q}_i}\right) = 0, \quad i = 1, \cdots, n_T, \tag{13a}$$

$$\upsilon + \mu t_i \geq \frac{t_i}{1 + t_i\widetilde{q}_i}, \quad i = 1, \cdots, n_T. \tag{13b}$$

The complementary slackness condition $\lambda_i\widetilde{q}_i = 0$ for $i = 1, 2, \cdots, n_T$ states that λ_i is zero unless the i-th inequality constraint is active at the optimum. Therefore, from (13a) we obtain optimum $\widetilde{q}_i$

$$\widetilde{q}_i = \begin{cases} \frac{1-\mu}{\upsilon+\mu t_i} - \frac{\upsilon}{t_i(\upsilon+\mu t_i)}, & t_i > \frac{\upsilon}{1-\mu}; \\ 0, & \text{otherwise}, \end{cases}$$

where υ and μ are constants chosen to satisfy two power constraints $\sum_{i=1}^{n_T} \max\left(0, \frac{1-\mu}{\upsilon+\mu t_i} - \frac{\upsilon}{t_i(\upsilon+\mu t_i)}\right) = P_T$ and $\sum_{i=1}^{n_T} t_i \max\left(0, \frac{1-\mu}{\upsilon+\mu t_i} - \frac{\upsilon}{t_i(\upsilon+\mu t_i)}\right) = P_T$, and $\widetilde{\boldsymbol{Q}} = \mathrm{diag}(\widetilde{q}_1, \widetilde{q}_2, \cdots, \widetilde{q}_{n_T})$. Therefore, the optimum input signal covariance matrix $\boldsymbol{Q} = \boldsymbol{U}_T\widetilde{\boldsymbol{Q}}\boldsymbol{U}_T^\dagger$. From (1), the linear spatial precoder

$$F = \sqrt{\frac{P_T}{n_T}}\boldsymbol{U}_T\widetilde{\boldsymbol{Q}}^{1/2}\boldsymbol{U}_n^\dagger,$$

where $\boldsymbol{U}_n$ is an arbitrary unitary matrix. Here we take $\boldsymbol{U}_n = \boldsymbol{I}_{n_T}$.

B. Numerical Results

We now present numerical results to illustrate the capacity improvements obtained from the spatial precoder derived in the previous section. The performance of the precoder is compared with the equal power loading scheme.

We consider a MIMO system with n_T transmitter antennas constrained within a scatter-free circular region of radius $r_T = 0.5\lambda$ and a large number of uncorrelated receiver antennas for a total power budget of $P_T = 10$dB. Fig. 1 shows the capacity results for Uniform Circular Arrays (UCA) and Uniform Linear Arrays (ULA) using the linear spatial precoder $\boldsymbol{F}$ and equal power allocation scheme $\boldsymbol{Q} = (P_T/n_T)\boldsymbol{I}_{n_T}$ for increasing the number of transmitter antennas in the transmit region. Also shown is the maximum achievable capacity from the transmit region when all the n_T antennas are placed optimally such that the spatial correlation is zero between all the antennas. In this case, the maximum achievable capacity from the transmit region is given by [7, Eq. 35],

$$C_{max}(r_T) = n_{sat}(r_T)\log\left(1 + \frac{P_T}{n_{sat}(r_T)}\right), \tag{14}$$

where $n_{sat}(r_T) = 2N_T + 1$ is the antenna saturation point for the region which also corresponds to the number of effective modes in the scatter-free transmit region. In our case, from

[5]If $g(x) \leq \upsilon$ is a constraint inequality, then a variable λ with the property that $g(x) + \lambda = \upsilon$ is called a slack variable.

(4), $n_{sat}(r_T = 0.5\lambda) = 11$, which is shown by the vertical dashed line in Fig. 1.

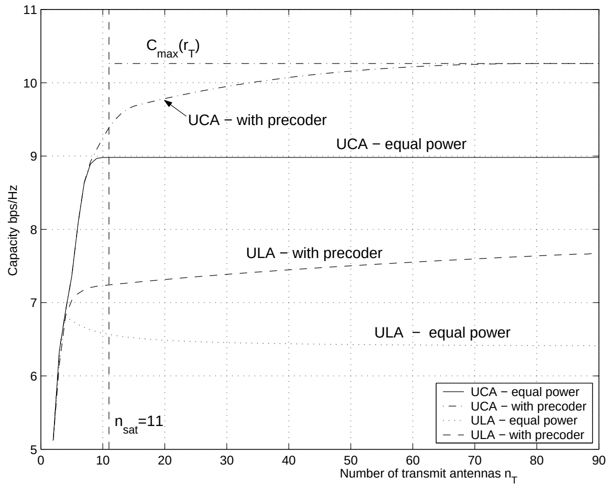

Fig. 1. Capacity comparison between spatial precoder and equal power loading ($\boldsymbol{Q} = (P_T/n_T)\boldsymbol{I}_{n_T}$) schemes for uniform circular arrays and uniform linear arrays in a rich scattering environment with transmitter aperture radius $r_T = 0.5\lambda$ and a large number of uncorrelated receiver antennas ($r_R = \infty$) for an increasing number of transmitter antennas. Also shown is the maximum achievable capacity (14) from the transmit region.

It is observed that with the equal power loading scheme, capacity performance of both the UCA and ULA does not reach the maximum achievable capacity $C_{max}(r_T)$ from the region as the number of antennas is increased. This is because both the UCA and ULA do not optimally place the antennas within the given region. Furthermore, with this scheme capacity is saturated even before n_T approaches n_{sat} for both antenna configurations. In fact the capacity achieved with this scheme corresponds to a region of smaller radius with optimally placed antennas within. Let $\widetilde{n}_{sat}(< n_{sat})$ be the new antenna saturation point for a given antenna configuration. Therefore, with equal power loading one cannot achieve further capacity gains by increasing the number of antennas beyond $\widetilde{n}_{sat}$.

In contrast, spatially precoded systems give significant capacity improvements as the number of antennas are increased beyond $\widetilde{n}_{sat}$. For $n_T > 80$, we see the capacity of the precoded UCA system reaches $C_{max}(r_T)$, which corresponds to 1.2bps/Hz capacity gain over the equal power loading scheme. In this case, spatial precoder virtually arranges the antennas into an optimal configuration as such the spatial correlation is zero between all the antenna elements. In the case of precoded ULA, it requires a large number of transmit antennas to achieve $C_{max}(r_T)$. However, as we can see, the spatial precoder still provides significant capacity gains over the equal power loading scheme for any $n_T > \widetilde{n}_{sat}$. We also observed that precoding does not provide significant capacity gains for lower number of transmit antennas. This is mainly due to the low spatial correlation between antenna elements in the transmit array for lower number of antennas.

In the next section we compare the average power allocated to modes in the transmit region for the two power loading schemes we considered and follow with some analysis.

C. Transmit Modes and Power Allocation

Let $\boldsymbol{x} = [x_1, x_2, \cdots, x_{n_T}]^T$ be the column vector of baseband transmitted signals from n_T transmitter antennas over a single signalling interval, then the signal leaving the scatter-free transmit region along direction $\hat{\boldsymbol{\phi}}$ is given by

$$\Phi(\hat{\boldsymbol{\phi}}) = \sum_{t=1}^{n_T} x_t e^{ik\boldsymbol{u}_t \cdot \hat{\boldsymbol{\phi}}}. \tag{15}$$

As before, we consider a 2-D scattering environment, then the 2-D modal expansion of the plane wave $e^{ik\boldsymbol{u}_t \cdot \hat{\boldsymbol{\phi}}}$ is given by [20, page 67],

$$e^{ik\boldsymbol{u}_t \cdot \hat{\boldsymbol{\phi}}} = \sum_{n=-\infty}^{\infty} \overline{\mathcal{J}_n(\boldsymbol{u}_t)} e^{in\phi}, \tag{16}$$

where $\mathcal{J}_n(\boldsymbol{u}_t)$ is the *spatial-to-mode* function (3), $\boldsymbol{u}_t \equiv (\|\boldsymbol{u}_t\|, \phi_t)$ location of the t-th transmitter antenna and $\hat{\boldsymbol{\phi}} \equiv (1, \phi)$. Substitution of (16) into (15), gives

$$\Phi(\hat{\boldsymbol{\phi}}) = \sum_{n=-\infty}^{\infty} \sum_{t=1}^{n_T} x_t \overline{\mathcal{J}_n(\boldsymbol{u}_t)} e^{in\phi}, \tag{17a}$$

$$= \sum_{n=-\infty}^{\infty} a_n e^{in\phi}, \tag{17b}$$

where $a_n = \sum_{t=1}^{n_T} x_t \overline{\mathcal{J}_n(\boldsymbol{u}_t)}$ is the n-th transmit mode excited by n_T antennas. Note that sum (17b) in fact is the Fourier series expansion of signal $\Phi(\hat{\boldsymbol{\phi}})$ with Fourier coefficients a_n. The average power allocated to the n-th transmit mode is then given by

$$\sigma_n^2 = E\left\{|a_n|^2\right\} = \sum_{t=1}^{n_T} \sum_{t'=1}^{n_T} E\left\{x_t x_{t'}\right\} \overline{\mathcal{J}_n(\boldsymbol{u}_t)} \mathcal{J}_n(\boldsymbol{u}_{t'}), \tag{18}$$

where $E\{x_t x_{t'}\}$ is the (t, t')-th entry of $\boldsymbol{Q}$. For the equal power loading scheme, (18) simplifies to

$$\sigma_n^2 = \frac{P_T}{n_T} \sum_{t=1}^{n_T} J_n^2(k\|\boldsymbol{u}_t\|).$$

As described in Section II-A, the number of effective modes excited by a spatially constrained antenna array is limited by the size of the aperture and is independent of number of antennas packed into the aperture. Fig. 2 shows the average power allocation to the first 11 effective transmit modes for the two antenna configurations considered in the previous section. The results shown here are for $n_T = 80$ and $P_T = 10$dB.

In this work we assumed that the receiver has the full knowledge of the channel matrix $\boldsymbol{H} = \boldsymbol{J}_R \boldsymbol{H}_S \boldsymbol{J}_T^\dagger$ and the transmitter has the knowledge of antenna configuration matrix $\boldsymbol{J}_T$ only. Since the scattering channel matrix $\boldsymbol{H}_S$ is not known to the transmitter, the maximum capacity will occur for equal power allocation to the full set of uncorrelated transmit modes available for the given region, i.e., $\sigma_n^2 = P_T/(2N_T + 1)$. From Fig. 2, for both antenna configurations, equal power loading scheme assigns different power levels to modes in

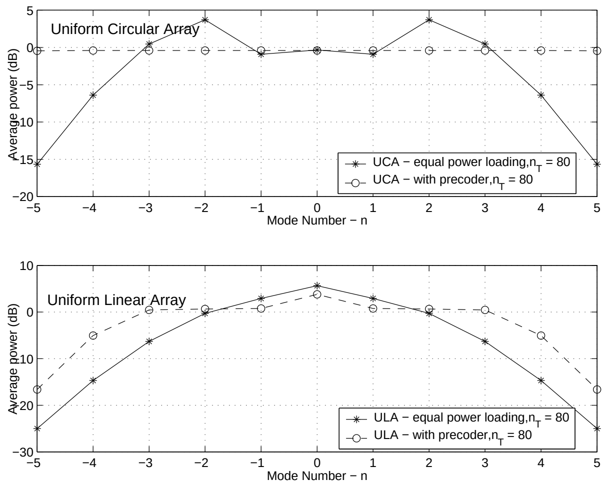

Fig. 2. Average power allocated to each transmit mode for the UCA and ULA antenna configurations, within a circular aperture of radius 0.5λ. $P_T = 10$dB and $n_T = 80$.

the transmit region, and as a result, both configurations fail to achieve the maximum capacity available from the region (Fig. 1). However, in the case of spatially precoded UCA, precoder assigns equal power to all available modes in the transmit region. In this case, precoder makes the transmitter scatter-free matrix product $\boldsymbol{J}_T \boldsymbol{J}_T^\dagger = \boldsymbol{I}$ by correctly allocating power into each transmit antenna and utilizes the full set of uncorrelated communication modes between regions to achieve the theoretical maximum capacity $C_{max}(r_T)$. With spatially precoded ULA, we see that lower order modes (except the 0-th order mode) receive equal power while higher order modes receive unequal power. However, for a large number of transmit antennas, spatial precoder assigns equal power to all effective modes in the transmit region and thus achieves the theoretical maximum capacity $C_{max}(r_T)$.

V. Discussion

In this paper, by considering the spatial dimension of a MIMO channel we have derived a fixed linear spatial precoder which eliminates the detrimental effects of non-ideal antenna placement and improves the capacity performance of spatially constrained dense MIMO systems. We also showed that unlike the equal power loading scheme, with a large number of antennas this spatial precoder is capable of achieving the theoretical maximum capacity available for a fixed region of space. Furthermore, numerical results suggest that spatial precoding can provide significant capacity gains by adding two to three more antennas in to the region than the number which saturates the nonprecoded scheme. Since the design is based on readily available antenna configuration details (antenna spacing and placement), the precoder is fixed and transmitter does not require any feedback of channel state information from the receiver. This is an added advantage over the other precoding schemes (or water filling strategies) found in the literature. In reality wireless channels experience non-isotropic scattering at both end of the channel. However, we observed that this spatial precoding scheme still provides significant capacity gains in the presence of scattering correlation, and will be reported elsewhere.

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References

[1] I.E. Telatar, "Capacity of multi-antenna gaussian channels," *Tech. Repo., AT&T Bell Labs*, 1995.

[2] G.J. Foschini and M.J. Gans, "On limits of wireless communications in a fading environment when using multiple antennas," *Wireless Personal Communications*, vol. 6, pp. 311–335, 1998.

[3] Da-Shan Shiu, G.J. Foschini, M.J. Gans, and J.M. Kahn, "Fading correlation and its effect on the capacity of multielement antenna systems," *IEEE Trans. Commun.*, vol. 48, no. 3, pp. 502–513, 2000.

[4] T.S. Pollock, T.D. Abhayapala, and R.A. Kennedy, "Introducing space into MIMO capacity calculations," *Journal on Telecommunications Systems (Kluwer Academic Publishers)*, vol. 24, no. 2, pp. 415–436, 2003.

[5] C. Chen-Nee, D.N.C. Tse, J.M. Kahn, and R.A. Valenzuela, "Capacity scaling in MIMO wireless systems under correlated fading," *IEEE Trans. Info. Theory*, vol. 48, no. 3, pp. 637–650, 2002.

[6] L. Hanlen and M. Fu, "MIMO wireless communication systems: Capacity limits for sparse scattering," in *3rd Australian Communcations Theory Workshop*, Canberra, Australia, 2002.

[7] T.S. Pollock, T.D. Abhayapala, and R.A. Kennedy, "Fundamental limits of MIMO capacity for spatially constrained arrays," in *4th Australian Communications Theory Workshop*, Melbourne, Australia, February 2003, pp. 7–12.

[8] D. Gesbert, T. Ekman, and N. Christophersen, "Capacity limits of dense palm-sized MIMO arrays," in *Globecom'02*, Taipei, Taiwan, 2002.

[9] S. Wei, D. Goeckel, and R. Janaswamy, "On the capacity of fixed length antenna arrays under bandlimited correlated fading," in *Conf. on Inf. Sciences and Systems, CISS'02*, Princeton, 2002, pp. 1088–1093.

[10] A. Goldsmith, S. A. Jafar, N. Jindal, and S. Vishwanath, "Capacity limits of MIMO channels," *IEEE Journal on Selected Areas in Communications*, vol. 21, no. 5, pp. 684–702, June 2003.

[11] L. Hanlen and A. Grant, "On capacity of ergodic multiple-input multiple-output channels," in *6th Australian Communcations Theory Workshop*, Brisbane, Australia, Feb. 2005, pp. 121–124.

[12] T.D. Abhayapala, T.S. Pollock, and R.A. Kennedy, "Spatial decomposition of MIMO wireless channels," in *The Seventh International Symposium on Signal Processing and its Applications*, Paris, France, July 2003, vol. 1, pp. 309–312.

[13] T.D. Abhayapala, T.S. Pollock, and R.A. Kennedy, "Characterization of 3D spatial wireless channels," in *IEEE Vehicular Technology Conference (Fall), VTC2003*, Orlando, Florida, USA, Oct. 2003, vol. 1, pp. 123–127.

[14] T.S. Pollock, T.D. Abhayapala, and R.A. Kennedy, "Spatial limits to MIMO capacity in general scattering environments," in *7th International Symposium on DSP for Communication Systems*, Coolangatta, Australia, Dec. 2003, pp. 49–54.

[15] H.M. Jones, R.A. Kennedy, and T.D. Abhayapala, "On dimensionality of multipath fields: Spatial extent and richness," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing, ICASSP'2002*, Orlando, Florida, May 2002, vol. 3, pp. 2837–2840.

[16] N. Chiurtu, B. Rimoldi, and I.E. Telatar, "Dense multiple antenna systems," in *IEEE Information Theory Workshop*, Cairns, Australia, 2001.

[17] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, Dover Publications, Inc., New York, 1972.

[18] W.C. Jakes, *Microwave Mobile Communications*, John Wiley, New York, 1974.

[19] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004.

[20] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, vol. 93, Springer, New York, 2 edition, 1998.