Mutual Information of Non-Coherent Rayleigh Fading Channels with Gaussian Input

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***Abstract*— In this paper, the mutual information of a discrete time Rayleigh fading channel is considered, where neither the transmitter nor the receiver has the knowledge of the channel state information. We derive a closed form expression for the mutual information of single input single output Rayleigh fading channel when the input distribution is complex Gaussian. We compare the channel capacity and the mutual information attained with Gaussian input using the derived closed form expressions. Furthermore, we argue that the mutual information is bounded by the signal to noise ratio with the simulation results showing the sub optimality of Gaussian signalling in non-coherent Rayleigh fading channels.**

***Index Terms*— Channel capacity, mutual information, Rayleigh fading, Gaussian distribution, Gaussian quadrature.**

I. INTRODUCTION

The independent and identically distributed (iid) Gaussian is the optimal input distribution in the non fading channel [1], and the Rayleigh fading channel when the receiver has the perfect channel state information (CSI) [2]. The Gaussian input achieves capacity when the transmitter has perfect CSI in addition to the receiver with optimal power distribution over the channel, usually called *water filling* at the transmitter [2], [3]. However, when neither the receiver nor the transmitter has CSI (non-coherent), the optimal input distribution which achieves channel capacity is not Gaussian [4]. It is of interest to find out the mutual information in non-coherent Rayleigh fading channels for this input distribution. In this paper, we derive a closed form expression for the mutual information when the input distribution is complex Gaussian in non-coherent Rayleigh fading channels and show the sub optimality of having such inputs.

A large amount of literature is available on the capacity and the achievable rates over fading channels e.g. [5], [6]. Fading channel capacity with perfect receiver CSI was originally shown by Ericson [7] and later by Lee [8] who showed the degradation of the channel capacity in a Rayleigh fading environment compared to the AWGN channel and use of diversity schemes to improve. In particular, [6] describes the statistical models of fading channels and focuses on the information theory of fading channels, by emphasizing capacity as the most important measure. Even when the capacity or its supremum is known, it is difficult to identify the input distribution which provides the capacity based on the input constraints [9]. Even though, the Gaussian input offer the channel capacity in coherent Rayleigh fading channels, the capacity achieving input distribution in non-coherent Rayleigh fading channels is discrete with finite number of mass points [4], [9]. A rigorous proof is given in [4] for the discreteness of the optimal input in non-coherent Rayleigh fading channels. Further, the capacity is calculated numerically using convex optimisation identifying the number of discrete mass points, their probabilities and locations. Unfortunately, there is no form of analytical or tabulated numerical values to find the capacity for a given signal to noise ratio (SNR) compared to coherent Rayleigh fading channels.

The effect on Gaussian input in non-coherent Rayleigh fading channel is shown by Lapidoth [10] who claimed that the mutual information is bounded by the SNR. The CSI is obtained by training with known pilot symbols inserted in the transmitted sequence. Due to the presence of noise, the receiver is provided with imperfect CSI and the performance of the channel depends on its quality [10]. Considering the worst case scenario, the channel can become non-coherent with Gaussian input which is optimal with perfect CSI. Therefore, a closed form expression is imperative in which the ultimate lower bound is shown with the imperfect CSI at the receiver.

The interest in Gaussian input is studied in [11], which shows the numerical analysis of the mutual information of non-coherent Rayleigh fading channel with an analytical lower bound. Here we extend this work and provide a closed form expression for the mutual information using Gaussian quadrature formulas and demonstrate the performance of Gaussian signalling at any SNR. This work supersedes the result given in [10] for the asymptotic value since the closed form expression derived in this work can be used for any SNR. Furthermore, our result has a significant impact in understanding the performance of Gaussian signalling in non-coherent Rayleigh fading channels where the eventual lower bound is available for coherent channels when the imperfect CSI is received by the receiver.

II. SYSTEM MODEL

Consider the Rayleigh fading channel,

$$y = ax + n \tag{1}$$

where x and y are the complex channel input and output respectively. The random variables a and n represent the

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fading and noise components associated with the channel and the time index is omitted for simplicity. It is assumed that both a and n are independent zero mean circular complex Gaussian random variables with equal unit variance in each dimension. The random variables a, x, and n are considered to be independent of each other. The input $x \epsilon X$ is average power limited: $\int x^2 p_X(x)dx = \Omega_x \leq P$. The constant $\gamma = -\int_0^\infty e^{-y}\log y dy \approx 0.5772...$, denotes the Euler's constant.

It is assumed that neither the receiver nor the transmitter has the knowledge of channel state information other than the statistics.

III. MUTUAL INFORMATION

The Mutual information between the input and output of a Rayleigh fading channel can be expressed as [9]

$$I(X;Y) = \int_0^\infty \int_0^\infty p_{Y|X}(y|x)p_X(x) \times\log\left[\frac{p_{Y|X}(y|x)}{\int_0^\infty p_{Y|V}(y|v)p_V(v)dv}\right]dxdy \tag{2}$$

using the probability distribution of the magnitudes of the random variables X and Y. The output conditional probability density function (pdf), $p_{Y|X}(y/x)$ [9], [12] for the channel model (1) with complex random variables a and n having a unit variance in each dimension can be written as [11]

$$p_{Y|X}(y|x) = \frac{y}{1+x^2}\exp\left[\frac{-y^2}{2(1+x^2)}\right]. \tag{3}$$

Without loss of generality, the magnitude sign is removed in (3) and the same notation will be used throughout the rest of this paper.

The mutual information $I(X;Y) = h(Y) - h(Y|X)$ of the channel (2) can be evaluated using

$$h(Y) = -\int_0^\infty p_Y(y)\log p_Y(y)dy, \tag{4}$$

and

$$h(Y|X) = \frac{1}{2}\mathrm{E}_x[\log(1+x^2)] - \frac{1}{2}\log 2 + (1+\frac{\gamma}{2}). \tag{5}$$

The equations (4) and (5) were first used by Taricco [9] to apply Lagrange optimization to obtain the channel capacity.

A. Output Conditional Entropy

When the input distribution is complex Gaussian, both the real and imaginary of x are independent and Gaussian, and the pdf of $|x|$ is Rayleigh and given by

$$p_X(x) = \frac{2x}{\Omega_x}\exp\left(\frac{-x^2}{\Omega_x}\right),\ x \geq 0, \tag{6}$$

where Ω_x is the average mean squared input power. The output conditional entropy when the input distribution is complex Gaussian is given by [11]

$$h(Y|X) = \frac{1}{2}C_{\text{rcsi}} - \frac{1}{2}\log 2 + (1+\frac{\gamma}{2}), \tag{7}$$

where

$$C_{\text{rcsi}} = -\exp\left(\frac{1}{snr}\right)\mathrm{Ei}\left(\frac{-1}{snr}\right), \tag{8}$$

is the channel capacity when the CSI is perfectly known at the receiver [6], [7], [8], snr is the signal to noise ratio, and $\mathrm{Ei}(x) = -\int_{-x}^\infty e^{-t}/t\,dt$ is the exponential integral.

B. Output Entropy

For the input distribution (6), we obtain the output pdf

$$p_Y(y) = \int_0^\infty p_X(x)p_{Y|X}(y|x)dx \tag{9a}$$

$$= \int_0^\infty \frac{2x}{\Omega_x}\exp\left(\frac{-x^2}{\Omega_x}\right)\frac{y}{1+x^2}\exp\left[\frac{-y^2}{2(1+x^2)}\right]dx. \tag{9b}$$

Substituting (9) in (4) we get the output entropy

$$h(Y) = -\int_0^\infty \int_0^\infty \frac{2xy}{\Omega_x(1+x^2)}\exp\left[\frac{-x^2}{\Omega_x} - \frac{y^2}{2(1+x^2)}\right]dx \times\log\left\{\int_0^\infty \frac{2xy}{\Omega_x(1+x^2)}\exp\left[\frac{-x^2}{\Omega_x} - \frac{y^2}{2(1+x^2)}\right]dx\right\}dy. \tag{10}$$

Since this integral can not be evaluated analytically, we will show the use of Hermit-Gauss quadrature to arrive at a closed form expression.

C. Gaussian Quadrature and Hermit Polynomials

Gaussian quadrature formulas are useful in numerical integrations and provides the fast and accurate results [13]. The common investigated method for approximating a definite integral is $\int_a^b \omega(x)f(x)dx \simeq \sum_{i=1}^q A_i f(x_i)$, assuming the moments are defined and finite or bounded of the function $\omega(x)$. The Gaussian quadrature formula has a degree of precision or exactness m if the solution is exact whenever $f(x)$ is a polynomial of degree $\leq m$ or equivalently, whenever $f(x) = \{1, x, ..., x^m\}$ and it is not exact for $f(x) = x^{m+1}$. The x_i are called the nodes of the formula and A_i are called coefficients (or weights). If $\omega(x)$ is non negative in $[a,b]$, then n points and coefficients can be found to make the solution exact for all polynomials of degree $\leq 2q-1$ and it is the highest degree of precision which can be obtained using q points [13].

If the $\omega(x)$ is in the form of e^{-x^2}, the solution to the integral can be found using the roots (nodes) of the Hermit polynomial $H_q(x)$ [14] and the weights given by

$$\omega_i = \frac{2^{q-1}q!\sqrt{\pi}}{q^2[H_{q-1}(x_i)]^2}.$$

The roots and the weights are excessively given in [14] for $a=0$ and $b=\infty$ with $q=15$.

D. Output entropy $h(y)$ in Closed form

Define $t^2 = x^2/\Omega_x$, where $dx = \Omega_x dt$, then substitution into (9) gives

$$p_Y(y) = \int_{t=0}^{t=\infty} e^{-t^2} \frac{2ty}{(1+\Omega_x x^2)} \exp\left[\frac{-y^2}{2(1+\Omega_x x^2)}\right] dt. \tag{11}$$

This integral is in the form of $\int_a^b \phi(v)\omega(v)dv$ where $\omega(v) \equiv e^{-t^2}$. Therefore it can be evaluated using Hermit polynomials in the form of $p_Y(y) = \sum_{j=1}^{q} \omega_j f(v_j)$. The quantities v_j and ω_j are the roots and the weights of the Hermit polynomials respectively. Applying these weights and roots in (11) we get

$$p_Y(y) = \sum_{j=1}^{q} \omega_j \frac{2v_j y}{(1+\Omega_x v_j^2)} \exp\left[\frac{-y^2}{2(1+\Omega_x v_j^2)}\right]. \tag{12}$$

Using this result, we get the output entropy

$$h(Y) = -\int_0^\infty \left\{ \sum_{j=1}^{q} \omega_j \frac{2v_j y}{(1+\Omega_x v_j^2)} \exp\left[\frac{-y^2}{2(1+\Omega_x v_j^2)}\right] \right\} \times \log\left\{ \sum_{j=1}^{q} \omega_j \frac{2v_j y}{(1+\Omega_x v_j^2)} \exp\left[\frac{-y^2}{2(1+\Omega_x v_j^2)}\right] \right\} dy. \tag{13}$$

Taking the integration inside and substituting $t^2 = y^2/(2(1+\Omega_x v_j^2))$ in (13), we obtain the ℓ^{th} term where $\ell = 1, ..., q$

$$h(Y)_\ell = -\int_0^\infty e^{-t^2}(4\omega_\ell v_\ell t) \log\left\{ \sum_{j=1}^{q} \frac{2\sqrt{2}\omega_j v_j t}{(1+\Omega_x v_j^2)} \times \sqrt{(1+\Omega_x v_j^2)} \exp\left[-t^2 \frac{(1+\Omega_x v_\ell^2)}{(1+\Omega_x v_j^2)}\right] \right\} dt. \tag{14}$$

The integral in this l^{th} term has the form $\int_a^b \phi(v)\omega(v)dv$ where $\omega(v) \equiv e^{-t^2}$. Using Hermit polynomials to evaluate the integral in (14), we get

$$h(Y)_\ell = \sum_{i=1}^{r} \omega_i (4\omega_\ell v_\ell v_i) \log\left\{ \sum_{j=1}^{q} \frac{2\sqrt{2}\omega_j v_j v_i}{(1+\Omega_x v_j^2)} \times \sqrt{(1+\Omega_x v_\ell^2)} \exp\left[-v_i^2 \frac{(1+\Omega_x v_\ell^2)}{(1+\Omega_x v_j^2)}\right] \right\}. \tag{15}$$

Taking the summation for $\ell \in \{1, ..., q\}$, we obtain

$$h(Y) = -\sum_{\ell=1}^{q} h(Y)_\ell \tag{16a}$$

$$= -\sum_{\ell=1}^{q} \sum_{i=1}^{r} (4\omega_i v_i \omega_\ell v_\ell) \log\left\{ \sum_{j=1}^{q} \frac{2\sqrt{2}\omega_j v_j v_i}{(1+\Omega_x v_j^2)} \times \sqrt{(1+\Omega_x v_\ell^2)} \exp\left[-v_i^2 \frac{(1+\Omega_x v_\ell^2)}{(1+\Omega_x v_j^2)}\right] \right\}. \tag{16b}$$

The $h(Y)$ presented in closed form in (16) using Hermit-Gauss quadrature is very useful in finding the mutual information for any SNR. The computational time is much less than the numerical integrations to be carried out with high accuracy. The mutual information $I(X;Y) = h(Y) - h(Y|X)$ can be found subtracting (7) from (16).

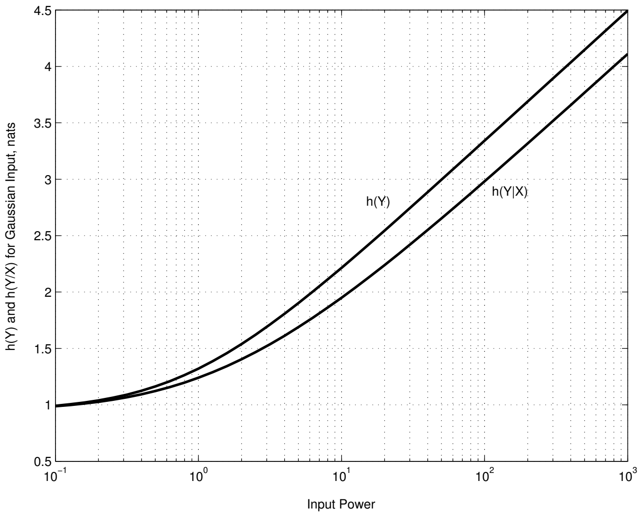

Fig. 1. Comparison of the output entropy $h(Y)$ and the output conditional entropy $h(Y|X)$ of non-coherent Rayleigh fading channel when the input distribution is complex Gaussian.

IV. NUMERICAL RESULTS AND ANALYSIS

Fig. 1 shows the numerical values of $h(Y)$ and $h(Y|X)$ plotted using the closed form solutions derived in (7) and (16). The output entropy (16) is plotted using 15 roots and weighting factors given in [14], and therefore, $r = q = 15$. The mutual information using the closed form expressions derived in this paper for output entropies when the input distribution is complex Gaussian is shown in Fig. 2. These expressions are very useful since the roots v and the weights ω are available in tabulated form for various degrees of m of the polynomial interested in [13]. The computation time is negligible when compared to the numerical integrations to be solved and the accuracy is very high and typically known as exact on appropriate selection of the numbers of roots and the weighting factors [14].

The capacity of non-coherent Rayleigh fading channel [4] attained with a discrete input is plotted in Fig. 2 to compare with the mutual information obtained when the input is Gaussian distributed. Also, Fig. 2 has the asymptotic value of the mutual information, $\lim_{\Omega_x^2 \to \infty} (C_{\text{cnf}} - C_{\text{rcsi}}) = \gamma$ shown in [10], when the input distribution is complex Gaussian. The difference between the channel capacity and the mutual information obtained with Gaussian input is shown as a percentage of the channel capacity in Fig. 3 which reaches 44% at high SNR. This analysis shows that the mutual information obtained with Gaussian distributed input is very low compared to the channel capacity in the absence of CSI indicating the sub-optimality of Gaussian signalling in non-coherent Rayleigh fading channels.

V. CONCLUSIONS

The mutual information of a non-coherent Rayleigh fading channels when the input distribution is complex Gaussian is expressed in closed form using Gaussian quadrature formulas with very high accuracy. The mutual information is bounded by the SNR and much lower than the channel capacity

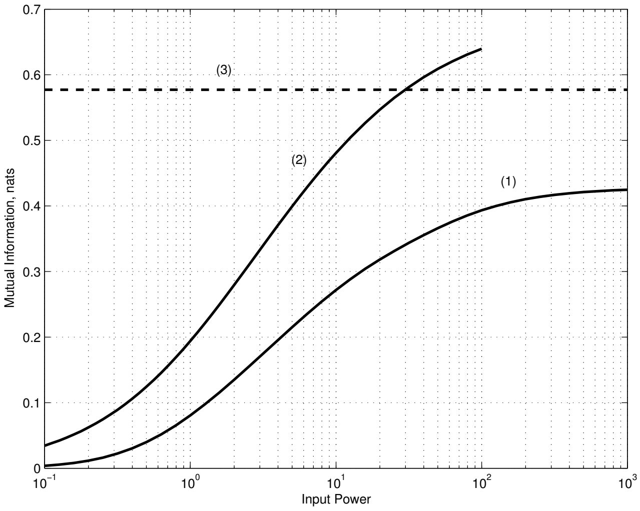

Fig. 2. The mutual information (1) of non-coherent Rayleigh fading channel when the input is Gaussian distributed vs the channel capacity (2) simulated with two discrete inputs. The dashed line (3) shows the limit of the mutual information when SNR approaches infinity.

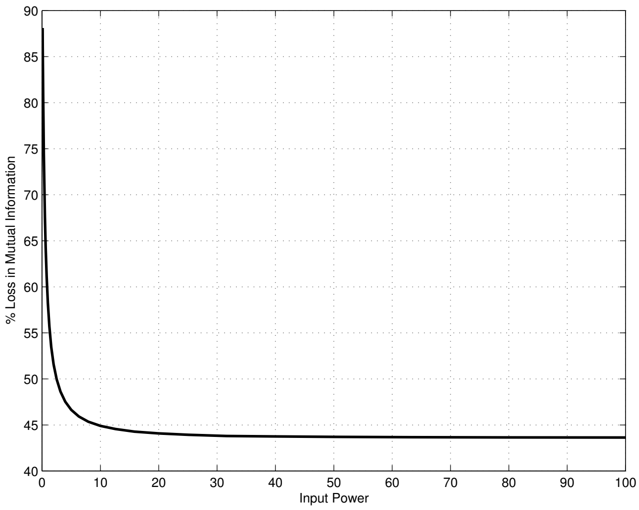

Fig. 3. The % loss in mutual information achieved with the Gaussian distributed input compared to the channel capacity.

achieved with discrete input. The results of this paper can be used as a lower bound of the coherent Rayleigh fading channels when the receiver is provided with imperfect CSI in the presence of the high noise or the failure in proving such information by the pilot symbols inserted in the transmitted sequence.

VI. ACKNOWLEDGEMENTS

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