Cramér-Rao Lower Bounds for the Synchronization of UWB Signals

J. Zhang, R. A. Kennedy, T. D. Abhayapala

***Abstract*— In this paper, we present Cramér-Rao lower bounds (CRLBs) for the synchronization of UWB signals which should be tight lower bounds for the theoretical performance limits of UWB synchronizers. The CRLBs are investigated for both single pulse systems and time-hopping systems in AWGN and multipath channels. Insights are given into the relationship between CRLBs for different Gaussian monocycles. An approximation method of the CRLBs is discussed when nuisance parameters exist. CRLBs in multipath channels are studied and formulated for three scenarios depending on the way multipath interference is treated. We find that larger number of multipath implies higher CRLBs and inferior performance of synchronizers, and multipath interference on CRLBs can not be eliminated completely except in very special cases. As every estimate of time delay could not be perfect, the least influence of synchronization error on the performance of receivers is quantified.**

***Index Terms*— Ultra Wideband, Synchronization, Cramér-Rao lower bounds**

I. INTRODUCTION

Ultra Wideband (UWB) is a promising technique in the application of short-range high-speed wireless communication and precise location tracking. Typically, ultra narrow pulses, such as Gaussian monocycles [1], are modulated to transmit information. These pulses could be narrower than 1 nanosecond. This brings very stringent synchronization requirements.

A UWB signal is basically a baseband signal without phase and carrier information, hence time delay estimation is the main task of a synchronizer. This synchronizer could be one in a simple single-pulse UWB system, however, due to the power limitation imposed by FCC [2], UWB pulses are generally combined with spread spectrum techniques, especially time-hopping (TH), to achieve multiuser access, to ensure sufficient received energy and to mitigate interference to existing wireless systems. Similar to traditional spread spectrum systems, the synchronization of a time-hopping UWB system is accomplished in two steps: *code acquisition* followed by *code tracking*. The former, involving the optimization of search strategies, tries to determine the phase of the incoming pseudo-noise (PN) sequence within a fraction of chip width. The latter refers to the process of achieving and maintaining fine alignment of the chip boundaries of the incoming and locally generated PN sequences.

As UWB pulses are very narrow, very stringent synchronization requirements are incurred, and timing errors usually imply marked degradation of receiver performance [3]. Abundant research on the design and performance of synchronization systems have been reported in the literature, e.g., [4]–[6]. These techniques can be transplanted into UWB systems with some modifications to meet the stringent timing requirement, as discussed in [7]–[10]. Different to them, in this paper, we try to find some general performance limitations for UWB synchronizers, and provide guidelines for the system design within acceptable performance region.

It is known that in the presence of noise, perfect synchronization cannot be achieved. For UWB systems with stringent timing requirements, it is of special interest to characterize this synchronization error and its influence on the performance of detectors. This task becomes even more urgent when we realize that the radiated power of UWB signals is so low that the channel estimates could contain large errors and the performance of synchronizers could be largely deteriorated. Under these conditions, is it still possible for UWB synchronizers to reach a satisfying accuracy of timing locking? Some common performance parameters to evaluate synchronizers are *tracking time*, *S-curve behavior* and *probability of success*. However, in order to provide benchmarks for actual UWB synchronizers, we are more interested in understanding their theoretical performance limits. In the theory of parameter estimation, Cramér-Rao Lower Bound (CRLB) is most widely used in evaluating the performance of estimates.

The CRLB [11] is a fundamental lower bound on the variance of any unbiased estimator. The analysis of CRLB for traditional systems is well founded [5], [12]–[19], but for UWB, there is no systematic work reported yet to our knowledge. The evaluation of the CRLB is generally mathematically quite difficult when the observed signal contains, besides the parameter to be estimated, also some nuisance parameters that are unknown [14], [19]. These nuisance parameters could be the transmitted data and sometimes, multipath gains and delays which arise in fading channels. When the nuisance parameters are present, the modified CRLB (MCRB) [13]–[15], and the asymptotic CRLB (ACRLB) [14], are good approximations to the true CRLB at higher signal-to-noise ratio (SNR), and the lower-SNR limit of the CRLB is approximated in [18] by applying a Taylor Series expansion.

This paper is concerned with evaluating the CRLB for UWB signals. Both single-pulse systems and time-hopping systems are considered. For time-hopping, the CRLB should be a lower bound for the performance of code tracking. The structure of this paper is as follows. In Section II, the problem is formulated. In Section III, considering AWGN channels, the CRLBs for single-pulse systems are investigated in both cases of known and unknown transmitted data. Some insights

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into the relationship between CRLBs for different Gaussian monocycles are given explicitly. We also highlight an oversight in the lower-SNR approximation method [18] and provide a possible solution to remedy this problem by tightly locating the range of SNR γ_s. These results can be readily extended to a TH UWB system in AWGN channels with minor modifications. In Section IV, we extend this work to more practical multipath channels while considering an unmodulated TH system. Depending on the way multipath interference is treated in a practical synchronizer, three scenarios are analyzed when multipath interference contributes as an increase of noise variance or multiple synchronization parameters. In Section V, the influence of synchronization error on the performance of receiver is quantified, which may be the least influence a UWB correlator receiver can expect. Finally, numerical results are given in Section VI to verify some analytical results and illustrate the effect of pulse truncation on CRLBs.

II. Problem Formulation

Binary pulse position modulation (BPPM) and binary phase shift keying modulation (BPSK, or antipodal modulation) are considered here. Let $s(t)$ be the transmitted UWB signal. In a single-pulse system, $s(t) = \sum_i b_i \omega(t - iT_s)$ for BPSK, and $s(t) = \sum_i \omega(t - iT_s - b_i T_d)$ for BPPM, where $\omega(t)$ is a UWB pulse, $b_i \in \{-1, +1\}$ is the i^{th} transmitted data, T_s is the symbol period, and T_d is the time offset of BPPM. In a unmodulated time-hopping system, $s(t) = \sum_i s_i(t) = \sum_i \sum_{j=1}^{N_f} \omega(t - iT_s - jT_f - c_j T_c)$ where $s_i(t)$ is the ith transmitted symbol, T_f is the frame width, N_f is the number of frames in a symbol, T_c is the chip width, and c_j are the time-hopping codes.

The UWB pulses considered are series of Gaussian monocycles $\omega(t; n, t_p)$, which are scaled and/or differentiated versions of the basic Gaussian waveform $\omega_0(t) = \exp(-2\pi t^2)$, that is, $\omega(t; n, t_p) = \omega_0^{(n)}(t/t_p)$, where the superscript ${}^{(n)}$ stands for n-order differentiation with respect to t, and t_p parameterizes the width of the pulse.

To ensure equal energy of monocycles, a coefficient $\varepsilon(n, t_p)$ is introduced, and let $\omega(t) = \varepsilon(n, t_p)\omega(t; n, t_p)$. Denote the energy of $\omega(t)$ as E_p and symbol SNR as γ_s, then $\varepsilon(n, t_p)$, depending on n and t_p, satisfies

$$\varepsilon^2(n, t_p) = \frac{E_p}{\int_{-\infty}^{+\infty} \omega^2(t; n, t_p) dt}. \tag{1}$$

When passing through a pure AWGN channel $n(t)$, the received signal $r(t)$ becomes

$$r(t) = s(t - \tau) + n(t), \tag{2}$$

where every sample of $n(t)$ is Gaussian distributed with zero mean and variance σ_0^2, and τ is the timing delay to be estimated.

When passing through a frequency selective fading channel, $h(t) = \sum_{\ell=1}^{L} a_\ell \delta(t - \tau_\ell)$, the received signal is given by

$$r(t) = \sum_{\ell=1}^{L} a_\ell s(t - \tau_\ell) + n(t), \tag{3}$$

where a_ℓ and τ_ℓ are real multipath gains and delays, respectively. Note the time delay τ between transmitter and receiver is merged into τ_ℓ.

Due to the low duty cycle of UWB signals, we assume the received signal is free of intersymbol interference (ISI) unless indicated otherwise. For the effect of ISI and the design of training sequence accordingly, the readers can refer to [20], [21].

For the AWGN model in (2), for the purpose of forming estimates based on K independent observations, the received signal can be represented as a vector model

$$\mathbf{r} = \mathbf{s}(\mathbf{b}, \tau) + \mathbf{n}, \tag{4}$$

where $\mathbf{r} = [r_1, \cdots, r_K]$, $\mathbf{s} = [s_1, \cdots, s_K]$ and $\mathbf{n} = [n_1, \cdots, n_K]$ are the sample vectors of the received signal $r(t)$, the transmitted signal $s(t - \tau)$ and the noise $n(t)$, respectively, and $\mathbf{b} = \{b_i\}$ are the transmitted data sequence.

Suppose an unbiased estimate $\hat{\tau}$ of the time delay τ can be generated from (4), then the estimation error variance is lower bounded by the CRLB $\mathrm{E}_{\mathbf{r}}[(\hat{\tau} - \tau)^2] \geq \mathrm{CRLB}(\tau)$, where

$$\mathrm{CRLB}(\tau) = \Big(\mathrm{E}_{\mathbf{r}|\tau}\big[-\frac{d^2}{d\tau^2}\ln(p(\mathbf{r}|\tau))\big]\Big)^{-1}. \tag{5}$$

In (5), the conditional pdf $p(\mathbf{r}|\tau)$ is the likelihood function of τ, and the expectation $\mathrm{E}_{\mathbf{r}|\tau}[\cdot]$ is with respect to $p(\mathbf{r}|\tau)$.

The likelihood function $p(\mathbf{r}|\tau)$ can be obtained by averaging the joint likelihood function $p(\mathbf{r}|\mathbf{b}, \tau)$ over the *a priori* distribution of the data $\mathbf{b}$: $p(\mathbf{r}|\tau) = \mathrm{E}_{\mathbf{b}}[p(\mathbf{r}|\mathbf{b}, \tau)]$. When $\mathbf{b}$ is known, $p(\mathbf{r}|\tau) = p(\mathbf{r}|\mathbf{b}, \tau)$.

Since the additive noise $n(t)$ is white and zero mean, the joint conditional pdf $p(\mathbf{r}|\mathbf{b}, \tau)$ can be expressed as

$$\begin{aligned} p(\mathbf{r}|\mathbf{b}, \tau) &= \prod_{k=1}^{K} \frac{1}{\sqrt{2\pi}\sigma_0} \exp(-\frac{1}{2\sigma_0^2}(r_k - s_k)^2) \\ &= (\frac{1}{\sqrt{2\pi}\sigma_0})^K \exp(-\frac{1}{2\sigma_0^2}\sum_{k=1}^{K}(r_k - s_k)^2). \end{aligned} \tag{6}$$

Applying the signal orthogonal expressions [6, p.335] or letting the number of samples K go to infinity [11, p.274] (or from the standpoint of generating sufficient statistics), we have

$$\sum_{k=1}^{K}(r_k - s_k)^2 = \int_{T_o} [r(t) - s(t - \tau)]^2 dt, \tag{7}$$

where T_o is the observation period.

Now, a continuous-time equivalent of $p(\mathbf{r}|\mathbf{b}, \tau)$ can be developed. Considering the subsequent operations of logarithm and differentiation, only terms related to $\mathbf{b}$ and τ will be retained. Then the evaluation of $p(\mathbf{r}|\mathbf{b}, \tau)$ is equivalent to evaluating the likelihood function

$$\Lambda(\mathbf{b}, \tau) = \exp\big(\frac{1}{2\sigma_0^2}(2\int_{T_o} r(t)s(t - \tau)dt - \int_{T_o} s^2(t - \tau)dt)\big). \tag{8}$$

The process from (4) to (8) can be applied to the multipath model (3) with minor modifications.

III. CRLB FOR SINGLE-PULSE SYSTEMS IN AWGN CHANNELS

A. CRLB with Known Transmitted Data

The CRLB with known $\mathbf{b}$, further derived from (8) or directly from [15], has the form

$$\mathrm{CRLB}(\tau;\mathbf{b}) = \frac{\sigma_0^2}{\int_{T_o} \dot{s}^2(t-\tau)dt}, \quad (9)$$

where $\dot{s}(t-\tau)$ denotes first partial differentiation with respect to τ.

Assuming that the pulse is strictly restricted within a symbol period, and $T_o = NT_s$, where N is the number of symbols contained in the observation period (one pulse per symbol in this case), then for both BPSK and BPPM, the denominator in (9) equals $N\int_{T_s}\dot{\omega}^2(t-\tau)dt$. For a specific monocycle, the lower variance bound becomes

$$\mathrm{CRLB}(\tau;\mathbf{b}) = \frac{1}{N\gamma_s}\frac{\int_{T_s}\omega^2(t-\tau;n,t_p)dt}{\int_{T_s}\dot{\omega}^2(t-\tau;n,t_p)dt}, \quad (10)$$

where the symbol SNR is $\gamma_s = E_p/\sigma_0^2$.

If the symbol period T_s is large enough so that most of the energy of the pulse concentrates within T_s, we can express (10) in frequency domain

$$\mathrm{CRLB}(\tau;\mathbf{b}) = \frac{1}{N\gamma_s}\frac{\int_{-\infty}^{+\infty}|W(f;n,t_p)|^2 df}{\int_{-\infty}^{+\infty} f^2|W(f;n,t_p)|^2 df}, \quad (11)$$

where $W(f;n,t_p)$ is the Fourier Transform of $\omega(t;n,t_p)$.

According to the properties of the Fourier Transform of derivatives of functions, we find explicit relationships exist between the CRLBs of monocycles with different n but same t_p, that is,

$$\begin{aligned}&\frac{\mathrm{CRLB}(\tau;\mathbf{b})_n}{\mathrm{CRLB}(\tau;\mathbf{b})_{n+1}}\\ &= \frac{\int_{-\infty}^{+\infty}|W(f;n,t_p)|^2 df \cdot \int_{-\infty}^{+\infty} f^4|W(f;n,t_p)|^2 df}{\left(\int_{-\infty}^{+\infty} f^2|W(f;n,t_p)|^2 df\right)^2} \quad (12)\\ &> 1, \quad (13)\end{aligned}$$

where the inequality follows from an application of Schwarz's inequality. This inequality implies that monocycles with higher order differentiation have the potential for better performance in the sense of lower synchronization error variance.

For monocycles with different t_p but same n, the ratio between their CRLBs can be found as

$$\frac{\mathrm{CRLB}(\tau;\mathbf{b})_{t_{p1}}}{\mathrm{CRLB}(\tau;\mathbf{b})_{t_{p2}}} = \left(\frac{t_{p1}}{t_{p2}}\right)^2, \quad (14)$$

which implies that monocycles with smaller t_p (narrower effective pulse width) have the potential for better synchronization performance.

B. CRLB with Unknown Randomly Transmitted Data

For PPM, the uncertainty of time jitter introduced by modulation will cause large synchronization error when the transmitted data is random and unknown. When further methods are adopted to solve this problem, the CRLB analysis in these cases will usually be based on a system model similar to the one with known data. Hence we only consider BPSK UWB signals in this subsection.

For BPSK, the likelihood function in (8) becomes

$$\Lambda(\mathbf{b},\tau) = \exp\Big(\sum_{i=1}^{N}\frac{1}{\sigma_0^2}\big(b_i y(\tau) - b_i^2\gamma_s\big)\Big), \quad (15)$$

where $y(\tau) = \int_{T_s} r(t)\omega(t-\tau)dt$.

Dropping the constant term $\gamma_s\sum_{i=1}^{N}(b_i^2) = N\gamma_s$, we obtain the log-likelihood function of $p(\mathbf{r}|\tau)$ as

$$\begin{aligned}\mathcal{L}(\mathbf{r};\tau) &= \ln p(\mathbf{r}|\tau) = \ln \mathrm{E}_{\mathbf{b}}\big[\Lambda(\mathbf{b},\tau)\big]\\ &= \sum_{i=1}^{N}\ln \mathrm{E}_{b_i}\big[\exp(\frac{1}{\sigma_0^2}b_i y(\tau))\big]\\ &= N\ln\cosh(\frac{1}{\sigma_0^2}y(\tau)). \quad (16)\end{aligned}$$

By differentiating $\mathcal{L}(\mathbf{r};\tau)$ twice with respect to τ, we get

$$\begin{aligned}\frac{\partial^2\mathcal{L}(\mathbf{r};\tau)}{\partial\tau^2} &= \frac{N}{\sigma_0^2}\tanh(\frac{y(\tau)}{\sigma_0^2})\ddot{y}(\tau) +\\ &\frac{N}{\sigma_0^4}\big(1-\tanh^2(\frac{y(\tau)}{\sigma_0^2})\big)\dot{y}^2(\tau), \quad (17)\end{aligned}$$

where $\dot{y}(\tau)$ and $\ddot{y}(\tau)$ denote first and second derivative of $y(\tau)$ with respect to τ.

Due to the nonlinear function $\tanh(\cdot)$ in (17), an analytical solution for $\mathrm{E}_{\mathbf{r}|\tau}[\partial^2\mathcal{L}(\mathbf{r};\tau)/\partial\tau^2]$ is infeasible.

Since the pulse energy is restricted to be very low by the FCC [2] (the maximum power of a transmitted pulse with bandwidth 7GHz is only 0.5mW), one can refer to the lower-SNR limit of CRLB in [18], applying a Taylor extension of the likelihood function $p(\mathbf{r}|\mathbf{b},\tau)$, to obtain a similar result for UWB. One thing we wish to emphasize here is, in [18], the statistical property of the likelihood function $L(\mathbf{u},\tau)$ (original notation in [18]) is somewhat ignored. Due to $L(\mathbf{u},\tau)$ containing Gaussian variables with variance comparable to the reciprocal of symbol SNR, more care is needed when dropping the higher order terms in Taylor extension according to the lower symbol SNR assumption. A similar problem arises in an alternative method we introduce below, where this ambiguity is revealed further, and resolved by tightly locating the value of the symbol SNR.

The alternative method we suggest is also based on approximation. The basic idea is to find best-fitting functions for $\ln(\cosh(\cdot))$ in a piecewise fashion. To make analysis tractable, these functions are polynomials with order smaller than 3. But they should not be constructed by only considering the goodness of fit due to the succeeding expectation operation. This is because, $y(\tau)$ is a random variable and when we deal with the expectation operation, we have to make sure that all the possible samples of $y(\tau)$ are involved. Although integrating these polynomials in segments is feasible, it can not produce a closed form result and is still a numerical method. Instead, we try to construct each polynomial in which the variable space supports the sampling space. It seems impossible as the pdf of $y(\tau)$ distributes in the entire one-dimension real space. We overcome this obstacle by assuring that most of samples (say, 99%) are located in the interval of interest.

With this criterion in mind, we find a three-segments approximation is a good choice by studying the shape of the waveform $\ln(\cosh(x))$. A detailed discussion is shown in Appendix A. Examples of such three lower order (≤ 2) polynomials are

$$\ln(\cosh(x)) \approx \begin{cases} 0.3x^2 + 0.14x - 0.018, & |x| < 1.5 \\ 0.000034x^2 + x - 0.69, & 1.5 \leq |x| \leq 2.5 \\ x - 0.69, & |x| > 2.5. \end{cases} \tag{18}$$

The root mean squared approximation errors are 0.0081, 0.0091, 0.0031 for the three pieces, respectively. The ranges of corresponding SNR γ_s are $[-\infty, -6.25]$dB, $[10.3, 10.8]$dB and $[10.8, +\infty]$dB, respectively, which can be determined according to the way addressed in Appendix A.

Due to non-existence of a polynomial with goodness of fit and a fully covered sampling space simultaneously, an appropriate interval corresponding to SNR range of $(-6.25, 10.3)$dB can not be found.

Represent a general 2-order polynomial function as $\ln(\cosh(x)) \approx f(x) = ax^2 + b|x| + c\,|_{x=y(\tau)/\sigma_0^2}, |x| \in [x_1, x_2]$, where $0 \leq x_1 < x_2$, we derive the CRLB based on it below.

The reciprocal of the CRLB can be calculated as

$$-\mathrm{E}_{\mathbf{r}|\tau}[\frac{\partial^2 \mathcal{L}(\mathbf{r};\tau)}{\partial \tau^2}] = -N\,\mathrm{E}_{\mathbf{r}|\tau}[2ax\ddot{x} + 2a\dot{x}^2 + b\ddot{x}], \tag{19}$$

where we utilize

$$\frac{d^2\,|x|}{d\,\tau^2} = \frac{d^2}{d\,\tau^2}(\sqrt{x^2}) = \frac{d}{d\,\tau}\dot{x} = \ddot{x}. \tag{20}$$

As shown in Appendix B, these expectations are given by

$$\mathrm{E}_{\mathbf{r}|\tau}[\dot{x}^2] = \frac{1}{\sigma_0^2}\int_{T_s} \dot{\omega}^2(t-\tau)dt, \tag{21}$$

$$\mathrm{E}_{\mathbf{r}|\tau}[x\ddot{x}] = -\frac{\gamma_s + 1}{\sigma_0^2}\int_{T_s} \dot{\omega}^2(t-\tau)dt, \tag{22}$$

$$\mathrm{E}_{\mathbf{r}|\tau}[\ddot{x}^2] = -\frac{1}{\sigma_0^2}\int_{T_s} \dot{\omega}^2(t-\tau)dt. \tag{23}$$

Then for a specific monocycle $\omega(t; n, t_p)$, the CRLB is

$$\mathrm{CRLB}(\tau) = \frac{1}{N(2a\gamma_s + b)\gamma_s}\frac{\int_{T_s}\omega^2(t-\tau;n,t_p)dt}{\int_{T_s}\dot{\omega}^2(t-\tau;n,t_p)dt}. \tag{24}$$

By substituting a and b with the coefficients of polynomials in (18), the CRLBs for different γ_s are readily obtainable. It is clear that the relationship between CRLBs for monocycles with different n or t_p is identical to that when the transmitted data is known.

By comparing (10) and (24), we find $\mathrm{CRLB}(\tau;\mathbf{b})/\mathrm{CRLB}(\tau) = 2a\gamma_s + b$. Referring to (18), it is obvious that the CRLB with unknown data is always larger than that with known data for the lower SNR case, and converges to $\mathrm{CRLB}(\tau;\mathbf{b})$ in the higher SNR case, which coincides with the attributes of ACRB given in [14].

IV. CRLB for Time-Hopping UWB Systems in Selective-Fading Channels

When the channel is AWGN, the analysis and results in Section III can be applied to time-hopping UWB systems with minor modifications. The change can be merged into the symbol SNR γ_s, that is, γ_s equals to the ratio between the energy of N_f pulses and the noise variance σ_0^2 for TH UWB systems. In this section, we will focus on selective fading channels.

Synchronization in selective fading channels is a challenging task. The performance largely depends on the schemes and algorithms. Based on the way multipath signals are treated, these systems can be divided into three categories. Accordingly, we consider the CRLB for each of them. Since CRLB with unknown data is straightforward but computationally complex as derived in Section III, we only consider the case of known data $\mathbf{b}$ here.

A. Passive methods: Regarding Multipath signals as Interference

This refers to methods that apply general synchronizers, such as early/late gates, while treating multipath signals as interference [22], [23], or partly utilizing multipath energy [24], or using a whitening filter before a synchronizer [25]. The effect of multipath interference on synchronizers has been studied in [22], [23], [26]–[29]. From the viewpoint of CRLB, all these methods can be generalized to a model in which only a specific multipath is of interest. Mathematically, we can represent this model as

$$r(t) = a_m s(t-\tau_m) + \underbrace{\sum_{\ell=1,\ell\neq m}^{L} a_\ell s(t-\tau_\ell)}_{\text{interference}} + n(t), \tag{25}$$

where a_m and τ_m are the parameters to be estimated.

Generally, the received signal $r(t)$ first passes through a PN code correlator $s_i(t-\hat{\tau}_m)$, where $\hat{\tau}_m$ is the pre-estimated delay, so that the energy of all pulses in a symbol are collected to make an estimation. Then the model in (25) can be further written as

$$r_f(\hat{\tau}_m) = a_m \sum_{i=1}^{N}\phi(\hat{\tau}_m + iT_s - \tau_m) + \underbrace{\sum_{i=1}^{N}\sum_{\ell=1,\ell\neq m}^{L} a_\ell \phi(\hat{\tau}_m + iT_s - \tau_\ell)}_{n_a} + n_f(\hat{\tau}_m), \tag{26}$$

where

$$r_f(\hat{\tau}_m) = \sum_{i=1}^{N}\int_{iT_s} r(t)s_i(t-\hat{\tau}_m)dt, \tag{27}$$

$$\phi(v) = \int_{iT_s} s(t-v)s_i(t)dv, \tag{28}$$

$$n_f(\hat{\tau}_m) = N\int_{iT_s} n(t)s_i(t-\hat{\tau}_m)dt. \tag{29}$$

Successful detection requires sampling $r_f(t)$ at $\hat{\tau}_m = iT_s + \tau_m$ accurately.

Each sample of $n_f(\hat{\tau}_m)$ is Gaussian distributed with zero mean and variance $\sigma^2_{n_f} = NN_fE_p\sigma_0^2$. The component n_a, containing interchip interference and ISI, is hard to model and evaluate without prior knowledge of TH codes and multipath delay. To make the analysis mathematical tractable, here we assume n_a is Gaussian distributed[1] with mean m_{n_a} and variance $\sigma^2_{n_a}$. In Appendix C, more information is given on the parameters of this distribution.

Recall that when considering the CRLB for TH UWB synchronizers in the phase of code tracking, it is reasonable to assume $\phi(t-\tau_m)|_{t=\hat{\tau}_m}$ is restricted in an interval $[-T_\phi, T_\phi]$ where T_ϕ is smaller than half of the frame period ($T_\phi < T_f/2$), then the sum of $\phi(t-\tau_m)$ for N symbols, $\sum_{i=1}^{N}\phi(t+iT_s-\tau_m)$, equals to $N\phi(t-\tau_m)$. Now, the estimation problem can be reformed as

$$r_f(t) = a_m N\phi(t-\tau_m) + n_a + n_f(t), \tag{30}$$

which is a problem of multiple parameters estimation in a Gaussian interference.

Although a_m and τ_m are correlated via the mean power profile of fading, they are usually treated as unknown and deterministic parameters and nonrandom parameter estimation techniques are applied, as the statistical relationship between them are hardly predictable. This means they are not a function of each other any more. Strictly speaking, τ_m is the only synchronization parameter, and CRLB(τ_m) can be obtained when regarding a_m as a nuisance parameter. However, it is known that joint estimation of τ_m and a_m usually gives lower CRLB for τ_m than that in a separate-estimation case [11], [14], [19]. Hence we will focus on joint estimation and generate CRLB(a_m) as a byproduct.

Representing (30) as a vector form $\mathbf{r_f} = a_m N\Phi + \mathbf{n_a} + \mathbf{n_f}$ and applying the similar process from (6) to (8), the joint log-likelihood function $\mathcal{L}(\mathbf{r_f}; a_m, \tau_m) = \ln p(\mathbf{r_f}; a_m, \tau_m)$ can be obtained as

$$\begin{aligned}\mathcal{L}(\mathbf{r_f}; a_m, \tau_m) = &-\frac{N}{2(\sigma^2_{n_a}+\sigma^2_{n_f})}\int_{2T_\phi}\big(Na_m^2\phi^2(t-\tau_m)\\ &- 2a_m r_f(t)\phi(t-\tau_m) + 2m_{n_a}a_m\phi(t-\tau_m)\big)dt. \end{aligned}\tag{31}$$

Lower bounds on the variances of estimates for the components of a_m and τ_m are given in terms of the diagonal elements of the inverse of the Fisher information matrix $\mathbf{J}^{-1}$ [11]. In this example, the elements of $\mathbf{J}$ equal

$$\mathbf{J} = \begin{pmatrix} -\mathrm{E}[\frac{\partial^2\mathcal{L}(\mathbf{r_f};a_m,\tau_m)}{\partial a_m^2}] & -\mathrm{E}[\frac{\partial^2\mathcal{L}(\mathbf{r_f};a_m,\tau_m)}{\partial a_m\partial\tau_m}] \\ -\mathrm{E}[\frac{\partial^2\mathcal{L}(\mathbf{r_f};a_m,\tau_m)}{\partial a_m\partial\tau_m}] & -\mathrm{E}[\frac{\partial^2\mathcal{L}(\mathbf{r_f};a_m,\tau_m)}{\partial \tau_m^2}] \end{pmatrix}, \tag{32}$$

where the expectation E[$\cdot$] is with respect to $p(\mathbf{r_f}; a_m, \tau_m)$.

Note $\phi(t-\tau_m)$ and a_m are mutually independent, the elements of $\mathbf{J}$ can be calculated as

$$J_{11} = C\int_{2T_\phi}\phi^2(t-\tau_m)dt, \tag{33}$$

$$J_{12} = J_{21} = Ca_m\int_{2T_\phi}\phi(t-\tau_m)\dot{\phi}(t-\tau_m)dt, \tag{34}$$

$$J_{22} = Ca_m^2\int_{2T_\phi}\dot{\phi}^2(t-\tau_m)dt, \tag{35}$$

where C is a constant defined as $C \triangleq N^2/(\sigma^2_{n_a}+\sigma^2_{n_f})$.

The cross terms J_{12} and J_{21} will vanish if we extend the observation period T_ϕ till $\phi(T_\phi) \approx 0$. Then the CRLBs for τ_m and a_m are

$$\mathrm{CRLB}(\tau_m) = 1/J_{22} = \big(Ca_m^2\int_{2T_\phi}\dot{\phi}^2(t-\tau_m)dt\big)^{-1}, \tag{36}$$

$$\mathrm{CRLB}(a_m) = 1/J_{11} = \big(C\int_{2T_\phi}\phi^2(t-\tau_m)dt\big)^{-1}. \tag{37}$$

It is clear that the estimation of time delay τ_m depends on the amplitude of the multipath given that C is the same for all multipath signals, while the estimation of a_m could be independent of τ_m supposing we extend the observation period appropriately. For the performance of synchronizer, the multipath interference contributes as an increased estimate variance[2].

Depending on the Gaussian approximation for the multipath interference n_a, C may be related to a specific monocycle, hence the relationship between CRLB for different monocycles can not be claimed directly.

Finally, we wish to say a little more on the relationship between our model in this section and practical systems. In the literature on synchronizers for spread spectrum systems such as CDMA, we can always find the terms of *fading bandwidth*, *tracking loop bandwidth* and *predetection bandwidth* and discussions on how the relationship between them affecting the performance of synchronizers in a multipath channel (e.g. [26]–[28]). Briefly, the relationship between these bandwidths determines the degree of multipath interference entering the final decision part of the synchronizer. Considering our model, the effect can be regarded as a reduction of noise variance $\sigma^2_{n_a}$.

B. Positive Joint-Detection of Multipath Signals

We refer to the method of jointly detecting fading amplitude and delay of all the multipath signals as a positive one. For CDMA, this method has been well studied in literature such as [16], [17], [25]. And the derivation of CRLB for CDMA systems can been found in [16], [17], [30]. Here, following the process in Section III, we study the CRLB using joint detection for a UWB system where the parameters $\mathbf{a} = [a_1, \ldots, a_\ell, \ldots, a_L]_{1\times L}$ and $\boldsymbol{\tau} = [\tau_1, \ldots, \tau_\ell, \ldots, \tau_L]_{1\times L}$ to be estimated, are treated as unknown but deterministic.

[1] In [11, p309], a general equation is provided for the CRLB of any unbiased estimate in colored noise. But a closed form solution is not readily available.

[2] The multipath interference also very much likely cause a biased estimation according to [27].

Beginning with (3), similar to the derivation from (4) to (8), we can obtain the log-likelihood function $\mathcal{L}(\mathbf{r};\boldsymbol{\tau},\mathbf{a})$ as

$$\mathcal{L}(\mathbf{r};\boldsymbol{\tau},\mathbf{a}) = \frac{1}{\sigma_0^2}\int_{T_o} r(t)\sum_{\ell} a_\ell s(t-\tau_\ell)dt - \frac{1}{2\sigma_0^2}\int_{T_o}[\sum_{\ell} a_\ell s(t-\tau_\ell)]^2 dt. \quad (38)$$

After some manipulation, the Fisher Information Matrix $\mathbf{J}$ has the structure

$$\mathbf{J} = \begin{pmatrix} J_{\boldsymbol{\tau}\boldsymbol{\tau}} & J_{\boldsymbol{\tau}\mathbf{a}} \\ J_{\mathbf{a}\boldsymbol{\tau}} & J_{\mathbf{a}\mathbf{a}} \end{pmatrix}, \quad (39)$$

where $J_{\boldsymbol{\tau}\boldsymbol{\tau}}$, $J_{\boldsymbol{\tau}\mathbf{a}}$, $J_{\mathbf{a}\boldsymbol{\tau}}$ and $J_{\mathbf{a}\mathbf{a}}$ are all $L \times L$ matrices with $[\ell, m]^{th}$ elements

$$J_{\boldsymbol{\tau}\boldsymbol{\tau}}[\ell, m] = \frac{1}{\sigma_0^2}\int_{T_o} a_\ell a_m \dot{s}(t-\tau_\ell)\dot{s}(t-\tau_m)dt, \quad (40)$$

$$J_{\mathbf{a}\mathbf{a}}[\ell, m] = \frac{1}{\sigma_0^2}\int_{T_o} s(t-\tau_\ell)s(t-\tau_m)dt, \quad (41)$$

$$J_{\boldsymbol{\tau}\mathbf{a}}[\ell, m] = J_{\mathbf{a}\boldsymbol{\tau}}[m, \ell] = -\frac{1}{\sigma_0^2}\int_{T_o} a_\ell \dot{s}(t-\tau_\ell)s(t-\tau_m)dt, \quad (42)$$

respectively.

The CRLB for τ_m is just the m^{th} diagonal element of the inverse of $\mathbf{J}$. Use $m = 1$ as an example and rewrite the matrix $\mathbf{J}$ as

$$\mathbf{J} = \begin{pmatrix} J_{11} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}, \quad (43)$$

we have

$$\begin{aligned} \text{CRLB}(\tau_1) &= J_{11}^{-1} + J_{11}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{C}J_{11}^{-1}\mathbf{B})^{-1}\mathbf{C}J_{11}^{-1} && (44)\\ &= J_{11}^{-1} + J_{11}^{-2}\mathbf{B}\widetilde{J_{11}}^{-1}\mathbf{C} && (45)\\ &\geq J_{11}^{-1} && (46) \end{aligned}$$

where $\widetilde{J_{11}}$ is called the *Schur complement* of J_{11} [31, p.175]. Since $\mathbf{J}$ is nonnegative definite, the Schur complement matrix $\widetilde{J_{11}}$ is also nonnegative definite, so is $\widetilde{J_{11}}^{-1}$. At the same time, $\mathbf{B}$ is the transpose of $\mathbf{C}$ since $\mathbf{J}$ is a symmetric matrix in this case. Thus we get $\mathbf{B}\widetilde{J_{11}}\mathbf{C} \geq 0$ and the inequality in (46) follows immediately. Whenever $J_{11} > 0$, we can get the inequality in (46) more readily according to

$$\text{CRLB}(\tau_1) = (J_{11} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} > J_{11}^{-1}. \quad (47)$$

As J_{11}^{-1} can be regarded as the CRLB in an AWGN channel with a known scalar of amplitude, this inequality implies the CRLB in joint detection is always larger than that in the single parameter estimation in an AWGN channel. Then an interesting question arises, whether more multipath means higher CRLB and inferior performance of synchronizer accordingly?

Let us consider a channel with $L-1$ multipath signals. The Fisher Information Matrix $\mathbf{J}'$ can be written as

$$\mathbf{J}' = \begin{pmatrix} J_{11} & \mathbf{B} \\ \mathbf{C} & \mathbf{D}' \end{pmatrix}, \quad (48)$$

with

$$\mathbf{D}' = \begin{pmatrix} \mathbf{D_1} & \mathbf{0} \\ \mathbf{0}^\dagger & 0 \end{pmatrix}, \quad (49)$$

where $\mathbf{0}$ is a $(L-2)\times 1$ zero vector and † stands for transpose operation. Then the CRLB with $L-1$ multipath is

$$\text{CRLB}(\tau_1)_{L-1} = (J_{11} - \mathbf{B}\mathbf{D'}^{-1}\mathbf{C})^{-1}. \quad (50)$$

Comparing $\mathbf{B}\mathbf{D}^{-1}\mathbf{C}$ and $\mathbf{B}\mathbf{D'}^{-1}\mathbf{C}$ gives

$$\begin{aligned} \mathbf{B}\mathbf{D}^{-1}\mathbf{C} - \mathbf{B}\mathbf{D'}^{-1}\mathbf{C} &= \mathbf{B}\big(\mathbf{D}^{-1} - \begin{pmatrix} \mathbf{D_1}^{-1} & \mathbf{0} \\ \mathbf{0}^\dagger & 0 \end{pmatrix}\big)\mathbf{C} && (51)\\ &\geq 0, && (52) \end{aligned}$$

where the inequality in (52) yields from that $\mathbf{D}^{-1} - \mathbf{D'}^{-1}$ is a nonnegative definite matrix as can be proven according to the property of partitioned nonnegative definite matrices (e.g., see [31, p178] and let $\mathbf{D}^{-1} = \mathbf{A}$ in equation (6.10)).

Since $J_{11} > 0$, we have

$$\text{CRLB}(\tau_1)_L > \text{CRLB}(\tau_1)_{L-1}, \quad (53)$$

which shows that more multipath does lead to higher CRLB and inferior performance of synchronizer. Since the number of multipath is closely relevant to the bandwidth of monocycles, we conclude that narrower monocycles will very likely cause larger CRLBs. We did not say "absolutely" because all other variables besides $\mathbf{D}$ during this derivation are assumed unchanged, but it could be unrealistic when different monocycles are applied.

Another key factor with influence on CRLB is the choice of TH codes. When the autocorrelation of TH codes is ideal, both the CRLBs in this subsection and last subsection will be the same and similar to the one in an AWGN channel.

C. Active methods: Cancellation of interference?

From the last two subsections, we have seen that the performance of synchronizers is deteriorated by the multipath interference. It is natural to ask whether the multipath interference can be mitigated or fully eliminated before entering the decision part of a synchronizer?

As shown for CDMA systems in [27], it is possible to remove part of multipath interference in UWB systems. However, unless the correlation of TH codes is ideal, the total removal of multipath interference is impossible due to the existence of $n(t)$. This is because, from Section IV-B, we see that any estimate of parameters, including amplitude and delay, even though unbiased, may still have a nonzero variance in the present of noise. The CRLB can generally be achieved by Maximum Likelihood estimation asymptotically (when the number of observation samples goes to infinity), and the estimation error becomes Gaussian distributed with zero mean and variance equivalent to the CRLB [5], [11]. Therefore, the final signal with a pair of synchronization parameters of interest contains the sum of $2(L-1)$ Gaussian variables, which has a variance larger than the variance of $n(t)$. Since CRLB is proportional to the variance of (interference and) noise, the CRLBs for this pair of parameters will be larger than those in a single path channel. So no matter how perfect the structure and algorithm to remove multipath signal are, the effect of multipath interference can only be mitigated but can not be cancelled thoroughly. This result also partly explains why more multipath generally leads to higher CRLBs.

However, there are some special cases when multipath interference becomes negligible. For example, when the maximal multipath delay is smaller than the frame period in a single pulse system, multipath signals do not interfere with each other due to the low duty cycle of UWB signal structure.

V. Influence of Synchronization error on BER

As every estimate of time delay could not be perfect, we use an example to show the influence of synchronization error on the performance of receivers in UWB systems.

We consider a BPSK modulated single-pulse signal in an AWGN channel like that in Section III. A correlator receiver [32], [33] is used to detect the signal.

The conditional bit-error-ratio (BER), depending on the synchronization error e_τ, is given by

$$P_e(e_\tau) = Q\Big(\frac{\rho(e_\tau)}{\sqrt{E_p}\sigma_0}\Big), \tag{54}$$

where we have assumed that the observation period equals a symbol period such that $N = 1$, $Q(x) \triangleq \int_x^{+\infty} \exp(-t^2/2)/\sqrt{2\pi}dt$ and $\rho(e_\tau) = \int_{T_s} \omega(t)\omega(t-e_\tau)dt$.

Recall that the best theoretically achievable e_τ is Gaussian distributed with zero mean and variance equivalent to the CRLB (denoted by σ_c^2). In the best case, $\sigma_c^2 = \sigma_0^2/(N\int_{T_s} \dot\omega^2(t-\tau)dt)$ from (9) is the smallest. Averaging $P_e(e_\tau)$ over e_τ, we get the mean BER

$$\begin{aligned} P_e &= \mathrm{E}[P_e(e_\tau)] \\ &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma_c} Q\Big(\sqrt{\frac{\rho^2(e_\tau)}{E_p\sigma_0^2}}\Big) \exp\big(\frac{-e_\tau^2}{2\sigma_c^2}\big)\, de_\tau. \end{aligned} \tag{55}$$

Statistically, this is the best achievable performance under certain SNR. This equation can be evaluated numerically by Monte Carlo simulation which requires high computational complexity. Alternatively, we invoke the Hermite-Gauss quadrature [34], and P_e can be accurately approximated by

$$P_e \simeq \frac{1}{\sqrt{\pi}} \sum_{n=1}^{N_h} H_{x_n} Q\big(\frac{\rho(\sqrt{2}\sigma_c x_n)}{\sqrt{E_p}\sigma_0}\big), \tag{56}$$

where N_h is the order of the Hermite polynomial $H_{N_h}(\cdot)$, x_n and H_{x_n} are the zeros (abscissas) and weight factors of N_h-order Hermite polynomial, respectively. These values are tabulated in many mathematical handbooks (e.g., [35]). In experiments, we find 16 coefficients ($N_h = 16$) are enough to generate accurate approximation results.

Further define a variable η as the *degrading ratio* between P_e and $P_e(0) = Q(\sqrt{\gamma_s})$, which is the BER in the case of perfect synchronization. We show the values of η for different monocycles in Section VI to compare the synchronization error robustness of monocycles.

VI. Numerical results

Since for multipath channels, the CRLBs depend on the time-hopping codes and detailed fading channel models, we only show numerical results on the CRLBs in pure AWGN channels in this paper.

In Fig. 1 - Fig. 3, the CRLBs for different monocycles in the case of known data $\mathbf{b}$ are demonstrated. Since in practice, a transmitted monocycle is usually the truncated portion of a whole pulse $w(t; n, t_p)$, this effect of truncation is considered by varying the actual width of pulse in (10).

From Fig. 1, we can see CRLBs are inversely proportional to symbol SNR and the observation period NT_s. The relationship between CRLBs for monocycles with different order n coincides with the analytical results in (13). This can be further observed in Fig. 2, which also depicts the effect of truncated pulses on CRLB. The CRLBs change little even when the truncated portion narrows to $1.6t_p$ (symmetric with respect to $t = 0$). However, with the width of truncated pulse decreasing further, the CRLBs become orderless. Fig. 3 shows the effect of t_p on the CRLBs, and is a direct verification of (14).

Fig. 4 demonstrates the influence of synchronization error on the performance of receivers. It is plotted from (56) using Hermite Gaussian approximation. The influence is notable when the observation window in the stage of synchronization has small width (NT_s), and weakens with N increasing (CRLBs decreasing). The figure also indicates that synchronization errors of different monocycles have very close influence on BER, although the data in experiments shows the influence of monocycles with larger n is a little worse when SNR γ_s is small, and changes toward opposite with SNR increasing.

VII. Conclusions

We have derived the Cramér-Rao lower bounds (CRLBs) for the synchronization of UWB signals for both single pulse systems and time-hopping systems in AWGN and multipath channels. Insights are given on the relationship between CRLBs for different Gaussian monocycles. The CRLBs in AWGN channels are discussed in both cases of known and unknown transmitted data. An approximation method of the CRLB is introduced when nuisance parameters, unknown transmitted data, exist. An oversight in the lower-SNR approximation method [18] is highlighted, and a possible solution is provided by tightly locating the range of SNR γ_s. The CRLBs in multipath channels are studied for three scenarios depending on the way multipath interference treated in a practical synchronizer, where multipath interference contributes as an increase of noise variance or multiple synchronization parameters. It is found that larger number of multipath implies higher CRLBs and inferior performance of synchronizers, and multipath interference on CRLBs can not be eliminated completely except in very limited cases. The least influence of synchronization error on the performance of receivers is quantified. The influence is notable when observation window (NT_s) in the stage of synchronization is small, and weakens with N increasing (CRLBs decreasing). Synchronization errors of different monocycles have very close influence on BER.

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Appendix A: Approximation of $\ln(\cosh(y(\tau)/\sigma_0^2))$

Here we show how to approximate $\ln(\cosh(y(\tau)/\sigma_0^2))$ as low order polynomials in a piecewise fashion and determine the corresponding range of symbol SNR γ_s.

From $y(\tau) = \int_{T_s} r(t)\omega(t-\tau)dt$, $y(\tau)$ has Gaussian distribution $\mathcal{N}(E_1, \gamma_s \sigma_0^4)$, where $|E_1| = E_p$ in the case of perfect synchronization, otherwise $|E_1| < E_p$. As the estimate is usually clustered tightly around the true value in our case, and E_1 changes smoothly for UWB monocycles, we assume $|E_1| \approx E_p$ (This can also be obtained from the assumption of unbiased estimation of τ). Then $y(\tau)/\sigma_0^2$ is also Gaussian distributed $\mathcal{N}(\gamma_s, \gamma_s)$ or $\mathcal{N}(-\gamma_s, \gamma_s)$. For Gaussian distribution, we know that when the distance between a sample and the mean is larger than about $2.6\sqrt{\text{variance}}$, the probability of appearance can be assumed to be zero. Let the interval of interest be $[x_1 \le y(\tau)/\sigma_0^2 \le x_2]$, to ensure most of samples be in this interval, γ_s should satisfy the following equations

$$\begin{cases} -2.6\sqrt{\gamma_s} + \gamma_s \ge |x_1| \\ 2.6\sqrt{\gamma_s} + \gamma_s \le |x_2| \\ |x_1| + 5.2\sqrt{\gamma_s} \le |x_2|. \end{cases} \tag{A-1}$$

Briefly, two guidelines for determining interval $[x_1, x_2]$ are: 1. to ensure this variable space be larger than the sampling space for a specific polynomial and SNR γ_s, and cover the range of γ_s as widely as possible; 2. Although two intervals can overlap, each interval should be fully covered by a single polynomial.

Considering the waveform of $\ln(\cosh(x))$, from $x = 0$ to a samll x_2, it has a very different shape with other segments and has to be approximated separately by a polynomial. This implies there is only one interval covering the segment containing the point zero. For this interval, only x_2 need be determined since $\ln\cosh(\cdot)$ is an even function, and the distributions $\mathcal{N}(\gamma_s, \gamma_s)$ and $\mathcal{N}(-\gamma_s, \gamma_s)$ are symmetric with respect to 0. For the same reason, it is enough to consider the positive value of x_1 and x_2 for other segments hereafter. Note γ_s should be at least 6.76 for $x_1 > 0$, this implies $x_2 > 13.52$.

A well known fact is that $\ln(\cosh(x))$ can be accurately approximated by $x^2/2$ when $|x| \ll 1$, and by $|x| - 0.69$ when $|x| \gg 1$. But this simple scheme is not good enough to be realistic. For example, for a value x_2 as large as 0.5, the approximation error is already 0.005, while the corresponding maximum SNR γ_s is only $0.0324 = -15$dB which is of little interest in practice.

Summarize the description above, we find a three-segments approximation is a good choice. Although the construction of these approximations is not unique, they can be represented as a general 2-order polynomial function $f(x)$, which leads to a general CRLB expression as shown in (24).

Appendix B: Derivation of $\mathrm{E}_{\mathbf{r}|\tau}[\cdot]$

First we derive the autocorrelation of $r(t)$ which will be used in subsequent calculation.

$$\begin{aligned} &\mathrm{E}_{\mathbf{r}|\tau}[r(t_1)r(t_2)] \\ &= \mathrm{E}_{\mathbf{r}|\tau}[(s(t_1-\tau)+n(t_1))(s(t_2-\tau)+n(t_2))] \\ &= \mathrm{E}_{\mathbf{r}|\tau}[s(t_1-\tau)s(t_2-\tau)] + \sigma_0^2\delta(t_1-t_2), \end{aligned} \tag{B-1}$$

where in the last equality, we utilize the assumption that signal and noise are mutually independent and $n(t)$ is AWGN with zero mean and covariance $\sigma_0^2\delta(t_1-t_2)$. Note the expectation with respect to $p(\mathbf{r}|\tau)$ is equivalent to average over the data $\mathbf{b}$ and noise $n(t)$. Recall that the convolution between $r(t)$ and $\omega(t)$ in $y(\tau)$ is only within one symbol period T_s, in the case of ISI-free, we have

$$\mathrm{E}_{\mathbf{r}|\tau}[s(t_1-\tau)s(t_2-\tau)] = \omega(t_1-\tau)\omega(t_2-\tau), \tag{B-2}$$

and

$$\mathrm{E}_{\mathbf{r}|\tau}[r(t_1)r(t_2)] = \sigma_0^2\delta(t_1-t_2) + \omega(t_1-\tau)\omega(t_2-\tau). \tag{B-3}$$

Then expectations on $y(\tau)$ can be calculated as

$$\begin{aligned} &\mathrm{E}_{\mathbf{r}|\tau}[y(\tau)\ddot{y}(\tau)] \\ &= \int_{T_s}\int_{T_s} \mathrm{E}_{\mathbf{r}|\tau}[r(t_1)r(t_2)]\omega(t_1-\tau)\ddot{\omega}(t_2-\tau)dt_1dt_2 \\ &= [\int_{T_s}\omega^2(t)dt + \sigma_0^2]\cdot\int_{T_s}\omega(t-\tau)\ddot{\omega}(t-\tau)dt \\ &= (\gamma_s+1)\sigma_0^2\int_{T_s}\omega(t-\tau)\ddot{\omega}(t-\tau)dt, \end{aligned} \tag{B-4}$$

$$\begin{aligned} &\mathrm{E}_{\mathbf{r}|\tau}[\ddot{y}(\tau)] \\ &= \int_{T_s}\mathrm{E}_{\mathbf{r}|\tau}[r(t)]\ddot{\omega}(t-\tau)dt \\ &= \int_{T_s}\omega(t-\tau)\ddot{\omega}(t-\tau)dt, \end{aligned} \tag{B-5}$$

and

$$\begin{aligned} &\mathrm{E}_{\mathbf{r}|\tau}[\dot{y}^2(\tau)] \\ &= \int_{T_s}\int_{T_s}\mathrm{E}_{\mathbf{r}|\tau}[r(t_1)r(t_2)]\dot{\omega}(t_1-\tau)\dot{\omega}(t_2-\tau)dt_1dt_2 \\ &= \sigma_0^2\int_{T_s}\dot{\omega}^2(t-\tau)dt + [\int_{T_s}\omega(t-\tau)\dot{\omega}(t-\tau)dt]^2. \end{aligned} \tag{B-6}$$

Assume that the energy of a pulse outside T_s is negligible, these results can be further simplified due to

$$\int_{T_s}\omega(t-\tau)\dot{\omega}(t-\tau)dt = 0 \tag{B-7}$$

and

$$\begin{aligned} &\int_{T_s}\omega(t-\tau)\ddot{\omega}(t-\tau)dt \\ &= -\int_{T_s}\omega(t-\tau)d(\dot{\omega}(t-\tau)) \\ &= \underbrace{-\omega(t-\tau)\dot{\omega}(t-\tau)|_{T_s}}_{=0} - \int_{T_s}\dot{\omega}^2(t-\tau)dt \\ &= -\int_{T_s}\dot{\omega}^2(t-\tau)dt. \end{aligned} \tag{B-8}$$

According to the linear relationship between x and $y(\tau)$, the expectations in terms of x are straightforward.

APPENDIX C: GAUSSIAN APPROXIMATION OF MULTIPATH INTERFERENCE

The key assumption we make is, for each multipath with index $\ell_{\ell \neq m}$, $\phi(t)_{t\neq 0}$ is identically independently distributed with mean m_ϕ and variance σ_ϕ^2. As the number of multipath L in a dense UWB channel is very large, we invoke the *Central Limit Theorem* so that every sample variable of $n_a(t)$ is Gaussian distributed with

$$\begin{array}{ll} \text{mean} & m_{n_a} = \sum_{\ell=1,\ell\neq m}^{L} a_\ell m_\phi \\ \text{variance} & \sigma_{n_a}^2 = \sum_{\ell=1,\ell\neq m}^{L} a_\ell^2 \sigma_\phi^2. \end{array} \tag{C-1}$$

The distribution of $\phi(t)_{t\neq 0}$ and the values of m_ϕ and σ_ϕ^2 can be determined according to a general model describing each sample and probability in detail or some specifically chosen TH codes and multipath delays.

REFERENCES

[1] R. A. Scholtz, "Multiple access with time-hopping impulse modulation," in *Proc. Military Communications Conf.*, Boston, MA, Oct. 1993, vol. 2, pp. 447–450.

[2] Federal Communication Committee, *FCC: First Report and Order*, Federal Communications Commission, U.S., April 22, 2002.

[3] W. M. Lovelace and J. K Townsend, "The effects of timing jitter and tracking on the performance of impulse radio," *IEEE J. Select. Areas Commun.*, vol. 20(9), pp. 1646–1653, Dec. 2002.

[4] M. K. Simon, J. K. Omura, R. A. Scholtz, and B. K. Levitt, *Spread Spectrum Communications, Volume I-III*, Computer Science Press, Maryland, 1985.

[5] H. Meyr, M. Moeneclaey, and S. A. Fechtel, *Digital Communication Receivers: Synchronization, Channel Estimation, And Signal Processing*, A Wiley-Interscience Publication, John Wiley & sons, Inc, New York, 1998.

[6] J. G. Proakis, *Digital Communications*, McGraw-Hill, third edition, 1995.

[7] E. A. Homier and R. A. Scholtz, "Rapid acquisition of Ultra-Wideband signals in the dense multipath channel," in *Proc. IEEE Conf. on UWB Systems and Technologies (UWBST)*, May 2002.

[8] N. Rinaldi et al., "U.C.A.N.'s ultra wide band system: Baseband algorithm design," in *Proc. International Workshop on UWB Systems*, June 2003.

[9] I. Maravic and M. Vetterli, "Low-complexity subspace methods for channel estimation and synchronization in Ultra-Wideband systems," in *Proc. International Workshop on UWB Systems*, Oulu, Finland, June 2003.

[10] S. Soderi, J. Iinatti, and M. Hamalainen, "CLPDI algorithm in UWB synchronization," in *Proc. International Workshop on UWB Systems*, June 2003.

[11] Harry L. Van Trees, *Detection, Estimation, and Modulation Theory*, John Wiley & Sons, Inc., New York, 1968.

[12] M. Moeneclaey, "A simple lower bound on the linearized performance of practical symbol synchronizers," *IEEE Trans. Commun.*, vol. com-31(9), pp. 1029–1032, Sep. 1983.

[13] M. Moeneclaey, "A fundamental lower bound on the performance of practical joint carrier and bit synchronizers," *IEEE Trans. Commun.*, vol. com-32(9), pp. 1007–1012, Sep. 1984.

[14] M. Moeneclaey, "On the true and the modified Cramer-Rao bounds for the estimation of a scalar parameter in the presence of nuisance parameters," *IEEE Trans. Commun.*, vol. 46(11), pp. 1536–1544, Nov. 1998.

[15] A. N. D'Andrea, U. Mengali, and R. Reggiannini, "The modified Cramer-Rao bound and its application to synchronization problems," *IEEE Trans. Commun.*, vol. 42(2/3/4), pp. 1391–1399, Feb./Mar./April 1994.

[16] E. G. Ström, S. Parkvall, S. L. Miller, and B. E. Ottersten, "Propagation delay estimation in asynchronous Direct-Sequence Code-Division multiple access systems," *IEEE Trans. Commun.*, vol. 44(1), pp. 84–92, Jan. 1996.

[17] E. G. Ström and F. Malmsten, "A maximum likelihood approach for estimating DS-CDMA multipath fading channels," *IEEE J. Select. Areas Commun.*, vol. 18(1), pp. 132–140, Jan. 2000.

[18] H. Steendam and M. Moeneclaey, "Low-snr limit of the Cramer-Rao bound for estimating the time delay of a PSK, QAM, or PAM waveform," *IEEE Commun. Lett.*, vol. 5(1), pp. 31–33, Jan. 2001.

[19] F. Rice, B. Cowley, B. Moran, and M. Rice, "Cramer-Rao lower bounds for QAM phase and frequency estimation," *IEEE Trans. Commun.*, vol. 49(9), pp. 1582–1591, Sep. 2001.

[20] S. C. White and N. C. Beaulieu, "On the application of the Cramer-Rao and detection theory bounds to mean square error of symbol timing recovery," *IEEE Trans. Commun.*, vol. 40(10), pp. 1635–1643, Oct. 1992.

[21] Y. Jiang, F. Sun, and J. S. Baras, "On the performance limits of data-aided synchronization," *IEEE Trans. Inform. Theory*, vol. 49(1), pp. 191–203, Jan. 2003.

[22] B. B. Ibrahim and A. Hamid Aghvami, "Direct sequence spread spectrum matched filter acquisition in frequency-selective rayleigh fading channels," *IEEE J. Select. Areas Commun.*, vol. 12(5), pp. 885–890, June.

[23] W. Sheen and G. L. Stuber, "Effects of multipath fading on delay-locked loops for spread spectrum systems," *IEEE Trans. Commun.*, vol. 42, pp. 1994–1956, Feb./Mar./April 1994.

[24] O. Shin and K. B. Lee, "Utilization of multipaths for spread-spectrum code acquistion in frequency-selective rayleigh fading channels," *IEEE Trans. Commun.*, vol. 49(4), pp. 734–743, April 2001.

[25] S. E. Bensley and B. Aazhang, "Maximum-likelihood synchronization of a single user for Code-Division Multiple-Access communication systems," *IEEE Trans. Commun.*, vol. 46(3), pp. 392–399, Mar. 1998.

[26] R. D. J. Van Nee, "Spread-spectrum code and carrier synchronization errors caused by multipath and interference," *IEEE Trans. Aerosp. Electron. Sys.*, vol. 29(4), pp. 1360–1365, Oct. 1993.

[27] G. Fock, J. Baltersee, P. Schulz-Rittich, and H. Meyr, "Channel tracking for Rake receivers in closely spaced multipath environments," *IEEE J. Select. Areas Commun.*, vol. 19, pp. 2420–2431, Dec. 2001.

[28] B. W.'t. Hart, R. D. J. Van Nee, and R. Prasad, "Performance degradation due to code tracking errors in spread- spectrum code-division multiple access systems," *IEEE J. Select. Areas Commun.*, vol. 14(8), pp. 1669–1679, Oct. 1996.

[29] Y. Ma, F. Chin, B. Kannan, and S. Pasupathy, "Acquisition performance of an Ultra Wideband communications system over a multiple-access fading channel," in *Proc. IEEE Conf. Ultra Wideband System and Tech.*, May 2002.

[30] M. C. Vanderveen, A. van der Veen, and A. Paulraj, "Estimation of multipath parameters in wireless communications," *IEEE Trans. Signal Processing*, vol. 46(3), pp. 682–690, March 1998.

[31] F. Zhang, *Matrix Theory: Basic Results and Techniques*, Springer-Verlag New York Inc., New York, 1999.

[32] M. Z. Win and R. A. Scholtz, "Ultra-wide bandwidth time-hopping spread-spectrum impulse radio for wireless multiple-access communications," *IEEE Trans. Commun.*, vol. 48(4), pp. 679–691, Apr. 2000.

[33] J. Zhang, T. D. Abhayapala, and R. A. Kennedy, "Performance of Ultra Wideband correlator receiver using gaussian monocycles," in *Proc. IEEE Int. Conf. on Communications (ICC)*, May 2003, vol. 3, pp. 2192 –2196.

[34] J. Zhang, R. A. Kennedy, and T. D. Abhayapala, "Performance of RAKE reception for Ultra Wideband signals in a lognormal fading channel," in *Proc. Int. Workship on Ultra Wideband Systems (IWUWBS)*, June 2003.

[35] A. H. Stroud and D. Secrest, *Gaussian Quadrature Formulas*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1966.

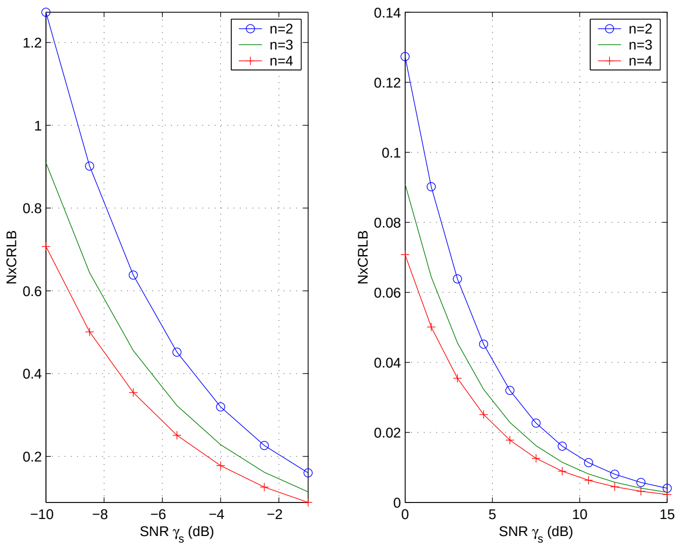

Fig. 1. CRLB versus symbol SNR γ_s for n-order monocycles with $t_p = 2$ns, $n = 2, 3, 4$.

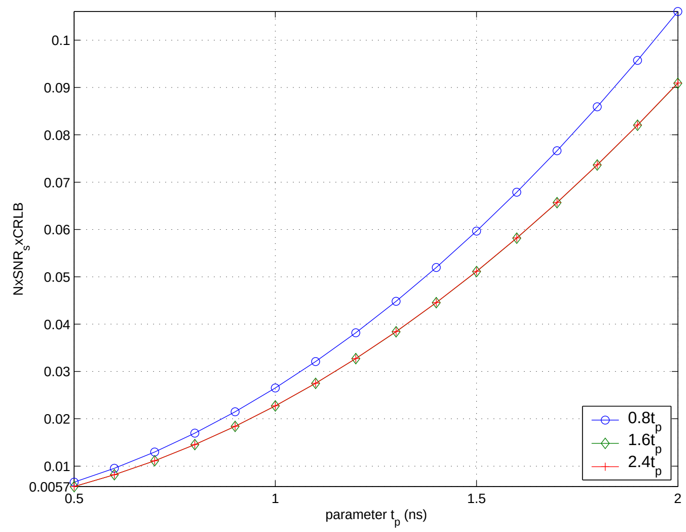

Fig. 3. CRLB for a 3 order ($n = 3$) monocycle with different parameter t_p; different lines correspond to different width of truncation.

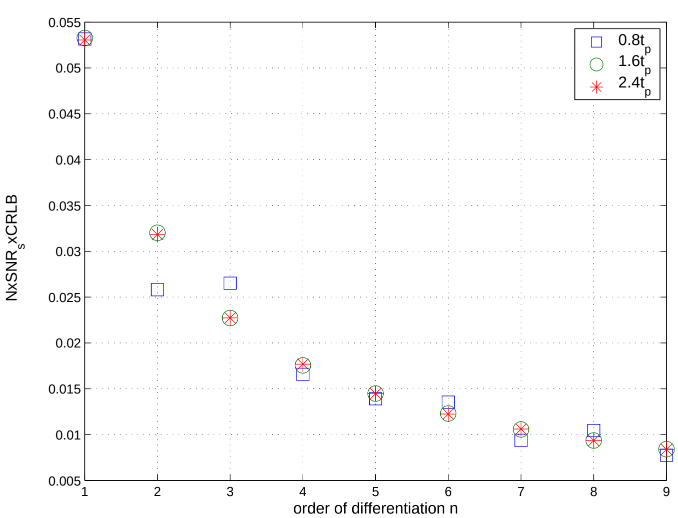

Fig. 2. CRLB versus order n for monocycles with $t_p = 1$ns; different lines correspond to different width of truncation.

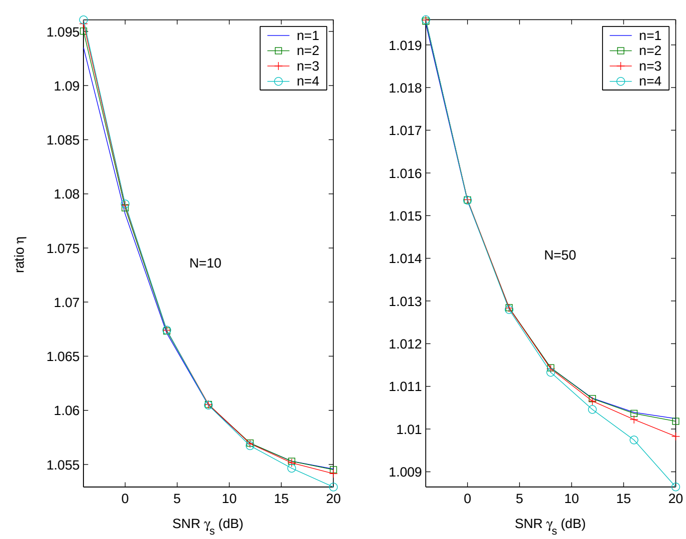

Fig. 4. The degrading ratio η versus SNR γ_s for monocycles. Two observation periods (NT_s) in a synchronizer are compared with $N = 10$ (left) and $N = 50$ (right). The time t is normalized with respect to t_p.