Spatial Limits on the Performance of Direction of Arrival Estimation

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***Abstract*— In this paper, we demonstrate that the performance of a direction of arrival (DOA) estimator is fundamentally limited by the size of the region over which we measure a wavefield. That is, even assuming continuous field measurements across the region, we still cannot achieve perfect performance. We use an approach based on modal decomposition of a spatially truncated field, and completely independent of sensor geometry, to derive the Cramér-Rao Bound (CRB) for spatially-limited DOA estimators. The model is validated by comparison with results from a uniform circular array (UCA) as the number of sensors goes to infinity. Simulations of the spatial CRB show how DOA performance improves as the measurement region expands. Simulations of the bound also indicate that P sources can only be effectively resolved once a certain threshold region size is reached.**

***Index Terms*— Direction of arrival estimation, spatial limits on resolution, sensor array processing.**

I. Introduction

Direction of arrival (DOA) estimation is an important problem in signal processing with direct applications in radar, imaging, and wireless communications. Conventional approaches to examining the performance limitations of DOA estimators have focussed on deriving resolution bounds based on sensor array geometry (size, shape and number of sensors).

In [1], the Cramér-Rao Bound (CRB) is derived for an array with arbitrary known geometry in white noise. The CRB lower bounds the covariance of any unbiased DOA estimator. This result has since been extended to a variety of other, more complicated, noise models [2], [3]. The weakness of this approach is that, due to the focus on the geometry of the sensor array, it is difficult to investigate more fundamental limitations on DOA performance.

In this paper, we disregard array geometry and show that DOA performance is fundamentally limited by the spatial extent of the array - in effect the size of the region in which we have knowledge of a wavefield. That is, even if the noisy field could be measured continuously over the region, there would still a limit to the DOA resolution that could be achieved.

Some limited work has been done in the area of spatial limits for DOA by Birkenes [4], [5]. These studies used a uniform circular array (UCA) with a fixed number of sensors, and examined the effect of changing array radius on DOA resolution. Whilst the study identifies the important role of spatial extent on DOA performance, the results are still affected by array geometry and inter-sensor spacing.

Our approach is based on a modal decomposition of the field. All of the information in a spatially limited field can be expressed using an infinite set of modal coefficients. We show that additive white field noise causes a variance in these coefficients which increases rapidly with order, and limits the information we can extract from the truncated field. Using this model, we derive a CRB which is dependent only on the spatial extent of the array.

The paper is organized as follows. Section II introduces a modal decomposition for wavefields. Section III presents a spatial measurement model for the DOA problem, and section IV presents a modal noise model. From these models, the spatially limited CRB is derived in section V. In section VI, the bound is simulated for varying source distributions and region sizes, and compared with the CRB for a uniform circular array (UCA) as the number of sensors is increased.

II. Modal Decomposition

Consider the 2D DOA problem, where $\mathbf{x} \in \mathbb{R}^2$ is a position vector. Let $F(\mathbf{x})$ denote a wavefield generated by P farfield narrowband sources, effectively plane waves. Then, $F(\mathbf{x})$ satisfies the 2D Helmholtz equation (the reduced wave equation)

$$\nabla^2 F(\mathbf{x}) + k^2 F(\mathbf{x}) = 0, \tag{1}$$

where $k \triangleq 2\pi/\lambda$ is the wavenumber, ∇^2 is Laplacian operator, and λ is the wavelength [6].

We can use this equation to define the space of valid Helmholtz wavefields, $\mathcal{H}$, a linear subspace of $\mathcal{L}^2(\mathbb{R}^2)$:

$$\mathcal{H} \triangleq \left\{ F(\mathbf{x}) \in \mathcal{L}^2(\mathbb{R}^2) : \nabla^2 F(\mathbf{x}) + k^2 F(\mathbf{x}) = 0 \right\}. \tag{2}$$

Any field in $\mathcal{H}$ can be expressed using a modal expansion [7],

$$F(\mathbf{x}) = \sum_{n=-\infty}^{\infty} f_n u_n(\mathbf{x}), \tag{3}$$

where the modes are defined using a polar coordinate representation of position $\mathbf{x} \equiv (|\mathbf{x}|, \theta_{\mathbf{x}})$ as,

$$u_n(\mathbf{x}) = j^n J_n(k|\mathbf{x}|) e^{jn\theta_{\mathbf{x}}}, \tag{4}$$

and $J_n(\cdot)$ is the integer order n Bessel function [8].

For fields made up of P distinct farfield sources, the expansion coefficients take a special form [9]

$$f_n = \sum_{p=1}^{P} \gamma_p e^{-jn\phi_p}, \tag{5}$$

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where γ_p and ϕ_p are, respectively, the amplitude and direction of the farfield sources.

In the spatially limited DOA problem, we only have knowledge of the wavefield within some small circular region of space, S, with radius r_S. This restriction of the knowledge is modeled using the spatial truncation operator P_S. This operator is defined such that for any function $M(\mathbf{x}) \in \mathcal{L}^2(\mathbb{R}^2)$,

$$(P_S M)(\mathbf{x}) = \begin{cases} M(\mathbf{x}) & \text{for } \mathbf{x} \in S \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

We define the space $\mathcal{G}$ of functions which are valid Helmholtz wavefields within S, and zero outside. The operator P_S is a projection operator from $\mathcal{H}$ onto $\mathcal{G}$. Any function $G(\mathbf{x})$ in $\mathcal{G}$ can be expressed using the modal expansion,

$$G(\mathbf{x}) = \sum_{n=-\infty}^{\infty} g_n v_n(\mathbf{x}), \tag{7}$$

where $v_n(\mathbf{x})$ is an orthonormal basis for $\mathcal{G}$ defined by

$$v_n(\mathbf{x}) = \begin{cases} \alpha_n^{-1/2} j^n J_n(k|\mathbf{x}|) e^{jn\theta_{\mathbf{x}}} & \text{for } \mathbf{x} \in S \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

and α_n is the normalizing constant defined by

$$\begin{aligned} \alpha_n &= \int_S \|u_n(\mathbf{x})\|^2 d\mathbf{x} \\ &= \int_0^{2\pi} d\theta_x \int_0^{r_S} |\mathbf{x}| J_n(k|\mathbf{x}|)^2 \, d|\mathbf{x}| \\ &= \pi {r_S}^2 \left[J_n(kr_S)^2 - J_{n+1}(kr_S) J_{n-1}(kr_S) \right]. \end{aligned} \tag{9}$$

Combining equations (4), (6) and (8) we can derive a relationships between $u_n(\mathbf{x})$ and $v_n(\mathbf{x})$,

$$\sqrt{\alpha_n} v_n(\mathbf{x}) = (P_S u_n)(\mathbf{x}). \tag{10}$$

III. Measurement Model

Consider the wavefield $F(\mathbf{x})$. We assume a measurement model whereby our observations of the field can only be made over a region S, and field is corrupted with noise,

$$G(\mathbf{x}) = \begin{cases} F(\mathbf{x}) + W(\mathbf{x}) & \mathbf{x} \in S \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

$$= (P_S F)(\mathbf{x}) + W(\mathbf{x}) \tag{12}$$

As $(P_S F)(\mathbf{x}) \in \mathcal{G}$, we can express it using a modal expansion with the form of (7):

$$(P_S F)(\mathbf{x}) = \sum_{n=-\infty}^{\infty} f_n (P_S u_n)(\mathbf{x}) = \sum_{n=-\infty}^{\infty} f_n \sqrt{\alpha_n} v_n(\mathbf{x}). \tag{13}$$

The noise field $W(\mathbf{x})$ is not a valid wavefield, and so $(P_S W)(\mathbf{x}) \notin \mathcal{G}$. We split the noise on S into two parts - the projection onto $\mathcal{G}$ (the part of the noise field which looks like a valid wavefield) and the projection onto $\perp\mathcal{G}$ (the rest of the noise field):

$$\begin{aligned} W(\mathbf{x}) &= W_{\mathcal{G}}(\mathbf{x}) + W_{\perp\mathcal{G}}(\mathbf{x}) \\ &= \sum_{n=-\infty}^{\infty} w_n v_n(\mathbf{x}) + W_{\perp\mathcal{G}}(\mathbf{x}). \end{aligned} \tag{14}$$

Substituting (13) and (14) into (12), we get,

$$G(\mathbf{x}) = W_{\perp\mathcal{G}}(\mathbf{x}) + \sum_{n=-\infty}^{\infty} (f_n \sqrt{\alpha_n} + w_n) v_n(\mathbf{x}) \tag{15}$$

We allow continuous information of the observed field $G(\mathbf{x})$ which allows us to extract modal coefficients h_n from the field,

$$h_n = \int_S G(\mathbf{x}) \overline{v_n(\mathbf{x})} d\mathbf{x} = f_n \sqrt{\alpha_n} + w_n. \tag{16}$$

Thus, the optimal unbiased estimates of the original field coefficients f_n are

$$\tilde{f}_n = \frac{h_n}{\sqrt{\alpha_n}} = f_n + \frac{w_n}{\sqrt{\alpha_n}} \tag{17}$$

Once we know how noise is projected into the w_n, we will have a model for how much information we can extract from a spatially limited field.

IV. Noise Model

In trying to represent noise in our measurement model, we have the problem that white noise is not an $\mathcal{L}^2$ function. The solution presented by Gallager in [10], is to model white noise by its projection into $\mathcal{L}^2(\mathbb{R})$. Under this model, the projection of the noise field, $N(\mathbf{x})$, onto any normalized function, $\psi(\mathbf{x})$, is:

$$z \triangleq \int_S N(\mathbf{x}) \overline{\psi(\mathbf{x})} d\mathbf{x}, \tag{18}$$

where z is a zero mean, Gaussian random variable with variance given by

$$\mathrm{E}\left\{|z|^2\right\} = \frac{N_0}{2} \int_S |\psi(\mathbf{x})|^2 d\mathbf{x} = \frac{N_0}{2}. \tag{19}$$

The constant $N_0/2$ is chosen to normalize the spatial power spectral density.

We generalize this model for the spatial case. Where the Gallager model defines a noise power independent of region size, we consider a noise field, $W(\mathbf{x})$ that is spatially white with a constant (albeit infinite) power per unit space. Thus, as the region expands, the noise variance will increase with the area of the region,

$$w_n \triangleq \int_S W(\mathbf{x}) \overline{v_n(\mathbf{x})} d\mathbf{x}, \tag{20}$$

where w_n is a zero mean, Gaussian random variable with variance

$$\mathrm{E}\left\{|w_n|^2\right\} = \mathrm{E}\left\{ \left| \int_S W(\mathbf{x}) \overline{v_n(\mathbf{x})} d\mathbf{x} \right|^2 \right\} \tag{21}$$

$$= \frac{N_0 \pi r_S^2}{2}. \tag{22}$$

It is easy to confirm that, using this formulation, the noise field is spatially invariant. That is, the noise field $W(\mathbf{x})$ exists for all $\mathbb{R}^2$, but is observable only within S.

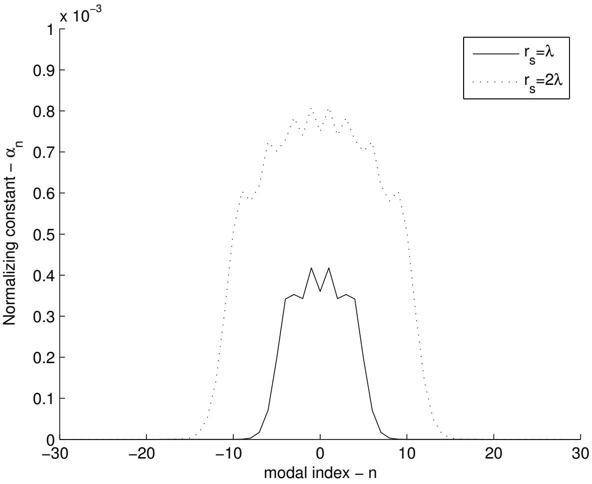

Fig. 1: Comparison of the behaviour of the normalizing constant α_n with increasing order for two different region radii.

Using this noise model, the optimal estimator of the field coefficients f_n from (17) will be a normally distributed Gaussian process,

$$\tilde{f}_n \sim \mathcal{N}\left(f_n, \frac{N_0 \pi r_S^2}{2\alpha_n}\right). \tag{23}$$

As has been shown, the coefficients $\tilde{f}_n$ contain all of the information that can be extracted from the field. This means that the problem of estimating directions of arrival from the spatially limited field can be reduced to the problem of estimating the ϕ_p from the infinite series $\tilde{f}_n$.

Note that although the noise field $W(\mathbf{x})$ is spatially white, the variance in the coefficients $\tilde{f}_n$ is clearly not white, and varies with $1/\alpha_n$. Fig. 1 shows the value of α_n from (9) for two different region sizes. Note in each case that α_n is basically zero for $n > N = \lceil \pi e r_S/\lambda \rceil$ [9]. This means that the higher order coefficient will rapidly become extremely noisy, and not very useful for DOA estimation. This idea is discussed more in the next section.

V. The Cramér-Rao Bound

In this section we derive the Cramér-Rao Lower Bound (CRB) for the spatially limited DOA problem. The derivation is based on how well we can estimate the directions of arrival, ϕ_p, from the noisy field coefficients $\tilde{f}_n$.

This problem is similar to the problem of estimating complex sine wave frequencies from single experiment data [1]. To make this clear, we put our problem into a particular matrix form, where we have infinite matrices for $n = [-\infty, \ldots, \infty]$

$$\tilde{\mathbf{F}} = \mathbf{A}(\mathbf{\Phi})\mathbf{X} + \mathbf{W}, \tag{24}$$

where

$$\tilde{\mathbf{F}} \triangleq [\ldots, f_{-1}, f_0, \ldots, f_n, \ldots]^{\mathrm{T}}, \tag{25}$$

$$\mathbf{\Phi} \triangleq [\phi_1, \ldots, \phi_P], \tag{26}$$

$$\mathbf{A}(\mathbf{\Phi}) \triangleq [\mathbf{a}(\phi_1), \ldots, \mathbf{a}(\phi_P)], \tag{27}$$

$$\mathbf{a}(\phi) \triangleq \left[\ldots, e^{j\phi}, 1, \ldots, e^{-jn\phi}, \ldots\right]^{\mathrm{T}}, \tag{28}$$

$$\mathbf{X} \triangleq [\gamma_1, \ldots, \gamma_P]^{\mathrm{T}}, \tag{29}$$

and

$$\mathbf{W} \triangleq [\ldots, w_{-1}, w_0, \ldots, w_n, \ldots]^{\mathrm{T}}. \tag{30}$$

From (23), we know that the noise in the coefficients is independent and non-uniform. Thus covariance matrix of $\mathbf{W}$ is diagonal,

$$\begin{aligned}\mathbf{Q} &= \mathrm{E}\left\{\mathbf{W}\mathbf{W}^{\mathrm{H}}\right\} \\ &= \frac{N_0 \pi {r_S}^2}{2} \mathrm{diag}\left\{\ldots, \frac{1}{\alpha_{-1}}, \frac{1}{\alpha_0}, \ldots, \frac{1}{\alpha_n}, \ldots\right\}.\end{aligned} \tag{31}$$

The deterministic CRB for this problem is derived in [3]:

$$\mathrm{CRB}_{\mathrm{Spatial}} = \frac{1}{2}\left\{\mathrm{Re}\left[\left(\tilde{\mathbf{D}}^{\mathrm{H}}\mathbf{P}_{\tilde{\mathbf{A}}}^{\perp}\tilde{\mathbf{D}}\right) \odot \hat{\mathbf{P}}^{\mathrm{T}}\right]\right\}^{-1}, \tag{32}$$

where $\odot$ stands for the Schur-Hadamard matrix product,

$$\mathbf{P}_{\tilde{\mathbf{A}}}^{\perp} = \mathbf{I} - \tilde{\mathbf{A}}\left(\tilde{\mathbf{A}}^{\mathrm{H}}\tilde{\mathbf{A}}\right)^{-1}\tilde{\mathbf{A}}^{\mathrm{H}}, \tag{33}$$

$$\tilde{\mathbf{A}} = \mathbf{Q}^{-1/2}\mathbf{A}, \tag{34}$$

$$\tilde{\mathbf{D}} = \mathbf{Q}^{-1/2}\mathbf{D}, \tag{35}$$

$$\mathbf{D} \triangleq \left[\left[\frac{d\mathbf{a}(\phi)}{d\phi}\right]_{\phi=\phi_1}, \ldots, \left[\frac{d\mathbf{a}(\phi)}{d\phi}\right]_{\phi=\phi_P}\right], \tag{36}$$

and

$$\hat{\mathbf{P}} = \mathbf{X}\mathbf{X}^{\mathrm{H}}. \tag{37}$$

Equation (32) is the spatial CRB - a performance bound for any unbiased DOA estimator with limited spatial extent.

The transformations (34) and (35) pre-whiten the non-uniform noise in the coefficients. This has the effect of 'blocking' the noisier high order coefficients as α_n goes to zero. From Fig. 1 and the discussion in the previous section, we see that coefficients for $n > N = \lceil \pi e r_S/\lambda \rceil$ will have almost no effect on the performance of DOA estimation. Previous work [9] models this effect through a truncation of coefficients for $n > N$. Here we see that we can avoid the need for this truncation by incorporating a more accurate noise model for coefficients.

A. Single Source

If there is only a single source ($P = 1$), the CRB represents a lower bound on the variance with which a single direction can be estimated. In this case, we can make considerable simplifications using the addition and recurrence relations for integer order Bessel functions [8],

$$\tilde{\mathbf{A}}^{\mathrm{H}}\tilde{\mathbf{A}} = \frac{2}{\pi {r_S}^2 N_0} \sum_{n=-\infty}^{\infty} \alpha_n = \frac{2}{N_0}, \tag{38}$$

$$\tilde{\mathbf{D}}^{\mathrm{H}}\tilde{\mathbf{D}} = \frac{2}{\pi {r_S}^2 N_0} \sum_{n=-\infty}^{\infty} n^2 \alpha_n = \frac{r_S^2 k^2}{2N_0}, \tag{39}$$

$$\tilde{\mathbf{D}}^{\mathrm{H}}\tilde{\mathbf{A}}\tilde{\mathbf{A}}^{\mathrm{H}}\tilde{\mathbf{D}} = \left(\frac{2}{\pi {r_S}^2 N_0} \sum_{n=-\infty}^{\infty} n\alpha_n \right)^2 = 0. \tag{40}$$

Simplifying, we are left with a simple closed form expression for the single source CRB,

$$\mathrm{CRB}_{P=1} = \frac{N_0}{{r_S}^2 k^2} = \frac{N_0 \lambda^2}{4\pi {r_S}^2} = \frac{N_0}{4\pi R^2}, \tag{41}$$

where $R = r_S/\lambda$ is the size of the region in wavelengths.

VI. Numerical Simulations

A. The Spatial CRB

In this section we examine the effect of varying region size, and number of sources on the spatial CRB. In calculating the bound from (32), the infinite matrices had to be truncated. Using matrices of over 2000 terms ensured that the truncation error was a trivial compared with the numerical precision of other calculations.

Symmetry considerations mean that the best simultaneous directional estimates for P sources will occur when these sources have equal amplitude and are spaced evenly in angle [5]. In less symmetric configurations, we may be able to estimate one source direction more accurately by increasing it's amplitude, or placing it away from other sources, but this will degrade the estimation of the other source directions.

Another advantage of this symmetric situation, is that all direction estimates will be unbiased by other sources, and we need only to consider one diagonal element of the CRB matrix to find a lower bound on the variance of all direction estimates.

Fig. 2 shows the effect of increasing the number of sources (with $N_0 = 0.1$). Obviously, the single source case has the form (41) from above, where the bound decreases with the square of radius. For large regions, the performance in estimating two directions is almost identical to that for estimating a single direction. As we decrease the radius, however, there comes a threshold size (about $r_S \approx 0.3\lambda$) where performance starts to rapidly diverge from the single source performance. A similar effect can be observed as more sources are added.

This result tends to suggest that P sources can only be effectively resolved once a certain region size is reached. As the region falls below this size, DOA performance degrades rapidly. This threshold may be related to the number of 'active' modes in the region [9]. A future paper will discuss these issues in more detail, and may provide an spatial alternative to the 'sensor-based' identifiability constraints set out in [11] and [12].

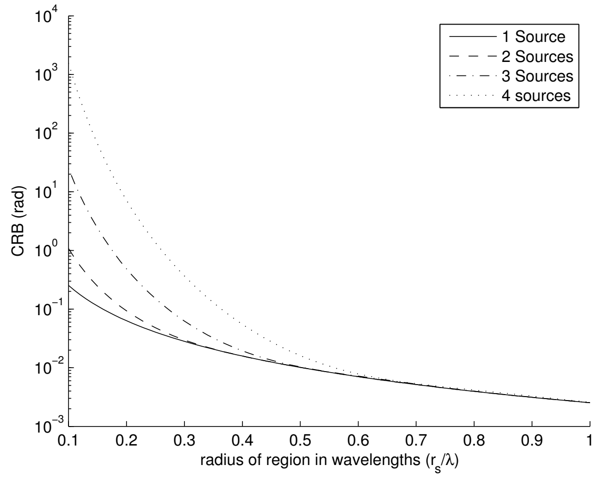

Fig. 2: Effect of varying region size, and number of sources on the spatial CRB.

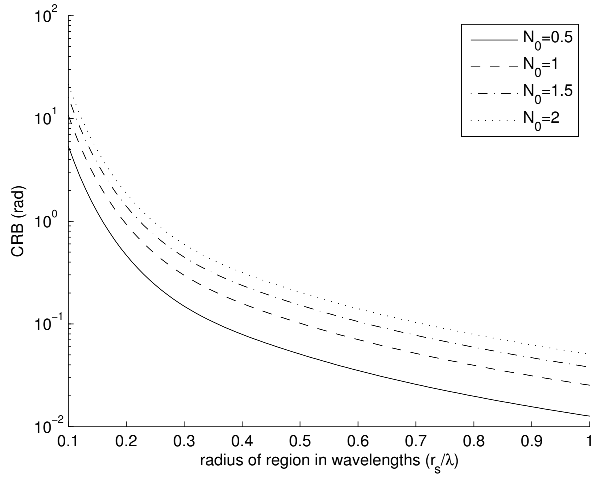

Fig. 3: Effect of varying noise levels on the spatial CRB.

Fig. 3 shows the effect of noise on the CRB for a DOA problem with $P = 2$ evenly spaced sources. As expected, the bound increases linearly with increasing noise power.

B. Comparison with Sensor Array CRB

Another interpretation of the spatial performance bound is that, no matter how many sensors we put into a region, DOA variance can never outperform the spatial CRB (32). In this section, we investigate this idea by comparing the spatial CRB to the bound derived for a uniform circular array (UCA) with the same spatial extent.

We assume a UCA with Q sensors, in a spatially white noise field with variance N_0. The field is generated by P perfectly correlated farfield sources, all with equal amplitude γ. This amplitude must be scaled to ensure that the received signal power remains constant as we increase the number of sensors [13]

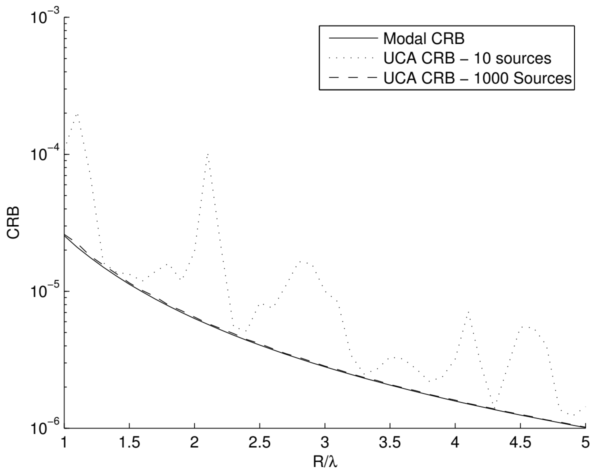

Fig. 4: Comparison of the spatial CRB with the CRB derived for two UCA's with the same radius, but different numbers of sensors.

$$\gamma = \frac{1}{\sqrt{Q}}. \quad (42)$$

The CRB for this situation is found in [5],

$$\mathrm{CRB}_{\mathrm{UCA}} = \frac{N_0}{2\gamma^2}\left\{\mathrm{Re}\left[\mathbf{D}^{\mathrm{H}}\mathbf{D} - \mathbf{D}^{\mathrm{H}}\mathbf{A}\left(\mathbf{A}^{\mathrm{H}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{H}}\mathbf{D}\right]\right\}^{-1}, \quad (43)$$

where Φ, $\mathbf{A}$ and $\mathbf{D}$ are defined as in equations (26), (27), and (36), and,

$$\mathbf{a}(\phi) \triangleq \left[e^{-jk\tau_1(\phi)}, \ldots, e^{-jk\tau_Q(\phi)}\right]^{\mathrm{T}}, \quad (44)$$

where

$$\tau_q(\phi) = -r_S \cos\left(\phi - \frac{2\pi(q-1)}{Q}\right). \quad (45)$$

We can now compare the spatial and sensor array CRBs, as shown in Fig. 4. Notice that for 10 sensors, the array CRB is clearly lower bounded by the spatial CRB, but displays wild behavior due to that fact that that there are not enough sensors to accurately model the wavefield in the region. When the number of sensors is increased to 100, the field is well modeled, and the array CRB sits only slightly above the modal CRB. The agreement between these two bounds seems to validate the noise model used in the modal decomposition.

These results seem to indicate that the array CRB is lower bounded by the spatial CRB. This is quite a powerful result. It shows that the the performance of an array based DOA estimator will be bounded by the maximal spatial extent of the array, independent of the number of sensors. In the limit of an infinite number of sensors in the region, the performance converges to the spatial CRB.

VII. Conclusion

We have demonstrated that the performance of a direction of arrival estimator is fundamentally limited by the size of the region over which we measure a wavefield. Further, we have derived the Cramér-Rao Bound for a spatially limited region in white noise. This bound is independent of array geometry, and number of sensors.

The spatial CRB was derived by a modal decomposition of the field, using a noise model which projects noise non-uniformly into the modal coefficients. An alternative CRB was derived using a UCA with the same spatial extent. As the number of sensors in this array were increased (and signal amplitude scaled to maintain SNR), the array performance converged towards the spatial CRB. This seems to indicate that the performance of DOA estimation from a sensor array is fundamentally limited by the spatial extent of that array, independent of the number of sensors.

Simulations of the bound showed that P sources can only be effectively resolved once a certain region size is reached. More work is needed to investigate the size of this threshold for various source distributions.

Another area for future work will be to generalize the spatial CRB to the case of multiple snapshots in time. This should be a trivial extension of the current work.

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