Maximal Ratio Combining Performance Analysis in Spatially Correlated Rayleigh Fading Channels

David B. Smith and Thushara D. Abhayapala

***Abstract*— New Results for the performance analysis of maximal ratio combining (MRC) using BPSK and M-PSK modulation in spatially correlated Rayleigh fading channels are presented in terms of antenna array configuration and parameters of scatterer distributions. Revised closed-form expressions for probabilities of bit error and symbol error of BPSK and M-PSK modulations respectively are given with MRC which allow for non-distinct eigenvalues (or closely spaced eigenvalues) from the correlation matrix at the receiver. The results of performance analysis assuming different receiver configurations and scattering scenarios give valuable insights into the performance of MRC in realistic Rayleigh fading scenarios for isotropic and non-isotropic scatterer distributions.**

***Index Terms*— Combining techniques, diversity, correlation, fading channels**

I. Introduction

Maximal ratio combining represents a theoretically optimal combiner over fading channels as a diversity scheme in a communication system. Theoretically, multiple copies of the same information signal are combined so as to maximise the instantaneous SNR at the output [1]. However system designs often assume that the fading is independent across multiple diversity channels. Physical constraints often restrain the use of antenna spacing that is required for independent fading across multiple antennas [2]. Therefore it is necessary to consider spatial correlation characteristics between the antennas.

Maximal ratio combining (MRC) of correlated fading signals with binary phase-shift keying (BPSK) has first been considered by Pierce and Stein in [3]. MRC of correlated fading signals with PSK modulation has been further considered in [4]–[7], and more recently in [8] where only one distribution function is considered for typical antenna configurations.

In [7] results are given for maximal ratio combining for complex Gaussian fading channels with correlated diversity for BPSK in Rician and Rayleigh fading which can be applied to a diversity scenario across space. This paper extends this work by establishing closed form expressions for performance in terms of the antenna array and scattering environment which can be applied to general distribution functions for arbitrary antenna configurations.

The expressions that are given in [9] for the spatial correlation of distributions of scatterers which can be applied to various non-isotropic scatterer distributions over multiple antennas are used. The spatial correlation formulation in [9] is applied with maximal ratio combining of PSK modulation to typical antenna configurations for 3 to 6 receive antennas in two typical non-isotropic Rayleigh fading environments, and a Rayleigh isotropic scattering scenario.

Related to this investigation a closed form expression for the bit error probability (BEP) for BPSK modulation is given, related to a generalised correlation function expression. Unlike the previous formulation, [3], [7], the closed form expression for the BEP allows for non-distinct eigenvalues in the correlation matrix. This is expanded to a closed form expression for the probability of symbol error (SEP) for M-PSK modulation where there may be non-distinct eigenvalues in the correlation matrix.

This paper is organised as follows. The next section describes the BEP of BPSK modulation with MRC where the correlation matrix can have distinct or non-distinct eigenvalues. Section III describes the symbol error probability (SEP) of M-PSK modulation with MRC for any given correlation matrix. Section IV provides the relevant space-time correlation formulation for arbitrarily spaced antennas. Section V provides analysis of MRC with standard antenna configurations as inputs in various scattering scenarios and section VI provides some conclusions.

II. BEP of BPSK with Maximal Ratio Combining

Let us consider the operation of a predetection L-branch maximal ratio combiner (MRC) on correlated Rayleigh fading channels. The received signals are assumed to be corrupted by additive white Gaussian noise (AWGN). The combiner input from the ℓth channel, $\ell = 1, \ldots, L$ is given by

$$y_\ell(t) = g_\ell(t)u(t) + n_\ell(t) \tag{1}$$

where $g_\ell(t)$ is the complex Gaussian channel gain, $u(t)$ is the transmitted information signal and $n_\ell(t)$ is the noise.

If we assume that the spatial correlation between two received signals from any two of the combiner inputs from antennas located at positions $\mathbf{x}_k$ and $\mathbf{x}_\ell$, following from [9] is

$$\rho(\mathbf{x_k}, \mathbf{x}_\ell) = \frac{E\{g_k(t)g_\ell^*(t)\}}{\sqrt{E\{g_k(t)g_k^*(t)\}E\{g_\ell(t)g_\ell^*(t)\}}}. \tag{2}$$

where $(\cdot)^*$ denotes the conjugate, and $E\{\cdot\}$ denotes the mathematical expectation.

The time index t will be discarded from this point because only spatial correlation is being considered and a quasi-static analysis is being applied. The spatial correlation $\rho(\mathbf{x}_k, \mathbf{x}_\ell)$ will be referred to as $\rho_{k,\ell}$ for simplicity.

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Following from [7], let $\mathbf{y} = [y_1 \;\; y_2 \;\; \ldots \;\; y_L]^T$ and the corresponding instantaneous bit SNR on the ℓth channel is $\gamma_\ell = |y_\ell|^2$. We are considering Rayleigh fading, thus $E[\mathbf{y}] = \mathbf{0}$ and the complex covariance matrix $\mathbf{\Sigma} = E[\mathbf{y}\mathbf{y}^\dagger]$. Now we can incorporate $\rho_{k,\ell}$ from (2) in (1) to obtain

$$\mathbf{\Sigma}(k,\ell) = \overline{\gamma_\ell}\rho_{k,\ell} \tag{3}$$

where $\overline{\gamma_\ell}$ is the average SNR per bit on the ℓth channel.

In the L-branch combiner all $\rho_{k,\ell}$ are contained within a spatial correlation matrix $\mathbf{R}$ whose (k,ℓ) element is $\rho_{k,\ell}$. Suppose that $\mathbf{R}$, has N distinct eigenvalues, $\lambda_1, \ldots, \lambda_N$ such that $N \leq L$, then following from [7] the bit-error-probability, P_b, is given by

$$P_b = \frac{1}{\pi}\int_0^{\pi/2} \sum_{n=1}^{N}\sum_{s=1}^{m_n} r_{s,n}\left(\frac{\overline{\gamma}_n\lambda_n}{\sin^2\theta}+1\right)^{-s} d\theta \tag{4}$$

where m_n is the multiplicity of each eigenvalue λ_n, such that $\sum m_n = L$ and, following from [10],

$$r_{s,n} = \frac{\left\{\frac{d^{m_n-s}}{dx^{m_n-s}}\prod_{i\neq n}\left(1-\frac{\lambda_i}{x}\right)^{-m_i}\right\}_{|x=\lambda_n}}{(m_n-s)!c^{m_n-s}} \tag{5}$$

where $c = \overline{\gamma}_n\lambda_n$.

Incorporating a result from [11], then from (4) we can obtain the following expression for P_b

$$P_b = \sum_{n=1}^{N}\sum_{s=1}^{m_n} r_{s,n}P_n(c)^s\sum_{k=0}^{s-1}\binom{s-1+k}{k}[1-P_n(c)]^k \tag{6}$$

where

$$P_n(c) = \frac{1}{2}\left(1-\sqrt{\frac{c}{1+c}}\right).$$

When there is I.I.D. Rayleigh channel fading (6) reduces to [12, eqn (27)].

III. Symbol-error-probability of M-PSK with MRC

The results for the bit-error-probability, BEP, of BPSK for MRC in a Rayleigh fading channel can be extended to the probability of symbol error (SEP) of M-PSK in a Rayleigh fading channel in a straightforward manner. Following from [7], the probability, P_s, can be written for M-PSK in a Rayleigh fading channel as,

$$P_s = \frac{1}{\pi}\int_0^{(M-1)\pi/M}\left[\det\left(\frac{\mathbf{\Sigma}\sin^2(\pi/M)}{\sin^2\theta}+\mathbf{I}\right)\right]^{-1} d\theta. \tag{7}$$

If it is assumed that there is a possible multiplicity of eigenvalues, then P_s can be written as

$$P_s = \frac{1}{\pi}\int_0^{(M-1)\pi/M}\sum_{n=1}^{N}\sum_{s=1}^{m_n} r_{s,n} \times\left(\frac{\overline{\gamma}_n\lambda_n\sin^2(\pi/M)}{\sin^2\theta}+1\right)^{-s} d\theta. \tag{8}$$

Using a result from [13, pp. 127-128], we obtain the following result for P_s

$$P_s = \frac{M-1}{M} - \sum_{n=1}^{N}\sum_{s=1}^{m_n} r_{s,n}\frac{1}{\pi}\sqrt{\frac{c}{1+c}}\left\{(\pi/2+\tan^{-1}\alpha) \times\sum_{k=0}^{s-1}\binom{2k}{k}\frac{1}{[4(1+c)]^k} + \sin(\tan^{-1}\alpha)\sum_{k=1}^{s-1}\sum_{i=1}^{k}\frac{T_{i,k}}{(1+c)^k}[\cos(\tan^{-1}\alpha)]^{2(k-i)+1}\right\} \tag{9}$$

where now $c = \overline{\gamma}_n\log_2 M.\lambda_n\sin^2\pi/M$, considering that $\overline{\gamma}_n$ is the average SNR per bit on each channel and SEP is being evaluated; $r_{s,n}$ is as defined in (5),

$$\alpha = \sqrt{\frac{c}{1+c}}\cot\frac{\pi}{M}$$

and

$$T_{i,k} = \frac{\binom{2k}{k}}{\binom{2(k-i)}{k-i}4^i[2(k-i)+1]}.$$

When there is I.I.D. Rayleigh channel fading (9) reduces to [12, eqn (21)].

IV. Spatial Correlation Formulation at Arbitrarily Spaced Antennas

A generalized spatial correlation, $\rho_{k,\ell}$, of antenna signals is defined for any two inputs, antennas, of an L-branch MRC, assuming that antennas are located in the azimuth plane. Conversely the scattered fields arriving at each MRC input are considered to be located in the azimuth plane only. Thus following from (2) and [9], using the 2-D modal expansion in [14],

$$\rho_{k,\ell} = \sum_{m=-\infty}^{m=\infty} j^m\gamma_m J_m(k\|\mathbf{x_k}-\mathbf{x_l}\|)e^{jm\phi_{k,\ell}} \tag{10}$$

where $J_m(\cdot)$ is the mth order Bessel function of the first kind, $k = 2\pi/\lambda$, $\phi_{k,\ell}$ is the angle of the vector connecting $\mathbf{x}_k$ and $\mathbf{x}_l$ and

$$\gamma_m = \int_0^{2\pi}\mathcal{P}(\phi)e^{-jm\phi}d\phi \tag{11}$$

where $\mathcal{P}(\phi)$ can be considered to be the normalised average power of a signal received from direction $\hat{\mathbf{y}}$. Note that when there are two inputs to the MRC, $L = 2$, $\phi_{k,\ell}$ can be considered to equal zero as stated in [9], however if $L > 2$, $\phi_{k,\ell}$ must be considered as non-zero.

Each spatial correlation, $\rho_{k,\ell}$, thus formulated, can be used to obtain eigenvalues of the correlation matrix $\mathbf{R}$, which can be used to find the corresponding bit-error-probability, P_b, using (6) for BPSK modulation, or symbol-error probability, P_s, for M-PSK modulation using (9).

V. Analysis of MRC with standard antenna configurations as inputs

In this section we analyze the performance of some standard antenna configurations as inputs to an $L = 3$ to $L = 6$ branch MRC in isotropic and some non-isotropic scattering scenarios based on the results of the previous sections. Some general performance guidelines are obtained on the basis of the BEP for BPSK modulation, and the SEP of 8-PSK modulation.

The standard antenna configurations acting as inputs to the MRC are the uniform circular array (UCA) and the uniform linear array (ULA). A range of antenna apertures in wavelengths (λ) are considered for the UCA, where the aperture is the diameter of the UCA. A range of antenna separations are considered for the ULA. For the non-isotropic distributions, various angular spreads, σ, defined as the square root of the variance of the particular distribution, are considered.

In all analysis we consider Rayleigh fading where the average bit SNR on each of the L channels is the same, such that $\overline{\gamma_l} = \overline{\gamma} = 10$ dB, $l = 1, \ldots, L$. In all analysis the angle of incidence from broadside is considered to be 60°. Fig. 1, Fig. 2 and Fig. 3 show the BEP with BPSK for an L = 3 branch, L = 4 branch and L = 6 branch MRC with different scattering scenarios using a UCA as inputs.

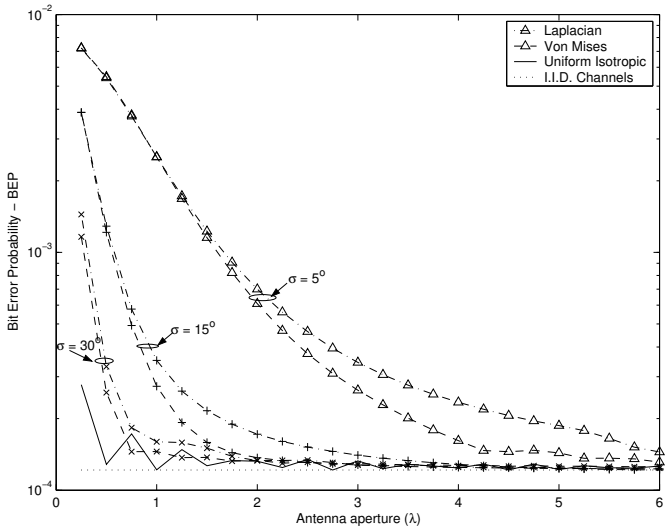

Fig. 1: Bit-Error-Probability (BEP) for an L=3 branch Maximal Ratio Combining (MRC) of BPSK modulation for Laplacian and Von Mises distributions of various angular spreads, σ, angle of incidence from broadside, $\beta = 60^o$, and a uniform isotropic distribution. Inputs to the MRC are from a Uniform Circular Array (UCA), with antenna aperture in wavelengths, λ . MRC for L I.I.D. channels is shown for reference. Average SNR per bit on the lth channel, $\overline{\gamma}_l$ = 10 dB .

It is clear that for both non-isotropic scattering scenarios over a range of angular spreads that there is a significant variation in the BEP. The BEP for L = 4 branch MRC shown in Fig. 2 is approximately 100 times greater for an angular spread $\sigma = 5^o$ compared with $\sigma = 30^o$ for both non-isotropic scatterer distributions at a UCA antenna aperture of 1λ. There is a similar trend of degradation from larger σ to smaller σ for Fig. 1 and Fig. 3.

It is demonstrated in Figs. 1–3 for the non-isotropic scattering scenarios, that even with a large angular spread that

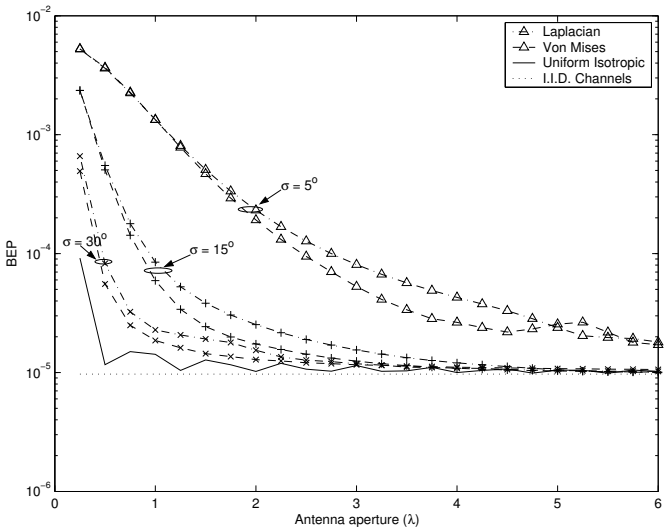

Fig. 2: BEP for L=4 branch MRC of BPSK modulation for Laplacian and Von Mises distributions of various σ, $\beta = 60^o$, and a uniform isotropic distribution, $\overline{\gamma}_l$ = 10 dB . Inputs to the MRC are from a UCA. MRC for L I.I.D. channels is shown for reference

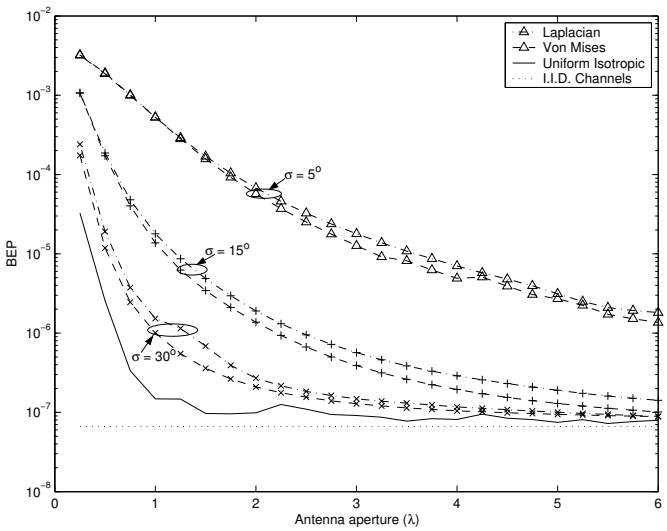

Fig. 3: BEP for L=6 branch MRC of BPSK modulation for Laplacian and Von Mises distributions of various σ, $\beta = 60^o$, and a uniform isotropic distribution, $\overline{\gamma}_l$ = 10 dB . Inputs to the MRC are from a UCA. MRC for L I.I.D. channels is shown for reference

there is degradation in the BEP when compared to that of a uniform isotropic distribution. The non-uniform improvement in the BEP of the uniform isotropic distribution as the antenna aperture increases, which is most pronounced in Fig. 2 for L = 4, may be attributable to the shape of the zeroth order Bessel function of the first kind correlation model used for this distribution.

The results for the variation of BEP with angular spread in Fig. 1–3 for the non-isotropic scatterer distributions may be directly attributable to a decrease in the spatial correlation. In [15] a decrease in spatial correlation is shown as antenna spacing and/or angular spread increases for non-isotropic distributions. The small variation of the BEP for the same angular spread is also explained in [15] by the distribution

variance dominating correlation, and not the choice of non-isotropic distribution.

The BEP for BPSK, with L = 6 branch MRC using a ULA as input, which has a range of antenna separation (the distance in wavelengths, λ, between adjacent elements of the ULA) this is shown in Fig. 4. Similar trends are displayed as compared with Figs. 1–3. However there are two noteworthy differences. Firstly for smaller array sizes Fig. 4 shows a larger variation in BEP between the non-isotropic scattering scenarios and the uniform scattering scenario. The second difference is that in this figure there is a uniform improvement in the BEP when there is a uniform isotropic distribution for the ULA as compared to the UCA.

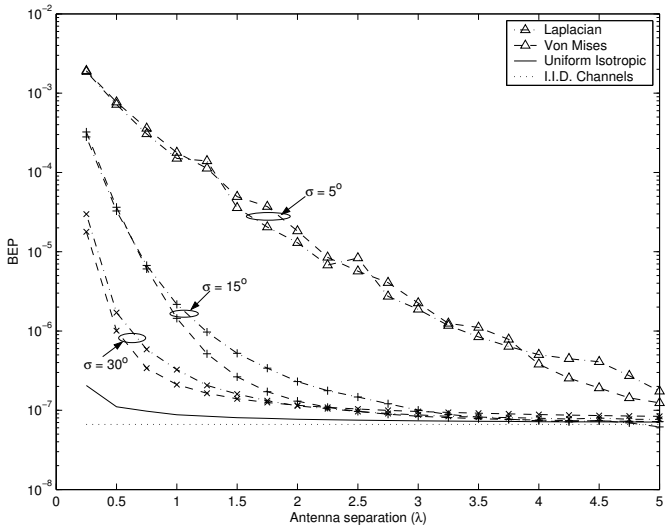

Fig. 4: BEP for L=6 branch MRC of BPSK modulation for Laplacian and Von Mises distributions of various σ, $\beta = 60^o$, and a uniform isotropic distribution, $\overline{\gamma}_l$ = 10 dB . Inputs to the MRC are from a ULA. MRC for L I.I.D. channels is shown for reference

Fig. 5 and Fig. 6 show very comparable trends for the symbol error probability, SEP for 8-PSK modulation, to the BEP for BPSK modulation, which is intuitive. Results using an L = 6 branch UCA are shown in Fig. 5, and results for an L = 6 branch ULA are shown in Fig. 6.

VI. Conclusions

Results of performance analysis for MRC using PSK modulation in Rayleigh fading channels assuming spatial correlation at the receiver have been presented. This performance analysis accounts for the antenna configuration at the receiver, the size and shape of the configuration, and the distribution of scatterers around the receiver. The results clearly demonstrate that there is a significant variation in performance when applying BPSK and M-PSK modulation over a range of angular spread for non-isotropic scattering scenarios. There is clear indication of the performance degradation due to spatial correlation at the receiver, dependent upon both the size and shape of the configuration, and whether there is isotropic or non-isotropic scattering in a Rayleigh fading channel.

Further investigation into the performance of other modulation schemes in non-isotropic scattering scenarios may emphasize what has been found from performance analysis

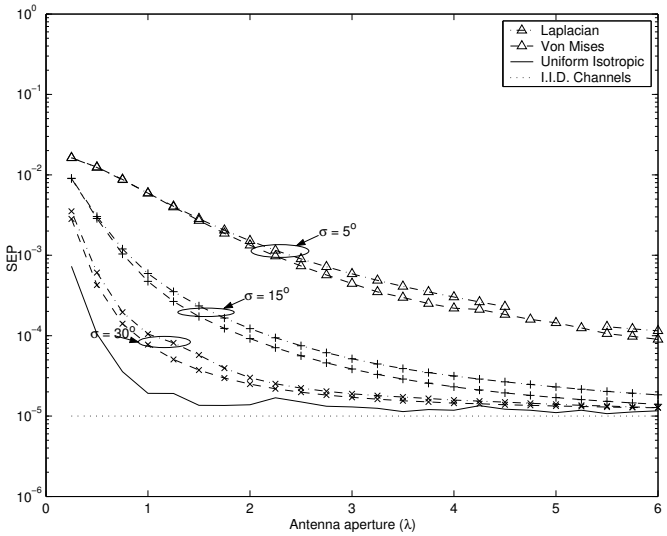

Fig. 5: SEP for L=6 branch MRC of 8-PSK modulation for Laplacian and Von Mises distributions of various σ, $\beta = 60^o$, and a uniform isotropic distribution, $\overline{\gamma}_l$ = 10 dB . Inputs to the MRC are from a UCA. MRC for L I.I.D. channels is shown for reference

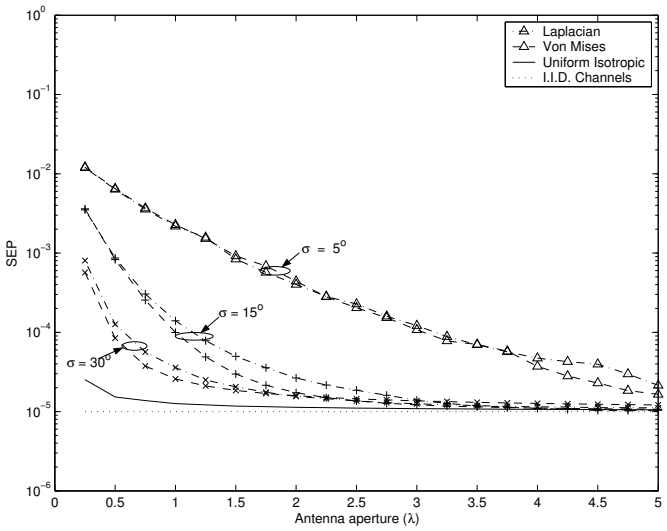

Fig. 6: SEP for L=6 branch MRC of 8-PSK modulation for Laplacian and Von Mises distributions of various σ, $\beta = 60^o$, and a uniform isotropic distribution, $\overline{\gamma}_l$ = 10 dB . Inputs to the MRC are from a ULA. MRC for L I.I.D. channels is shown for reference

in this paper, and perhaps give more insight into the general performance of MRC in Rayleigh fading channels.

The results in this paper give useful insight to aspects of practical implementation of MRC in Rayleigh fading channels with typical antenna configurations and modulation schemes. It is hoped that more understanding of the effects of non-isotropy, antenna shape and size may lead to better implementations of receivers for MRC in Rayleigh fading channels.

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