Source-field Wave-field Concentration and Dimension: Towards Spatial Information Content

Rodney A. Kennedy and Thushara D. Abhayapala
Department of Telecommunications Engineering, RSISE
The Australian National University and National ICT Australia[1]
Canberra ACT 0200 Australia

Abstract — **Using a spatial region as an information bearing resource is considered and an analogy drawn with the work on essentially time- and band-limited signals by Slepian, Landau and Pollack.**

I. Introduction

A central result in information theory relates to time-frequency concentration and the essential dimensionality of time-frequency signals governed by the Fourier Transform [1]. When constrained in both time and frequency there is a limit to the degree of concentration, as measure by fraction-out-of-band energy (FOBE), of energy simultaneously possible in the two domains. This is a form of uncertainty principle where the criterion for time-frequency concentration differs from the classical Heisenberg formulation which expresses signal concentration in terms of root mean square deviation (RMS). Here we deal with an analogy where we consider the fundamental limits to the use of free-space as an information bearing resource — a central concept for wireless communication which exploits space to achieve information transfer.

The extent to which we can exploit space is fundamentally constrained by the wave equation in free space. That is, the degree to which data, in an abstract sense, can be borne on information bearing wave-fields in a region of space is limited by the essential dimensionality of such wave-fields — a concept which builds on earlier work [2]. This means intuitively that in a given region of space there are only essentially a finite number of unit energy wave configurations given some error threshold. That is, waves in a finite region in space are essentially "spatially band-limited".

II. Spatial Analog to FOBE

FOBE time-frequency results center around the relationships between two non-intersecting linear subspaces – the subspace of time-limited signals and the subspace of band-limited signals. The geometrical relationship between these subspaces determines the limitations of how concentrated, in an energy sense, we can make a function simultaneously in an interval of time and an interval in frequency. This limitation leads to a notion of essential dimensionality which can be equated to the number of orthonormal functions which as a basis best represent any Fourier Transform pair within the time-frequency rectangle.

The question we pose and answer is: Is there an analogy in the spatial case? The relevance of this question is that it sheds light on the communication limits imposed by restricting attention to a region of space which in the practical case can be populated by a multiplicity of antennas.

A narrowband source-field $f(\boldsymbol{x})$ and the resulting wave-field $u(\boldsymbol{x})$, as a functions of space $\boldsymbol{x} \in \mathbb{R}^3$, are related through the time independent wave equation (Helmholtz equation)

$$\triangle u(\boldsymbol{x}) + k^2 u(\boldsymbol{x}) = f(\boldsymbol{x}), \tag{1}$$

where $\triangle$ is the Laplacian, and $k = 2\pi/\lambda$ is the wave number with λ the wavelength. These two fields related through the Helmholtz equation are the analogy of the time domain function and frequency domain function related through the Fourier Transform. Although (1) is a PDE, it is linear and when $f(\boldsymbol{x}) = 0$ within some source free region about the origin such as a ball of radius T, $\mathbb{B}_T^3 \triangleq \{\boldsymbol{x} \in \mathbb{R}^3 \colon |\boldsymbol{x}| \leq T\}$, the set of solutions forms a linear subspace. That is we have a linear subspace of (spatially high pass) space-limited source-fields

$$\mathcal{G}_S \triangleq \{u \in L^2(\mathbb{B}_S^3) \colon \triangle u + k^2 u = 0\}. \tag{2}$$

with corresponding orthogonal projection operator G_S.

A second linear subspace is the (spatially low pass) space-limited wave-fields and can be defined through a truncation operator, H_R, which sets the wave-field to zero outside some region of interest such as a sphere or radius R, $\mathbb{B}_R^3$. With $R < S$, these linear subspaces are non-intersecting and we can pursue the analogy of results in [1] such as: Which member(s) of the class of wave-fields generated by sources outside radius S, of unit energy in the region $|\boldsymbol{x}| \leq S$, have the maximum concentration of energy in the inner ball $|\boldsymbol{x}| \leq R$ where $R < S$:

Problem 1. *Determine wave-field $u \in \mathcal{G}_S$ — the closed subspace in $L^2(\mathbb{B}_S^3)$ satisfying $\triangle u + k^2 u = 0$ in $\mathbb{B}_S^3$ — which has the greatest concentration of energy in $\mathbb{B}_R^3$ where $R < S$. That is, find the $u \in \mathcal{G}_S$ which achieves*

$$\sup_{\substack{\|u\|^2_{\mathbb{B}_S^3}=1 \\ u \in \mathcal{G}_S}} \int_{\mathbb{B}_R^3} \left|u(\boldsymbol{x})\right|^2 \, dv(\boldsymbol{x}) \quad or \quad \sup_{w \neq 0} \frac{\|H_R G_S w\|^2_{\mathbb{B}_S^3}}{\|G_S w\|^2_{\mathbb{B}_S^3}} \tag{3}$$

This problem has solutions defined in terms of spherical Bessel functions analogous to the prolate spheroidal functions in [1], see [3]. In the case of isotropic sources the essential dimensionality of the wave-field scales with the area via $(e\pi R/\lambda)^2$.

References

[1] H. J. Landau and H. O. Pollack, "Prolate spheroidal wave functions, Fourier analysis and uncertainty - III: the dimension of the space of essentially time- and band-limited signals," *Bell System Tech. J.*, vol. 41, pp. 1295–1336, July 1962.

[2] H. M. Jones, R. A. Kennedy, and T. D. Abhayapala, "On Dimensionality of Multipath Fields: Spatial Extent and Richness," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing, ICASSP'2002*, Orlando, Florida, May 2002, vol. 3, pp. 2837–2840.

[3] R. A. Kennedy and T. D. Abhayapala, "Spatial Concentration of Wave-Fields: Towards Spatial Information Content in Arbitrary Multipath Scattering," in *Proc. 4th Australian Communications Theory Workshop*, Melbourne, Australia, February 2003, pp. 38–45.

[1]National ICT Australia is funded through the Australian Government's *Backing Australia's Ability* initiative, in part through the Australian Research Council. Research partially supported by ARC grant No: DP0343804.