Rank Reduced ESPRIT Techniques in the Estimation of Principle Signal Components

Jian Zhang, Rodney A. Kennedy and Thushara D. Abhayapala

***Abstract*— In this paper, we present novel rank reduced ESPRIT algorithms to estimate principle signal components with low computational complexity, which can typically be applied in the high resolution identification of closely spaced wireless multipath channels. These algorithms transform the generalized eigen-problem from an original high dimensional space to a lower dimensional space depending on the number of desired principle signals. As only principle singular values and vectors are required, fast algorithms such as the power method can be applied to greatly simplify the proposed algorithm and make it implementable in real time.**

***Index Terms*— Rank Reduced ESPRIT, Estimation of Principle Signal Components**

I. Introduction

ESPRIT is a signal subspace algorithm using a matrix-shifting approach to achieve high resolution identification of signals corrupted by noise [1], [2]. The applications of ESPRIT and its variants (ESPRITs) include harmonic retrieval [3], estimation of directions of arrival (DOA) in array signal processing [2], filter design [4] and identification of closely spaced wireless multipath channel [5]. In ESPRITs, the data matrix or a matrix of some statistics of the data (such as correlation) is decomposed into orthogonal signal and noise subspaces using singular value decomposition (SVD), and then the signal parameters are estimated using the rotation invariance over the signal subspace by generalized eigen-value decomposition (GED). Compared to the traditional least squares methods, these algorithms can normally provide much better performance on the accuracy and stability of estimates. However, the computation of SVD and GED of high dimensional matrices is often very intensive, and this associated high computational cost makes these techniques, as well as other subspace methods, less attractive for real-time implementation. In this paper, we develop some rank reduced ESPRIT algorithms to reduce the computational complexity.

Rank reduction is a general principle for finding the right tradeoff between model bias and model variance when reconstructing signals from noisy data. Abundant research has been reported, for example, in [6]–[9]. These rank reduction techniques usually try to find a low rank approximation of the original data matrix following some optimization criteria such as the least squares and the minimum variance criteria. In the SVD-based reduced rank methods, the low rank approximation matrix is a result of keeping dominant singular values while setting insignificant ones to zeros. Rank reduction technique is inherent in some improved ESPRIT algorithms, such as TLS-ESPRIT [2]. However, the reduced rank discussed in the literature is constrained to d, the number of signal sources, and d is usually required to be known as a prior or estimated by other techniques.

The approach we address in this paper removes the constraint on d, and is suitable for any rank $p \leq d$. Due to the rotational invariance property of the formed data matrix, we will show that there exists an approach which can greatly reduce the computation complexity, with negligible error introduced. Basically, this algorithm is designed to accurately identify the principle signal components with affordable complexity when the number of signal resources is very large, for example, the identification of part of multipath signals with largest power in a rich multipath environment. The computational cost is roughly associated with the number of the desired principle components.

The following notation is used in this paper. Matrices and vectors are denoted by boldface upper-case and lower-case letters, respectively. The conjugate transpose of a vector or matrix is denoted by the superscript $(\cdot)^*$, the transpose is denoted by $(\cdot)^T$, and the pseudo-inverse of a matrix is denoted by $(\cdot)^\dagger$. Finally, $\mathbf{I}$ denotes the identity matrix and diag$(\cdots)$ denotes a diagonal matrix.

II. Data Model and ESPRIT Algorithm

Since many applications can be generalized into a signal model similar to the harmonic retrieval problem, we use the latter as an example to address our algorithms. Consider a signal consisting of d harmonics with unknown constant amplitudes and phases, and an additive noise that is assumed zero-mean stationary complex white Gaussian random process (CAWGN). The signal can be represented as

$$x(k) = \sum_{i=1}^{d} s_i e^{jk\omega_i} + n(k), \tag{1}$$

where $\omega_i \in (-\pi, \pi)$ and s_i are the normalized frequencies and complex amplitudes of the ith harmonics, respectively, and $n(k)$ is the random additive noise with variance σ^2. The objective is to estimate the unknown but deterministic frequencies ω_i from the measured data.

J. Zhang is with the Research School for Information Science and Engineering, Australian National University, Canberra, ACT 0200, Australia. email: jian@syseng.anu.edu.au

R .A. Kennedy and T. D. Abhayapala are with National ICT Australia (NICTA) Limited, and Department of Telecommunications Engineering, Research School of Information Sciences and Engineering, The Australian National University, Canberra ACT 0200, Australia. email: {rodney.kennedy,thushara.abhayapala}@anu.edu.au

This work was partially supported by the Australian Research Council under Discovery Project Grant DP0343804.

Define several $m \times 1$ vectors of samples from (1) as follows,

$$\begin{aligned}\mathbf{x}(k) &= [x(k), \cdots, x(k+m-1)]^T, \\ \mathbf{y}(k) &= [x(k+1), \cdots, x(k+m)]^T, \\ \mathbf{n}(k) &= [n(k), \cdots, n(k+m-1)]^T, \end{aligned} \qquad (2)$$

where $m > d$[1]. Further define the autocorrelation matrix of the sampling vector $\mathbf{x}(k)$ as $\mathbf{R}_{xx} = E\big[\mathbf{x}(k)\mathbf{x}^*(k)\big]$, and the cross-correlation matrix of $\mathbf{x}(k)$ and $\mathbf{y}(k)$ as $\mathbf{R}_{xy} = E\big[\mathbf{x}(k)\mathbf{y}^*(k)\big]$. These two matrices can be expressed as

$$\mathbf{R}_{xx} = \mathbf{A}\mathbf{S}\mathbf{A}^* + \sigma^2\mathbf{I}, \qquad (3)$$

$$\mathbf{R}_{xy} = \mathbf{A}\mathbf{S}\mathbf{\Phi}^*\mathbf{A}^* + \sigma^2\mathbf{Z}, \qquad (4)$$

where

$$\begin{aligned}\mathbf{A}_{m\times d} &= \big[\mathbf{a}(\omega_1), \mathbf{a}(\omega_2), \cdots, \mathbf{a}(\omega_i), \cdots, \mathbf{a}(\omega_d)\big], \\ \mathbf{a}(\omega_i)_{m\times 1} &= \big[1, e^{j\omega_i}, e^{j2\omega_i}, \cdots, e^{j(m-1)\omega_i}\big]^T, \\ \mathbf{S}_{d\times d} &= \text{diag}\big(|s_1|^2, |s_2|^2, \cdots, |s_d|^2\big), \\ \mathbf{\Phi}_{d\times d} &= \text{diag}\big(e^{j\omega_1}, e^{j\omega_2}, \cdots, e^{j\omega_d}\big), \end{aligned} \qquad (5)$$

and $\mathbf{Z} = \sigma^{-2}E\big[n(k)n^*(k+1)\big]$ is an $m \times m$ matrix with ones on the first subdiagonal and zeros elsewhere.

The ESPRIT algorithm exploits the underlying deterministic nature of the harmonics in (1), which is basically a matrix shift invariance property. When the noise variance σ^2 is estimated via the insignificant singular values of the matrix $\mathbf{R}_{xx}$, the following two matrices are formed

$$\begin{aligned}\mathbf{C}_1 &= \mathbf{R}_{xx} - \sigma^2\mathbf{I} = \mathbf{A}\mathbf{S}\mathbf{A}^*, \\ \mathbf{C}_2 &= \mathbf{R}_{xy} - \sigma^2\mathbf{Z} = \mathbf{A}\mathbf{S}\mathbf{\Phi}^*\mathbf{A}^*. \end{aligned} \qquad (6)$$

The frequencies are then determined by the d generalized eigenvalues (GEs) of the matrix pencil $(\mathbf{C}_1 - \xi\mathbf{C}_2)$ that lie on the unit circle.

Substituting estimated frequencies into (1), the amplitudes s_i can be obtained by solving a Vandermonde system using least squares type algorithms [3], [8]. In ESPRIT, the power of harmonics can also be solved directly according to the generalized eigenvectors (GVs).

The estimates obtained by ESPRIT are asymptotically unbiased (when the number of samples goes to infinity). In practice, since the number of signal samples is finite, the noise components contained in the estimates of correlation matrices $\mathbf{R}_{xx}$ and $\mathbf{R}_{yy}$ always deviate from $\sigma^2\mathbf{I}$ and $\sigma^2\mathbf{Z}$, respectively. This deviation is one of the main sources causing errors in the estimation. Besides, when the dimension of matrices is large, the numerical error in the computation of SVD and GED frequently causes noticeable estimation error, especially for small singular values. This is due to the inherent instability of SVD and GED operation on singular or ill-conditioned matrices. Another error source, less noticed, is formed in the process of tracking the d GEs, locating on the unit circle, out of m GEs of the matrix pencil $\mathbf{C}_1 - \xi\mathbf{C}_2$.

Part of these errors are mitigated by some improved ESRPIT algorithms such as TLS-ESPRIT [2]. Combined with Total Least Squares (TLS), TLS-ESPRIT further applies SVD on the matrices $\mathbf{C}_1$ and $\mathbf{C}_2$ to extract components containing less noise, and transform the generalized eigen-problem in an instable $m \times m$ space to that in a more stable $d \times d$ space.

The computational load of ESPRITs is mainly from the SVD and GED operation in terms of a high dimension matrix. The basic ESPRIT algorithm needs one SVD and one GED of $m \times m$ matrices, while all published TLS-ESPRIT type algorithms require operations of at least two SVDs of $m \times m$ matrices and one GED of $d \times d$ matrices. For general $m \times m$ matrices, the complexity of SVD is about $4m^3$, and the complexity of GED is about $8m^3$ [10].

III. Reduced-Rank Approximation in ESPRIT Algorithm

From the analysis of error sources above, we see that a smaller space usually implies better stability and simplicity compared to a larger one. This motivates us to design stable and less complex algorithms in a smaller space when only p out of d harmonics need to be estimated.

A. Principle subspace and frequency estimation

Since C_1 in (6) is a Hermitian matrix, the SVD of $\mathbf{C}_1$ has the form

$$\begin{aligned}\mathbf{C}_1 &= \mathbf{U}\Lambda\mathbf{U}^* \\ &= [\mathbf{U}_p \ \mathbf{U}_r] \begin{bmatrix} \Lambda_p & \mathbf{0} \\ \mathbf{0} & \Lambda_r \end{bmatrix} [\mathbf{U}_p \ \mathbf{U}_r]^* \\ &= \mathbf{U}_p\Lambda_p\mathbf{U}_p^* + \mathbf{U}_r\Lambda_r\mathbf{U}_r^*, \end{aligned} \qquad (7)$$

where the $m \times m$ diagonal matrix Λ contains singular values in descending order, the unitary matrix $\mathbf{U}$ consists of left singular vectors, which equal right singular vectors since $\mathbf{C}_1$ is Hermitian. $\mathbf{U}_p$ and $\mathbf{U}_r$ are the left and right submatrices of $\mathbf{U}$, associated with the p principle and the remaining $m - p$ smaller singular values, respectively.

Multiply the matrix pencil $(\mathbf{C}_1 - \xi\mathbf{C}_2)$ by $\mathbf{U}_p^*$ from the left and by $\mathbf{U}_p$ from the right, we get a new $p \times p$ matrix pencil

$$(\Lambda_p - \xi\mathbf{U}_p^*\mathbf{C}_2\mathbf{U}_p), \qquad (8)$$

where we have utilized the orthogonality between the columns of $\mathbf{U}_p$ and $\mathbf{U}_r$, that is, $\mathbf{U}_p^*\mathbf{U}_p = \mathbf{I}$, $\mathbf{U}_p^*\mathbf{U}_r = \mathbf{0}$.

For the new matrix pencil, we have the following results.

Theorem 1: In ESPRIT-type algorithms, $\mathbf{C}_1$ and $\mathbf{C}_2$ are constructed as in (6), and the SVD of $\mathbf{C}_1$ is described as in (8). Then the matrix pencil $(\Lambda_p - \xi\mathbf{U}_p^*\mathbf{C}_2\mathbf{U}_p)$ has p distinctive generalized eigenvalues $\xi_i, i = 1, 2, \cdots, p$, and the angles of these GEs ξ_i are just the p frequencies ω_i, corresponding to p harmonics with largest power.

Proof: First, using (6), rewrite Λ_p as

$$\Lambda_p = \mathbf{U}_p^*\mathbf{C}_1\mathbf{U}_p = \mathbf{U}_p^*\mathbf{A}\mathbf{S}^{\frac{1}{2}}(\mathbf{S}^{\frac{1}{2}})^*\mathbf{A}^*\mathbf{U}_p \triangleq \mathbf{B}^*\mathbf{B}, \qquad (9)$$

where the $d \times p$ matrix $\mathbf{B} \triangleq (\mathbf{S}^{\frac{1}{2}})^*\mathbf{A}^*\mathbf{U}_p$. From

$$p = \text{Rank}(\Lambda_p) = \text{Rank}(\mathbf{B}^*\mathbf{B}) \leq \text{Rank}(\mathbf{B}), \qquad (10)$$

we know $\text{Rank}(\mathbf{B}) = p$. Then we can define the Moore-Penrose inverse (pseudo inverse) of $\mathbf{B}$ as

$$\mathbf{B}^\dagger = (\mathbf{B}^*\mathbf{B})^{-1}\mathbf{B}^* = \Lambda_p^{-1}\mathbf{B}^*. \qquad (11)$$

[1]Generally, $m \gg d$ for a good accuracy of estimates when the additive noise is not small

Using the expressions related to $\mathbf{B}$, the matrix pencil $\mathbf{U}_p^*(\mathbf{C}_1 - \xi_i\mathbf{C}_2)\mathbf{U}_p$ can be rewritten as

$$\begin{aligned}\mathbf{U}_p^*(\mathbf{C}_1 - \xi_i\mathbf{C}_2)\mathbf{U}_p =& \mathbf{U}_p^*\mathbf{A}\mathbf{S}^{\frac{1}{2}}(\mathbf{I} - \xi_i\mathbf{\Phi}^*)(\mathbf{S}^{\frac{1}{2}})^*\mathbf{A}^*\mathbf{U}_p \\ =& \mathbf{B}^*(\mathbf{I} - \xi_i\mathbf{\Phi}^*)\mathbf{B} \\ =& \Lambda_p\mathbf{B}^\dagger(\mathbf{I} - \xi_i\mathbf{\Phi}^*)\mathbf{B}. \end{aligned} \tag{12}$$

With the help of the SVD expressions of $\mathbf{B}$ and $\mathbf{B}^\dagger$, it can be further proved that if $\mathbf{I} - \xi_i\mathbf{\Phi}^*$ is full rank, $\mathbf{B}^\dagger(\mathbf{I} - \xi_i\mathbf{\Phi}^*)\mathbf{B}$ will have rank p, so will $\mathbf{U}_p^*(\mathbf{C}_1 - \xi_i\mathbf{C}_2)\mathbf{U}_p$. This contradicts the fact that $\mathbf{U}_p^*(\mathbf{C}_1 - \xi_i\mathbf{C}_2)\mathbf{U}_p$ is singular when ξ_i is a GE of the matrix pencil. Therefore, $\mathbf{I} - \xi_i\mathbf{\Phi}^*$ can only be singular which means ξ_i is also a GE of $(\mathbf{C}_1 - \xi\mathbf{C}_2)$. Notice that $\mathbf{I}$ and $\mathbf{\Phi}^*$ are both diagonal matrices, ξ_i must equal $e^{j\omega_i}$ which is the ith diagonal element of $\mathbf{\Phi}$. Then $\omega_i = \text{Angle}(\xi_i)$.

Because all frequencies are different from each other, ξ_i is a rank 1 reducing factor of the matrix $\Lambda_p - \xi\mathbf{U}_p^*\mathbf{C}_2\mathbf{U}_p$, corresponding to a one-dimension eigen-space. According to the relationship between the eigen-space and the matrix column space, (the dimension of a matrix column space equals the dimension of a whole eigen-space spanned by all linearly independent eigenvectors), we can further conclude that there must be p distinctive GEs, each corresponding to a eigen-space with one spanning basis.

Since the SVD of a matrix exhibits the spectral distribution of the comprised signal [6], the principle singular values and vectors reflect the information of the frequencies with largest power. This intuitively explains why the p GEs are associated with the p frequencies with largest power. ■

From the process of the proof, we can also find some relationships between the eigenvectors of the new matrix pencil and the old one as shown below.

Corollary 1: When $p = d$, corresponding to a common GE ξ_i, the generalized eigenvector of $\mathbf{C}_1 - \xi\mathbf{C}_2$ can be determined by $\mathbf{U}_p\mathbf{v}_i$, where $\mathbf{v}_i$ is the generalized eigenvector of $\Lambda_p - \xi\mathbf{U}_p^*\mathbf{C}_2\mathbf{U}_p$.

So far, the p principle signal components can be estimated without any performance loss, using reduced rank TLS-ESPRIT technique with a $4m^3 \times 2$ flops computation of two complete SVDs plus a $8p^3$ flops of one GED.

Discussion: Although our basic interest is always to track the strongest signals, it is possible to extend this algorithm to track any p harmonics rather than the principle ones by choosing singular values and vectors accordingly.

B. Power Estimation of the Harmonics

In the case when only p out of d frequencies are known, the estimates of p amplitudes obtained by solving under-determined linear equations of (1) will comprise large error. Alternatively, the power of these p harmonics can be estimated in a subspace method requiring no further information.

Let $\mathbf{v}_i$ be the generalized eigenvector corresponding to the generalized eigenvalue ξ_i. From the definition of generalized eigen-problem, using the expressions (6) in the matrix pencil $(\Lambda_p - \xi\mathbf{U}_p^*\mathbf{C}_2\mathbf{U}_p)$, we have

$$\mathbf{U}_p^*\mathbf{A}\mathbf{S}(\mathbf{I} - \xi_i\mathbf{\Phi}^*)\mathbf{A}^*\mathbf{U}_p\mathbf{v}_i = \mathbf{0}. \tag{13}$$

Left multiply (13) by $\mathbf{v}_i^*$, yields

$$(\mathbf{v}_i^*\mathbf{U}_p^*\mathbf{A})\mathbf{S}(\mathbf{I} - \xi_i\mathbf{\Phi}^*)(\mathbf{v}_i^*\mathbf{U}_p^*\mathbf{A})^* = 0. \tag{14}$$

As $\mathbf{S}(\mathbf{I} - \xi_i\mathbf{\Phi}^*)$ is a $d \times d$ diagonal matrix with only ith diagonal element equaling zero, the $1 \times d$ vector $\mathbf{v}_i^*\mathbf{U}_p^*\mathbf{A}$ has the form

$$\mathbf{v}_i^*\mathbf{U}_p^*\mathbf{A} = [0, \cdots, 0, \mathbf{v}_i^*\mathbf{U}_p^*\mathbf{a}(\omega_i), 0, \cdots, 0], \tag{15}$$

that is, except ith element, all others equal zero.

Notice that $\Lambda_p = \mathbf{U}_p^*\mathbf{C}_1\mathbf{U}_p$ and $\xi_i\mathbf{\Phi}^*$ is a diagonal matrix with ith diagonal element equaling one, (14) can be rewritten as

$$\begin{aligned}\mathbf{v}_i^*\Lambda_p\mathbf{v}_i &= (\mathbf{v}_i^*\mathbf{U}_p^*\mathbf{A})(\mathbf{S}\,\xi_i\mathbf{\Phi}^*)(\mathbf{v}_i^*\mathbf{U}_p^*\mathbf{A})^* \\ &= |s_i|^2\left|\mathbf{v}_i^*\Lambda_p\mathbf{a}(\omega_i)\right|^2. \end{aligned} \tag{16}$$

It follows that

$$|s_i|^2 = \frac{\mathbf{v}_i^*\Lambda_p\mathbf{v}_i}{\left|\mathbf{v}_i^*\Lambda_p\mathbf{a}(\omega_i)\right|^2}. \tag{17}$$

C. Approximation with Noisy Correlation Matrix

The preceding algorithm still has order m^3 computational complexity due to the SVD of $m \times m$ matrix. The main objective of the two SVDs is to estimate noise variance and the principle singular components of $\mathbf{C}_1$, respectively. Since only p out of m principle singular values and vectors are required, the computation can be simplified by applying fast algorithms with lower complexity, such as the power method [10]. For each dominant singular value and vector, the power method has a computational order of m^2 for a $m \times m$ matrix[2]. To be stated, the power method works only when the gap between two singular values is large enough. For the series of principle singular values, this condition can generally be satisfied. With the help of power method, two kinds of simplifications are available. The first one only applies power method in the step of computing $\mathbf{U}_p$ and Λ_p with little performance difference. This results in an algorithm of complexity $4m^3 + pm^2 + 8p^3$. The second one directly applies power method on the matrix $\mathbf{R}_{xx}$, instead of $\mathbf{C}_1$, which leads to a substitution of the principle singular vectors and values of $\mathbf{R}_{xx}$ for $\mathbf{U}_p$ and Λ_p, and $\mathbf{R}_{xy}$ for $\mathbf{C}_2$ in the matrix pencil $(\Lambda_p - \xi\mathbf{U}_p^*\mathbf{C}_2\mathbf{U}_p)$. Thus, the complexity is only in the order of $pm^2 + 8p^3$. This simplification seems to be a substitution of noisy matrices for noise-free ones, however, intuitively, only insignificant errors will be introduced in the final output. This is because the substitution happens between the noise-immune principle singular components, and it is known that principle singular values have good stability when small perturbation of noise matrix is present. In Section IV, it can be seen that this simplification introduce negligible performance loss in the case of medium to high signal to noise ratio (SNR). To be rigorous, the approximation effect can be analyzed based on matrix perturbation theorem [10].

[2] In power method, the times of iteration to achieve convergence depends on the ratio of two adjoint singular values. For symmetric/Hermitian matrix, it usually converges very fast and 2 or 3 iterations are good enough.

D. Summary of Reduced-Rank ESPRIT Algorithm

The second simplification method discussed above can very likely be implemented on-line for the retrieval of principle signal components within rich sources. This can be regarded as a rank reduced LS-ESPRIT algorithm as only one of two noisy matrices is truncated in the sense of least squares approximation. The key steps of the proposed algorithm may be summarized as follows.

1) Construct the correlation matrices $\mathbf{R}_{xx}$ and $\mathbf{R}_{xy}$ from the samples.
2) Apply the direct SVD or a fast algorithm (e.g., the power method) to solve p principle singular values and vectors of $\mathbf{R}_{xx}$, and form matrices Λ_p and $\mathbf{U}_p$.
3) Compute the p generalized eigenvalues ξ_i and eigenvectors $\mathbf{v}_i$ of the matrix pencil $\Lambda_p - \xi \mathbf{U}_p^* \mathbf{R}_{xy} \mathbf{U}_p$.
4) The p principle frequencies are the angles of ξ_i, and the power of these harmonics can be obtained by (17).

When the power method is applied, the total complexity of this algorithm is $pm^2 + 8p^3$.

E. Extension to General Shift Invariant Algorithms

The ideas presented above can be extended to general shift invariant algorithms, such as the Matrix pencil and the State-Space methods [9], [11]. The underlying data matrices can be the data samples directly, then they are usually not symmetric or Hermitian. Nevertheless, common GEs exist between the new reduced rank matrix pencil and the original one. This conclusion can be proven in a way similar to that in Theorem 1. However, due to the loss of the property of Hermitian, the power of harmonics can no longer be obtained in the way described in Section III-B. Unless working out all the parameters of d harmonics, the power/amplitude information can not be recovered by the linear equation method, either.

If the power/amplitude of harmonics are not required, an approach can be adopted to avoid the approximation of filtered matrices by noisy ones, such as $\mathbf{C}_1$ and $\mathbf{C}_2$ by $\mathbf{R}_{xx}$ and $\mathbf{R}_{xy}$ in the reduced-rank ESPRIT. This can be achieved by constructing two noise-free cross-correlation matrices from the sampled data as discussed in [3]. These two new correlation matrices contain no noise as the cross-correlation operation wipes out the noise components.

IV. SIMULATIONS

In our simulations, 20 harmonics with different frequencies are used. The amplitudes of all the harmonics are generated randomly using complex Gaussian function. To check the ability of our algorithms to automatically track those harmonics with largest power, all harmonics are unordered with respect to amplitude/power. The length of the sampling vector, m, is set to 300 in all experiments. Since m determines the resolution ability for frequencies when the noise variance is fixed, without loss of generality, the frequencies of these harmonics are set uniformly, with intervals varying in different experiments.Hereafter, we refer to the original ESPRIT in [1] as original ESPRIT, our proposed algorithm requiring the estimation of the noise variance in the first step as filtered ESPRIT, and our proposed algorithm approximating $\mathbf{C}_1$ and $\mathbf{C}_2$ with $\mathbf{R}_{xx}$ and $\mathbf{R}_{xy}$ as unfiltered ESPRIT. When the power method is applied, it is indicated.

Fig. 1 shows that the estimated frequencies and powers of the harmonics obtained by the three algorithms. The stars represent the true parameters of harmonics. The values displayed by diamonds, squares, and circles are the estimates generated by the original, filtered and unfiltered ESPRITs, respectively. The intervals between frequencies are 0.0408, and the SNR is set to 5dB. The results are averaged over five realizations. It is clear that both filtered and unfiltered ESPRIT provide satisfactory accuracy, and they do track the p harmonics with largest power. The large error in the original ESPRIT is caused by the factors discussed in Section III-A. When the SNR decreases, the accuracy of estimates slightly declines as can be seen in Fig. 2.

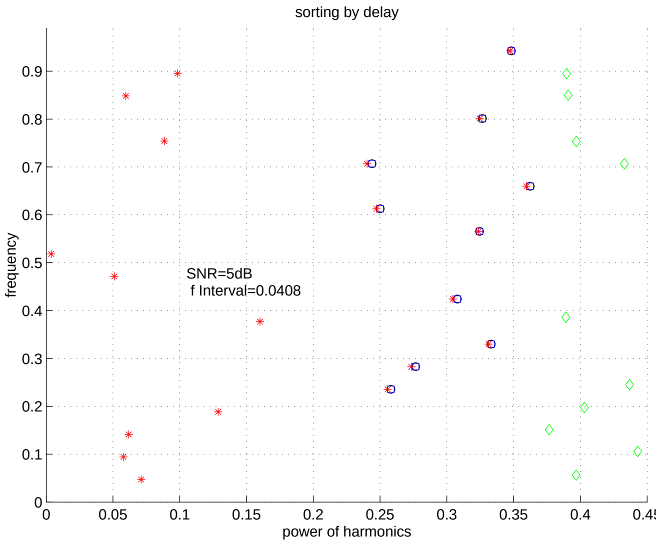

Fig. 1: Estimated frequency and power of harmonics when frequency interval is 0.0408, $p = 10$ and SNR=5dB.

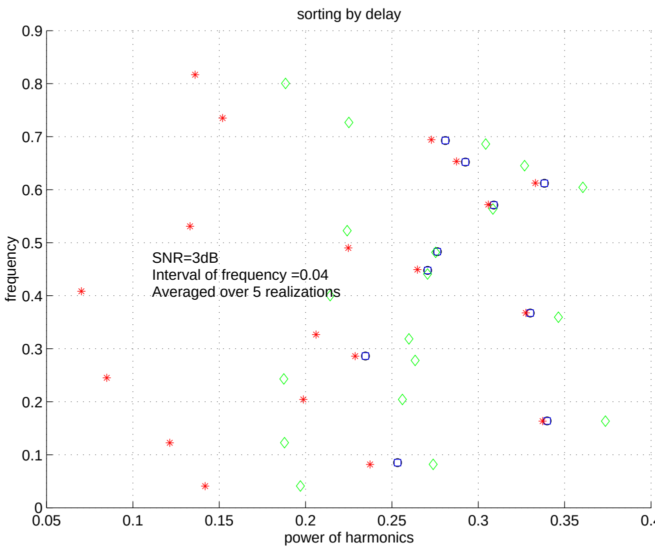

Fig. 2: Estimated frequency and power of harmonics when frequency interval is 0.04, $p = 10$ and SNR=3dB.

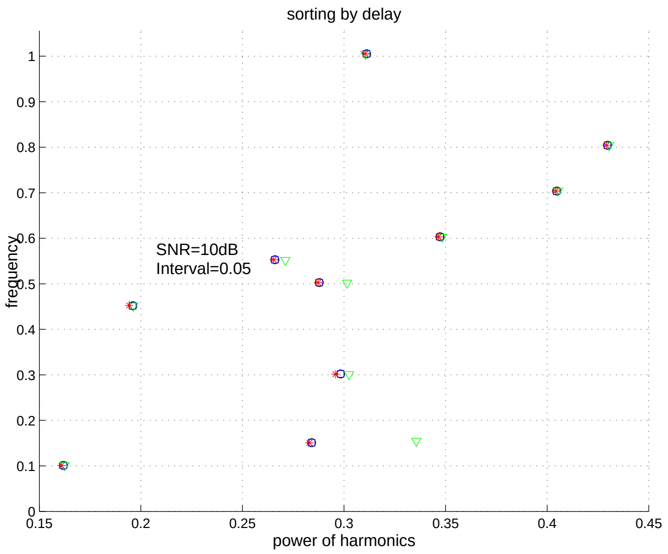

Fig. 3: Estimated frequency and power of harmonics obtained by unfiltered ESPRIT with Power method (marked with triangles) and direct SVD (marked with circles and squares) when frequency interval is 0.05, $p = 10$ and SNR=10dB.

In Fig. 3, the results compared are obtained by the unfiltered ESPRIT uisng the power method , the unfiltered ESPRIT and filtered ESPRIT using direct SVDs, marked with triangles, circles and squares, accordingly. In experiments, the times of iteration in the power method is set to 20 as the ratio between adjoint singular values becomes smaller with p increasing. However, this is still a notable saving of computation cost compared to the direct SVD. From the figure we can find that the performance difference achieved by the power method and direct SVDs is insignificant. With the SNR decreasing, the difference is enlarged gradually. However, the performance deterioration can be partly prevented by increasing the times of iteration.

V. Conclusions

We have shown that the rank reduction techniques can be well organized in the ESPRIT type algorithms, which leads to low-complexity algorithms in the estimation of principle components in rich signal sources. These algorithms transform the generalized eigen-problem in an original large space to that in a smaller space depending on the number of desired principle signal components. Proof is given on the transform, and an extension to general shift invariant techniques is possible, which can be proved in a similar way. As only principle singular values and vectors are required, fast algorithms such as the power method can be applied to greatly simplify these algorithms and make them implementable in real time.

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