ANTENNA SATURATION EFFECTS ON DENSE ARRAY MIMO CAPACITY

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ABSTRACT

We investigate the behaviour of MIMO capacity when the size of the antenna array is constrained. By increasing the number of antennas within a small region in space the antenna array becomes dense and spatial correlation inhibits capacity growth. A theoretically derived antenna saturation point is shown to exist for dense array MIMO systems, at which there is no capacity growth with increasing antenna numbers. We show this saturation point increases linearly with the radius of the region containing the antenna array and is independent of the number of antennas.

1. INTRODUCTION

Multiple-Input Multiple-Output (MIMO) communications systems using multi-antenna arrays simultaneously during transmission and reception have generated significant interest in recent years. Theoretical work of [1] and [2] showed the potential for significant capacity increases in wireless channels via spatial multiplexing with sparse antenna arrays. However, in reality the capacity is significantly reduced when the signals received by different antennas are correlated [3], corresponding to the antennas being placed close together.

In this paper we theoretically analyze the effect on capacity of increasing numbers of antennas in a uniform circular array of fixed radius. As the number of antennas grows the antenna array becomes dense and spatial correlation significantly limits the capacity. We argue that using a circular array is the best use of the space available since this topology maximizes the distance between each antenna and all its neighbors for *every* antenna. Under these conditions we derive a theoretical saturation point, where no further capacity gain is achieved with increasing numbers of antennas.

Recent independent works [4,5] have studied dense linear arrays, however, to the authors knowledge no work exists on 2D arrays, or given the number of antennas required to saturate the capacity, as addressed here.

2. CONVERGENCE OF ERGODIC CAPACITY

Consider a MIMO system consisting of S transmitters and Q receivers, let the transmitted signals be statistically independent equal power components each with a Gaussian distribution, then the ergodic channel capacity is given by [1],

$$\widetilde{C} = E\left\{\log\left|\boldsymbol{I}_Q + \frac{\eta}{QS}\boldsymbol{H}\boldsymbol{H}^\dagger\right|\right\} \quad (1)$$

where $\boldsymbol{H}$ is the normalized $Q \times S$ random flat fading channel matrix known at the receiver, η is the average signal-to-noise ratio (SNR), $\boldsymbol{I}_Q$ is the $Q \times Q$ identity matrix, $|\cdot|$ is the determinant operator, and $\dagger$ the Hermitian operator. The scaling factor $1/Q$ ensures the total received power remains independent of the number of receiver antennas [6].

Let $\boldsymbol{H} = [\boldsymbol{h}_1\boldsymbol{h}_2\cdots\boldsymbol{h}_S]$, where $\boldsymbol{h}_s$ is the $Q \times 1$ complex vector of channel gains corresponding to the sth transmit antenna, then the correlation matrix at the receiver is defined as $\boldsymbol{R}_Q \triangleq E\left\{\boldsymbol{h}_s\boldsymbol{h}_s^\dagger\right\}, \forall s$, where $\boldsymbol{R}_Q(p,q) = \rho_{pq}$ is the spatial correlation between two sensors p and q at the receiver.

Consider the situation where the transmit array has well separated antennas such that the transmitter covariance matrix $\boldsymbol{R}_S = \boldsymbol{I}_S$, corresponding to independent $\boldsymbol{h}_s$ vectors, then the sample correlation matrix at the receiver is given by $\widehat{\boldsymbol{R}}_Q \triangleq S^{-1}\sum_{s=1}^{S}\boldsymbol{h}_s\boldsymbol{h}_s^\dagger$ which converges to $\boldsymbol{R}_Q$ for large numbers of transmit antennas ($S \to \infty$). Observing that $\boldsymbol{H}\boldsymbol{H}^\dagger = \sum_{s=1}^{S}\boldsymbol{h}_s\boldsymbol{h}_s^\dagger$ then for a large number of well separated transmit antennas the ergodic capacity converges to the deterministic quantity C,

$$\lim_{S\to\infty}\widetilde{C} = C \triangleq \log\left|\boldsymbol{I}_Q + \frac{\eta}{Q}\boldsymbol{R}_Q\right|. \quad (2)$$

2.1. Maximum Theoretical Capacity

In the case of uncorrelated receiver antennas $\boldsymbol{R}_Q = \boldsymbol{I}_Q$ we get the maximum theoretical capacity

$$C_{\max} = Q\log(1 + \frac{\eta}{Q}) \quad (3)$$

which is identical to the identity channel case ($\boldsymbol{H} = \boldsymbol{I}$) shown in [2], therefore, for a large number of receivers we have

$$\lim_{Q\to\infty} C_{\max} = C_{\text{limit}} \triangleq \frac{\eta}{\ln(2)} \tag{4}$$

which is the absolute maximum capacity achievable for an ideal MIMO system.

3. CAPACITY OF A UNIFORM CIRCULAR ARRAY IN A 2D ISOTROPIC DIFFUSE FIELD

The capacity formula (2) can be expanded by the product of eigenvalues within the determinant, giving,

$$C = \sum_{m=0}^{Q-1} \log(1 + \frac{\eta}{Q}\lambda_m^{(Q)}) \tag{5}$$

where $\lambda_m^{(Q)} \in \sigma(\boldsymbol{R}_Q)$ are the Q eigenvalues of the spatial correlation matrix $\boldsymbol{R}_Q$. Therefore, we see that the capacity is governed by the eigenvalues of the spatial correlation matrix, and as such their properties dictate the behavior of the capacity given differing scattering environments, antenna numbers and placement.

Consider a uniform circular array (UCA) with radius r and Q receiver elements. Denote the set $\{d_\ell\}_{\ell=0}^{Q-1}$ as the distance between any element and the other $Q-1$ elements in the array (in a clockwise or anticlockwise direction), with $d_0 = 0$ being the distance between the element and itself, then

$$d_\ell \triangleq 2r\sin(\pi\ell/Q). \tag{6}$$

For the special case of scattering over all angles in the plane we have a 2D isotropic diffuse field (often referred to as a *rich* scattering environment) at the receiver and the spatial correlation between any element on the UCA and its ℓth neighbor is given by [7]

$$\rho_\ell \triangleq J_0(k\, d_\ell) \tag{7}$$

where $J_0(\cdot)$ are Bessel functions of the first kind, and $k = 2\pi/\lambda$ is the wavenumber. Due to UCA symmetry, for $\ell > 0$, $\rho_\ell = \rho_{Q-\ell}$, and the correlation matrix becomes a $Q \times Q$ symmetric circulant matrix,

$$\boldsymbol{R}_Q = \text{Circ}\left[\rho_0, \rho_1, \ldots, \rho_{\lceil\frac{Q-1}{2}\rceil}, \rho_{\lfloor\frac{Q-1}{2}\rfloor}, \ldots, \rho_2, \rho_1\right] \tag{8}$$

where $\lceil\cdot\rceil$ and $\lfloor\cdot\rfloor$ are the ceiling and floor operators respectively, and

$$\text{Circ}[x_1, x_2, \ldots, x_N] \triangleq \begin{bmatrix} x_1 & x_2 & \cdots & x_N \\ x_N & x_1 & \cdots & x_{N-2} \\ \vdots & & \ddots & \vdots \\ x_2 & x_3 & \cdots & x_1 \end{bmatrix} \tag{9}$$

defines the circulant matrix.

3.1. Eigenvalues of Spatial Correlation Matrix $\boldsymbol{R}_Q$

The eigenvalues of the symmetric circulant matrix $\boldsymbol{R}_Q$ are given by a simple closed form expression [8]

$$\lambda_m^{(Q)} = \sum_{\ell=0}^{Q-1} \rho_\ell e^{i2\pi m\ell/Q}. \tag{10}$$

For a UCA in a 2D isotropic diffuse field the correlation coefficients are real and symmetric, hence (10) represents the Discrete Cosine Transform (DCT) of the spatial correlation coefficients

$$\lambda_m^{(Q)} = \sum_{\ell=0}^{Q-1} \rho_\ell \cos(2\pi m\ell/Q). \tag{11}$$

Since $\boldsymbol{R}_Q$ is a positive-semidefinite Hermitian matrix and with the properties of the DCT it is easy to show $\lambda_m \in \mathbb{R}$, $\lambda \geq 0$, and $\lambda_{Q-m} = \lambda_m = \lambda_{-m}$, that is, the eigenvalues are real, non-negative and symmetric.

Theorem 1 (eigenvalue threshold). *For a UCA of radius r in a 2D isotropic diffuse field define the eigenvalue threshold:*

$$M \triangleq \lceil \pi e r/\lambda \rceil \tag{12}$$

then, for any $Q \geq 2M+1$ there exists a finite set of non-vanishing eigenvalues, $\{\lambda_m^{(Q)}\}_{m=-M}^{M}$, with set size independent of Q.

Before proving Theorem 1 we clarify its significance with the following interpretation:

For any UCA in a 2D isotropic diffuse field there is a finite set of significant spatial correlation matrix eigenvalues, where the set size increases linearly with the radius of the array and is independent of the number of antennas.

Proof (sketch). Substitution of (7) and (6) into (11) gives

$$\lambda_m^{(Q)} = \sum_{\ell=0}^{Q-1} J_0(2kr\sin(\pi\ell/Q))\cos(2m\pi\ell/Q) \tag{13}$$

letting $\xi = \pi\ell/Q$ and assuming a large number of antennas, we can approximate (13) with the integral

$$\lambda_m^{(Q)} \approx \frac{Q}{\pi}\int_0^\pi J_0(2kr\sin\xi)\cos(2m\xi)\,\mathrm{d}\xi \tag{14}$$

for $m \in [0, \lceil(Q-1)/2\rceil]$. Using the identity [9, p.32]

$$J_n^2(z) = \frac{1}{\pi}\int_0^\pi J_0(2z\sin\psi)\cos(2n\psi)\mathrm{d}\psi \tag{15}$$

then the eigenvalues can be expressed as

$$\lambda_m^{(Q)} \approx Q J_m^2(kr) \tag{16}$$

which is asymptotically equal to (13) with the antenna number.

Using the the following bound [10, p.362] on the bessel functions for $n \geq 0$

$$|J_n(z)| \leq \frac{|z|^n}{2^n \Gamma(n+1)} \tag{17}$$

the eigenvalues are then upper-bounded by

$$\lambda_m \leq Q \left(\frac{(\pi r/\lambda)^m}{\Gamma(m+1)} \right)^2. \tag{18}$$

Since Gamma function $\Gamma(m+1)$ increases faster than the exponential $(\pi r/\lambda)^m$ then (18) will rapidly approach 0 for some $m > 0$ for which $\Gamma(m+1) > (\pi r/\lambda)^m$. Using a relaxed Stirling lower bound[1] for $\Gamma(m+1)$, we wish to find m for which $(m/e)^m > (\pi r/\lambda)^m$, which is clearly satisfied when $m > \pi e r/\lambda$, asserting that m must be an integer we see that the eigenvalues vanish for $m > \lceil \pi e r/\lambda \rceil$, thus giving the eigenvalue threshold in (12).

Given the symmetric nature of the eigenvalues then for any number of antennas, $Q \geq 2M+1$, there is a finite set of $2M+1$ non-vanishing eigenvalues,

$$\boldsymbol{\lambda} = \{\lambda_{-M}, \lambda_{-M+1}, \cdots, \lambda_0, \cdots, \lambda_{M-1}, \lambda_M\} \tag{19}$$

whose number of elements grows only with the radius of the array, and is independent on the number of antennas. □

Fig. 1 shows the eigenvalues of the spatial correlation matrix $\boldsymbol{R}_Q$ for various UCA radii in a 2D isotropic diffuse field. Shown as a solid black line, it can be seen that the theoretical eigenvalue threshold derived in Theorem 1 defines the boundary between the significant and vanishing eigenvalues for each radius.

3.2. Capacity Growth Limits: Antenna Saturation

Due to the dependence of (5) on the eigenvalues of the spatial correlation matrix we see that Theorem 1 has significant implications on capacity growth with increasing antenna numbers. In this section we show that this fixed set size of eigenvalues, regardless of the number of antennas, leads to an antenna saturation effect on MIMO capacity.

Theorem 2 (antenna saturation point). *For a UCA of radius r in a 2D isotropic diffuse field define a saturation point Q_M as the minimum number of antennas required to generate a full set of significant eigenvalues $\lambda_m^{(Q_M)} \in \sigma(\boldsymbol{R}_{Q_M})$;*

$$Q_M \triangleq 2M+1 \tag{20}$$

[1] $\Gamma(z+1) > \sqrt{2\pi z}\, z^z e^{-z} > z^z e^{-z}$, $z > 0$

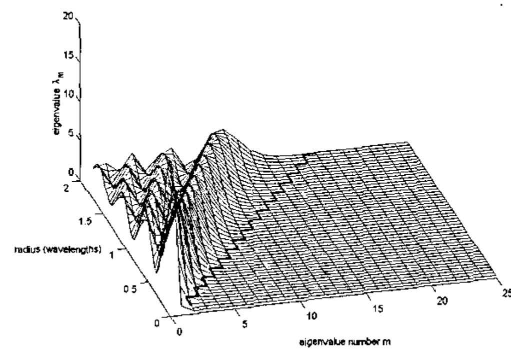

Fig. 1. The eigenvalues of the spatial correlation matrix for various UCA radii in a 2D isotropic diffuse scattering field. The dark solid line represents the theoretical eigenvalue threshold derived in Theorem 1, and clearly shows the boundary between the significant and vanishing eigenvalues of the spatial correlation matrix for each array radius.

then, for any $Q \geq Q_M$ the channel capacity is given by the constant

$$C \approx C_{sat} = \sum_{m=-M}^{M} \log\left(1 + \frac{\eta}{Q_M}\lambda_m^{(Q_M)}\right). \tag{21}$$

Before giving a proof of Theorem 2 we give the following interpretation:

For a MIMO system with a UCA in a 2D isotropic diffuse field there exists a saturation point in the number of antennas, which is dependent only on the radius of the array, after which the addition of more antennas gives no capacity gain.

Proof (sketch). Using the symmetric nature of the eigenvalues and assuming an odd number of antennas the capacity (5) can be written as[2]

$$C = \sum_{m=-(Q-1)/2}^{(Q-1)/2} \log\left(1 + \frac{\eta}{Q}\lambda_m^{(Q)}\right). \tag{22}$$

Consider the UCA placed in a 2D isotropic diffuse field, then as a direct result of Theorem 1 for $Q \geq 2M+1$ the channel capacity given by (22) is well approximated using the set of $2M+1$ non-vanishing eigenvalues, that is,

$$C \approx \sum_{m=-M}^{M} \log\left(1 + \frac{\eta}{Q}\lambda_m^{(Q)}\right). \tag{23}$$

[2] from Theorem 1 the case of even Q gives identical results, however to simplify notation we assume an odd number of antennas

Given two UCAs of equal radius r with antenna numbers $Q_1, Q_2 \geq 2M + 1$, and spatial correlation matrix eigenvalues $\lambda_m^{(Q_1)}$ and $\lambda_m^{(Q_2)}$ respectively, then from (16) we have the following relationship between the non-zero eigenvalues of systems with different numbers of receive antennas,

$$\frac{\lambda_m^{(Q_1)}}{Q_1} \approx \frac{\lambda_m^{(Q_2)}}{Q_2} \tag{24}$$

with the approximation asymptotically equal with the number of antennas. Define $Q_M \triangleq 2M + 1$ as the minimum number of antennas required to generate the full set of non-zero eigenvalues, then letting $Q_1 = Q_M$ and $Q_2 = Q$ we have

$$\lambda_m^{(Q)} \approx \frac{Q}{Q_M} \lambda_m^{(Q_M)} \tag{25}$$

where $\lambda_m^{(Q_M)} \in \sigma(\boldsymbol{R}_{Q_M})$ are the eigenvalues of the spatial correlation matrix $\boldsymbol{R}_{Q_M}$. Thus the non-zero eigenvalues for any UCA of radius r with number of antennas $Q \geq Q_M$ are simply scaled versions of the eigenvalues generated by an array with Q_M antennas. Substituting (25) into (23) gives

$$C \approx \sum_{m=-M}^{M} \log \left(1 + \frac{\eta}{Q_M} \lambda_m^{(Q_M)} \right) \tag{26}$$

which is independent of Q, hence the capacity growth becomes zero once the antenna number reaches the saturation point given by Q_M. □

It can be observed from Fig. 2 that the capacity (2) does indeed increase approximately with the maximum theoretical capacity (3) up until the theoretical saturation point defined in Theorem 2, after which no capacity gain is achieved with increasing antenna number.

4. DISCUSSION

We have derived a capacity saturation point, which depends only on the radius of the array, whereby further increases in the number of antennas fails to give further capacity gains. This result has significant implications for practical MIMO systems as the saturation point gives the minimum number of number of antennas required to achieve maximum capacity for a given region. Further to the UCA case, empirical studies using more general spatial correlation models [11] have shown that there are only ever $2\lceil \pi e r/\lambda \rceil + 1$ significant eigenvalues generated by arbitrarily placed antennas within a circular region of radius r. We believe the saturation point derived here for UCAs also holds for any antenna configuration within a circular region and we are currently developing theoretical results to support this.

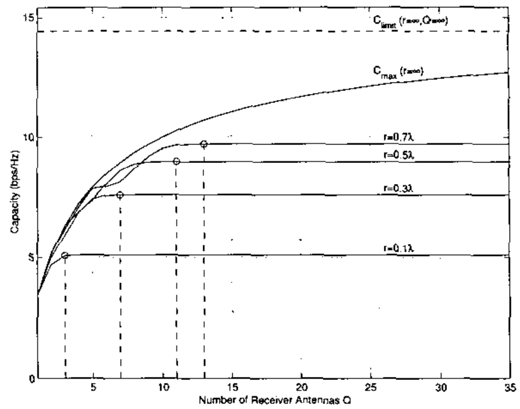

Fig. 2. Capacity of MIMO systems for various antenna numbers of a UCA with radii $r = 0.1, 0.3, 0.5$, and 0.7 wavelengths in a 2D isotropic diffuse scattering field, along with the theoretical limits. As indicated by the dashed lines for each radii, the Antenna Saturation Point theoretically derived in Theorem 2 gives a good indication where the MIMO system saturates and hence increasing antenna numbers gives no further capacity gain.

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