Introducing 'Space' into Space-Time MIMO Capacity Calculations: A New Closed Form Upper Bound

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Abstract

We present a new upper bound on capacity for multiple-input multiple-output (MIMO) wireless fading channels which is more general and realistic than previous capacity expressions. By including spatial information at the antenna arrays we derive a closed form upper bound on capacity which uses the physics of signal propagation combined with statistics of the scattering environment. This expression gives the capacity of a MIMO system in terms of antenna placement and scattering environment and leads to valuable insights into the factors determining capacity for a wide range of scattering models.

I. Introduction

Multiple-Input Multiple-Output (MIMO) communication systems using multi-antenna arrays simultaneously during transmission and reception have generated significant interest in recent years. Theoretical work of [1] and [2] showed the potential for significant capacity increases in wireless channels utilizing spatial diversity. However, in reality the capacity is significantly reduced when the signals received by different antennas are correlated, corresponding to a reduction of the effective number of sub-channels between transmit and receive antennas [2, 3]. Previous studies have given insights and bounds into the effects of correlated channels [3–5], however most have been for a limited set of channel realizations and antenna configurations. The restriction on most previous works has been the use of a random channel matrix model describing both the scattering environment and the antenna configurations.

These simulations or estimates based on channel matrix models, though likely to mimic reality, have several important problems:

1. It is difficult to relate channel models with realistic scattering environments.
2. The ergodic calculation limits analysis into the physical factors determining MIMO capacity.
3. No spatial information at either the transmitter or receiver is explicitly used.

In contrast, the contribution of this paper is an upper bound for MIMO capacity which overcomes these limitations, that is, with additional theory for modelling scattering environments which we refine here, we derive a model which can be readily reconciled with a multitude of scattering models and antenna configurations and allows us to derive a closed form upper bound for the MIMO capacity.

II. Ergodic Capacity of Multiple Antenna Systems

Consider a MIMO system consisting of S transmitting antennas with statistically independent power components each with Gaussian distributed signals, and Q receiving antennas. Let $\boldsymbol{x} = [x_1, x_2, \dots, x_S]^T$ be the vector of symbols sent by the S transmitting antennas, $\boldsymbol{n} = [n_1, n_2, \dots, n_Q]^T$ be the zero mean additive white gaussian noise vector each with variance σ^2, and $\boldsymbol{y} = [y_1, y_2, \dots, y_Q]^T$ be the vector of received symbols, where T denotes the vector transpose, then

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{n} \tag{1}$$

where $\boldsymbol{H}$ is a $Q \times S$ complex channel matrix, assumed to be constant over the symbol period. The channel capacity for the deterministic channel is then given by [2]

$$C_{\boldsymbol{H}} = \log\left|\boldsymbol{I}_Q + \frac{1}{\sigma^2}\boldsymbol{H}\boldsymbol{V}_{\boldsymbol{x}}\boldsymbol{H}^\dagger\right| \tag{2}$$

where $\boldsymbol{V}_{\boldsymbol{x}} = E\{\boldsymbol{x}\boldsymbol{x}^\dagger\}$ is the covariance matrix of the transmitted symbols $\boldsymbol{x}$, with $E\{\cdot\}$ the expectation operator, $|\cdot|$ is the determinant operator, and $\dagger$ the Hermitian operator. For a random channel model the channel matrix is stochastic hence the capacity given by (2) is also random. In this situation the mean (ergodic) capacity is obtained by taking the expectation of capacity $C_{\boldsymbol{H}}$ over all possible channel realizations,

$$C = E\{C_{\boldsymbol{H}}\} = E\left\{\log\left|\boldsymbol{I}_Q + \frac{1}{\sigma^2}\boldsymbol{H}\boldsymbol{V}_{\boldsymbol{x}}\boldsymbol{H}^\dagger\right|\right\}. \tag{3}$$

For the special case of uncorrelated Rayleigh MIMO fading channel a closed form expression for (3) is derived in [1], however, for correlated fading channels Monte Carlo simulations [3, 6], or asymptotic results for large number of antennas [7], are used to provide estimates of capacity. Some analytical lower and upper bounds on the ergodic capacity have been derived (e.g., see [2,3]), more recently, [4] gave an upper bound on ergodic capacity based on the correlation between each channel matrix element. However, these special cases, simulations, and bounds offer little insight and fail to provide a rigorous demonstration into factors determining capacity. Current channel matrix models do not include spatial information explicitly, although it is represented by the correlation between channel matrix elements it has no direct realizable physical representation and therefore does not easily lend itself to insightful capacity results. In particular, of interest is the effect on channel capacity of antenna placement at both the transmitter and receiver, particularly in the realistic case when antenna arrays are restricted in size, along with nonisotropic scattering around the receiver.

In contrast, we develop an upper bound on MIMO capacity based on the spatial correlation of the signals at the receiver and the statistics of the scattering environment. Our bound overcomes limitations of previous capacity calculations by showing the effects on MIMO capacity of antenna array geometry for any common scattering environment (e.g. von-Mises, gaussian, truncated gaussian, uniform, piecewise constant, polynomial, Laplacian, Fourier series expansion) around the receiver.

III. Upper Bound on Ergodic Capacity

We follow the approach first presented in [3] and more recently in [4], where Jensen's inequality is used to obtain an upper bound on $E\{C_H\}$, that is, since $\log|\cdot|$ is a concave function we have $E\{\log|\cdot|\} \leq \log|E\{\cdot\}|$, hence the ergodic capacity (3) is upper bounded as

$$C \leq \log\left|I_Q + \frac{1}{\sigma^2}E\left\{HV_xH^\dagger\right\}\right|. \qquad (4)$$

Let $r = [r_1, r_2, \ldots, r_Q]^T$ be the vector of noiseless signals received by the Q receiver antennas, i.e., $r = Hx$, then assuming independent transmitted symbols, we can write

$$V_r \triangleq E\{rr^\dagger\} \equiv E\left\{HV_xH^\dagger\right\}. \qquad (5)$$

Substitution of the received symbols covariance matrix (5) into the capacity bound (4) gives

$$C \leq \log\left|I_Q + \frac{1}{\sigma^2}V_r\right|. \qquad (6)$$

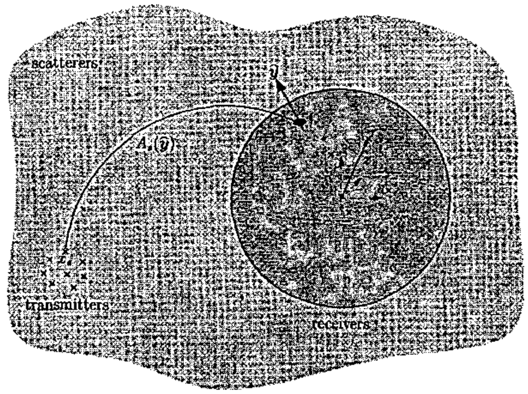

Figure 1: Proposed scattering model for a flat fading MIMO system. $A_s(\widehat{y})$ represents the total complex gain of the scatters for signal x_s arriving at the receiver array from direction $\widehat{y}$ via any number of paths through the scattering environment. The sphere surrounding the receive antennas contains no scatters and is assumed large enough that any scatters on its surface are considered farfield to the receiver sensors.

Therefore we see the covariance between signals at the receiver determines the upper bound on MIMO capacity. In the follow section we show this bound can be calculated for any scattering environment and is dependent only on the power distribution of the signals surrounding the receiver array and the array geometry.

IV. Receiver Spatial Correlation for General Distributions of Farfield Scatters

Consider the narrowband transmission of S independent symbols x through a general scattering environment with scatters assumed distributed in the farfield from the receiver sensors, as shown in Fig. 1, then the received signal at the pth sensor is given by

$$r_p = \sum_{s=1}^{S} x_s \int_\Omega A_s(\widehat{y}) e^{-ikz_p \cdot \widehat{y}} ds(\widehat{y}) \qquad (7)$$

where z_p is the location of the pth receiver sensor, $k = 2\pi/\lambda$ is the wavenumber with λ the wavelength, $\widehat{y}$ is a unit vector pointing in the direction of wave propagation, and $A_s(\widehat{y})$ is the total complex gain of the scatters for signal x_s arriving at the receiver array from direction $\widehat{y}$. The integration is over the unit sphere for a 3-dimensional multipath environment or the unit circle in the 2-dimensional case.

Define the normalized spatial correlation between the complex envelopes of the received signal at two sensors p and q as

$$\rho_{pq} \triangleq \frac{E\{r_p r_q^*\}}{E\{r_p r_p^*\}}. \tag{8}$$

From (7) we get the covariance between signals at sensors p and q as

$$\begin{aligned} E\left\{r_p r_q^*\right\} &= E\left\{\left[\sum_{s=1}^{S} x_s \int_\Omega A_s(\widehat{y}) e^{-ikz_p\cdot\widehat{y}} ds(\widehat{y})\right] \times \right. \\ &\left. \left[\sum_{s'=1}^{S} x_{s'}^* \int_\Omega A_{s'}^*(\widehat{y}') e^{ikz_q\cdot\widehat{y}'} ds(\widehat{y}')\right]\right\} \\ &= \iint_\Omega \sum_{s=1}^{S} \sigma_s^2 E\left\{A_s(\widehat{y}) A_s^*(\widehat{y}')\right\} \times \\ &\quad e^{-ik(z_p\cdot\widehat{y} - z_q\cdot\widehat{y}')} ds(\widehat{y}) ds(\widehat{y}') \end{aligned} \tag{9}$$

where $\sigma_s^2 = E\left\{x_s x_s^*\right\}$ is the transmitted power of antenna s, and we have assumed the transmitted symbols are independent across antennas and independent of the scattering environment. Assuming that the complex scattering gains from one direction is independent from another, i.e.,

$$E\left\{A_s(\widehat{y}) A_s^*(\widehat{y}')\right\} = \begin{cases} 0 & \text{if } \widehat{y} \neq \widehat{y}' \\ E\left\{|A_s(\widehat{y})|^2\right\} & \text{if } \widehat{y} = \widehat{y}' \end{cases} \tag{10}$$

then,

$$\begin{aligned} E\left\{r_p r_q^*\right\} = \int_\Omega \sum_{s=1}^{S} \sigma_s^2 E\left\{|A_s(\widehat{y})|^2\right\} \times \\ e^{-ik(z_p - z_q)\cdot\widehat{y}} ds(\widehat{y}) \end{aligned} \tag{11}$$

Hence the correlation between two receiver sensor positions is given by

$$\rho_{pq} = \frac{\int_\Omega \sum_{s=1}^{S} \sigma_s^2 E\left\{|A_s(\widehat{y})|^2\right\} e^{-ik(z_p - z_q)\cdot\widehat{y}} ds(\widehat{y})}{\int_\Omega \sum_{s=1}^{S} \sigma_s^2 E\left\{|A_s(\widehat{y})|^2\right\} ds(\widehat{y})} \tag{12}$$

Define

$$\mathcal{P}(\widehat{y}) \triangleq \frac{\sum_{s=1}^{S} \sigma_s^2 E\left\{|A_s(\widehat{y})|^2\right\}}{\int_\Omega \sum_{s=1}^{S} \sigma_s^2 E\left\{|A_s(\widehat{y})|^2\right\} ds(\widehat{y})} \tag{13}$$

as the normalized average power density distribution of signals from direction $\widehat{y}$, then the correlation between the received signals at two sensors p and q is given by

$$\rho_{pq} = \int_\Omega \mathcal{P}(\widehat{y}) e^{-ik(z_p - z_q)\cdot\widehat{y}} ds(\widehat{y}). \tag{14}$$

Here we see that the spatial correlation between two sensors at the receiver is dependent on the average power density distribution of surrounding scattering environment due to the transmitted symbols and the array geometry.

Similarly we can define

$$\mathcal{P}_s(\widehat{y}) \triangleq \frac{\sigma_s^2 E\left\{|A_s(\widehat{y})|^2\right\}}{\int_\Omega \sigma_s^2 E\left\{|A_s(\widehat{y})|^2\right\} ds(\widehat{y})} \tag{15}$$

as the normalized received average power density distribution over direction $\widehat{y}$ for signals only from transmitter s, then

$$\mathcal{P}(\widehat{y}) = \sum_{s=1}^{S} \frac{\sigma_{sr}^2 \mathcal{P}_s(\widehat{y})}{\sigma_r^2} \tag{16}$$

where

$$\sigma_r^2 = \int_\Omega \sum_{s=1}^{S} \sigma_s^2 E\left\{|A_s(\widehat{y})|^2\right\} ds(\widehat{y}) \tag{17}$$

is the total average power received at a sensor, and

$$\sigma_{sr}^2 = \int_\Omega \sigma_s^2 E\left\{|A_s(\widehat{y})|^2\right\} ds(\widehat{y}) \tag{18}$$

is the average power received at a sensor from transmitter s. Therefore the average power distribution $\mathcal{P}(\widehat{y})$ contains all information regarding the transmitter and surrounding scattering environment.

V. Capacity of a MIMO system with random scattering environment

From (5) and (8) we can then write

$$\frac{1}{\sigma^2} \boldsymbol{V}_r = \eta \boldsymbol{\Gamma}_Q \tag{19}$$

where $\eta = \sigma_r^2/\sigma^2$ is the average received signal-to-noise ratio (SNR) at each receiver branch, and $\boldsymbol{\Gamma}_Q$ is the $Q \times Q$ received signal spatial correlation matrix

$$\boldsymbol{\Gamma}_Q \triangleq \begin{bmatrix} \rho_{11} & \cdots & \rho_{1Q} \\ \vdots & \ddots & \vdots \\ \rho_{Q1} & \cdots & \rho_{QQ} \end{bmatrix} \tag{20}$$

where each ρ_{pq} depends on antenna placement and the power distribution of the scattering environment (incorporating any transmitter information) given by (14). The bound on capacity in (4) can now be expressed as

$$C \leq \log|\boldsymbol{I}_Q + \eta\boldsymbol{\Gamma}_Q| \tag{21}$$

which is the upper bound on capacity for the MIMO system given the scattering environment and transmitter configuration, described by the average power density distribution $\mathcal{P}(\widehat{y})$, and the receive antenna placement. Equality in (21) is achieved when the transmitter antenna elements are far apart from each other and the number of transmitter antennas is large ($S \rightarrow \infty$) [8]. Therefore, in the case of a large number of uncorrelated transmitters the left hand side of (21) gives the capacity achievable given the scattering environment power distribution $\mathcal{P}(\widehat{y})$ and the number of receiver antennas and their configuration.

VI. Capacity Growth Limits

For a large number of uncorrelated transmitters, the capacity given by (21) is maximized when there is no correlation between the receive antennas, i.e., $\mathbf{\Gamma}_Q = \boldsymbol{I}_Q$, giving,

$$C_{\max} = Q \log(1 + \eta). \tag{22}$$

Therefore, in the idealistic situation of zero correlation between both transmitter antennas and receiver antennas we see the best capacity growth achievable is linear in the number of antennas at the receiver. This result agrees with the traditional capacity formulation [1, 2] which is widely used to advocate the use of MIMO systems.

Conversely, when there is perfect correlation between each pair of antenna elements the correlation matrix becomes the $Q \times Q$ matrix of ones, $\mathbf{\Gamma}_Q = \mathbf{1}_Q$, and the capacity of the MIMO system will be minimized,

$$C_{\min} = \log(1 + Q\eta). \tag{23}$$

Here the logarithmic capacity growth is simply an effective increase in the average SNR of the single antenna receiver case, due to the assumption of independent noise at each antenna, and is often referred to as an antenna gain effect.

VII. Two Dimensional Scattering Environment

In the 2D scattering environment the signals arrive only from the azimuthal plane and we can use the two dimensional modal expansion of planes waves [9]

$$e^{ik\boldsymbol{z}\cdot\widehat{\boldsymbol{y}}} = \sum_{n=-\infty}^{\infty} i^n J_n(k\|\boldsymbol{z}\|) e^{in(\phi_z - \phi_y)} \tag{24}$$

where ϕ_z and ϕ_y are the angles to $\boldsymbol{z}$ and $\widehat{\boldsymbol{y}}$, respectively. Substitution of (24) into (14) gives the spatial correlation for a 2D environment as

$$\rho_{pq} = \sum_{n=-\infty}^{\infty} i^n \beta_n J_n(k\|\boldsymbol{z}_p - \boldsymbol{z}_q\|) e^{in\phi_{pq}} \tag{25}$$

where ϕ_{pq} is the angle of the vector connecting $\boldsymbol{z}_p$ and $\boldsymbol{z}_q$. The coefficients β_n characterize any possible scattering environment and are given by

$$\beta_n = \int_0^{2\pi} \mathcal{P}(\varphi) e^{-in\varphi} \mathrm{d}\varphi \tag{26}$$

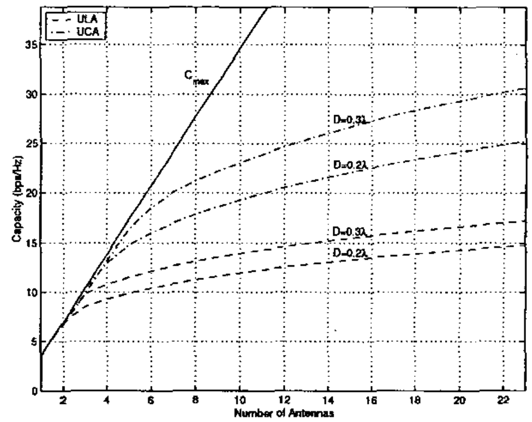

Figure 2: Theoretical upper bounds on capacity of MIMO systems for various receiver antenna numbers of a uniform linear (ULA) and uniform circular array (UCA) with aperture width (length/diameter) D=0.2, and 0.3 wavelengths in a 2D isotropic diffuse scattering field, along with the theoretical maximum capacity.

with $\mathcal{P}(\varphi)$ the average power density distribution over φ the angle to the scatters. For essentially all common choices of $P(\varphi)$: von-Mises, gaussian, truncated gaussian, uniform, piecewise constant, polynomial, Laplacian, Fourier series expansion, etc., there is a closed form expression for the β_n [10]. Therefore we have a closed form representation for the spatial correlation (25) and hence for the capacity bound of the system (21).

Example: Two Dimensional Omni-directional Diffuse Field

For the special case of scattering over all angles in the plane the received normalized average power reduces to $\mathcal{P}(\varphi) = 1/2\pi, \varphi \in [0, 2\pi)$, giving the spatial correlation between any two points in the plane as

$$\rho_{pq} = J_0(k\|\boldsymbol{z}_p - \boldsymbol{z}_q\|) \tag{27}$$

which is identical to the classical result in [11]. The capacity for the uniform linear and circular arrays of increasing receiver antenna numbers is shown in Fig. 2 for a two dimensional omni-directional diffuse field.

VIII. Three Dimensional Scattering Environment

In the case of a 3D scattering environment a similar derivation as above can be used, where instead of the 2D modal expansion a spherical harmonic expansion of the plane waves is used [9],

$$e^{ikz\cdot\hat{y}} = 4\pi \sum_{n=0}^{\infty} (-i)^n j_n(k\|z\|) \sum_{m=-n}^{n} Y_{nm}(\hat{z}) Y_{nm}^*(\hat{y}) \tag{28}$$

where $\hat{z} = z/\|z\|$, $j_n(\cdot)$ are the spherical Bessel functions, and

$$Y_{nm}(\hat{z}) \equiv Y_{nm}(\theta_z, \phi_z) \tag{29}$$

$$\triangleq \sqrt{\frac{2n+1}{4\pi} \frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta_z) e^{im\phi_z} \tag{30}$$

where θ_z and ϕ_z are the elevation and azimuth of the unit vector $\hat{z}$, respectively, and $P_n^m(\cdot)$ are the associated Legendre functions of the first kind. Substitution of (28) into (14) gives the spatial correlation for a 3D scattering environment as

$$\rho_{pq} = 4\pi \sum_{n=0}^{\infty} (-i)^n j_n(k\|z_p - z_q\|) \times \sum_{m=-n}^{n} \beta_{nm} Y_{nm}\left(\frac{z_p - z_q}{\|z_p - z_q\|}\right), \tag{31}$$

where coefficients β_{nm} describe the scattering environment,

$$\beta_{nm} = \int_{\Omega} \mathcal{P}(\hat{y}) Y_{nm}^*(\hat{y}) ds(\hat{y}). \tag{32}$$

As for the two dimensional scattering environment, for many common choices of $\mathcal{P}(\hat{y})$ there exists a closed form expression for the β_{nm} [10] and hence the capacity bound (21) can be computed for these scattering environments.

Example: Three Dimensional Omni-directional Diffuse Field

For the special case of scattering over all directions the β_{nm} are zero for all but $\beta_{00} = 1/4\pi$, hence the spatial correlation (31) becomes

$$\rho_{pq} = j_0(k\|z_p - z_q\|) = \mathrm{sinc}(k\|z_p - z_q\|) \tag{33}$$

which is identical to the classical result in [12] for a 3 dimensional scattering environment. The capacity for the uniform linear, circular, and spherical arrays of increasing receiver antenna numbers is shown in Fig. 2 for a three dimensional omni-directional diffuse field.

IX. Conclusions

We have derived a new upper bound on capacity for MIMO wireless fading channels by separating the effect of the scattering environment and antenna array configuration. Based on the spatial correlation of the signals at the receiver and the statistics of the scattering environment, our bound overcomes limitations of previous capacity calculations by showing the effects on MIMO capacity of antenna array geometry for any common scattering environment (von-Mises, gaussian, truncated gaussian, uniform, piecewise constant, polynomial, Laplacian, Fourier series expansion, etc.) around the receiver. This new bound offers significant insights into the factors determining capacity, in particular, those of the antenna array geometry and number of antennas, which are becoming increasingly more important for realistic MIMO system designs.

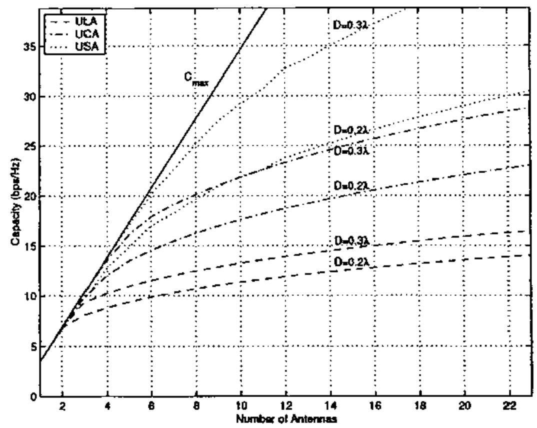

Figure 3: Theoretical upper bounds on capacity of MIMO systems for various receiver antenna numbers of a uniform linear (ULA), uniform circular (UCA), and uniform spherical array (USA) with aperture width (length/diameter) D=0.2, and 0.3 wavelengths in a 3D isotropic diffuse scattering field, along with the theoretical maximum capacity.

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