Modeling Multipath Scattering Environments Using Generalized Herglotz Wave Functions

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***Abstract*—We develop a general mathematical model for nearfield multipath scattering as a basis for studying the spatial limits imposed on multi-antenna wireless communication systems. This model generalizes the Herglotz Wave Function, which is an important tool in the study of inverse scattering problems, to a form where the scatterers can be nearfield. This permits the development of the most general form of spatial correlation which is known to be the principle factor in the determination of the capacity of wireless systems.**

***Index Terms*—Multipath channel modeling, Wireless channels, Herglotz wave functions, Nearfield sources.**

I. Introduction

Multipath modeling in wireless and mobile communication tends to be very naive and simplistic. Often multipath is represented as a superposition of a low number of paths usually with quite artificial scattering geometries such as having scatterers uniformly spaced on a circle. At the other extreme with a diffuse model there is often an incorrect use of identities due to confusion between two dimensional and three dimensional diffuse field identities. It is a fallacy to assert that any such multipath model simplifies the analysis of wireless systems or can reliably make predictions for realistic situations. One is confronted with either having a model that is far too restrictive and incapable of providing a compete description of the achievable physical range of channels, or having an unnecessarily complicated model which is neither complete nor parsimonious (that is, it overparametrizes a strict subset of physical channels).

In this paper we develop a general complete representation for multipath which suffers none of the above problems because it deals directly with the physics of the multipath problem. In our framework it is not necessary to distinguish between nearfield, farfield, specular or diffuse as all cases are handled transparently provided the region occupied by our sensors is physically separated from the sources. Further, the expressions we obtain lead naturally to parsimonious representations for multipath although this is not explored in this paper.

A standard model for a multipath field in $\mathbb{R}^3$ is to represent it as a superposition of plane waves from discrete directions:

$$u(\boldsymbol{x}) = \sum_p a_p \, e^{ik\boldsymbol{x}\cdot\widehat{\boldsymbol{y}}_p}, \tag{1}$$

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where the plane wave of index p has complex amplitude $a_p \in \mathbb{C}$, the propagation direction is denoted by the unit vector $\widehat{\boldsymbol{y}}_p$ and $\boldsymbol{x}\cdot\boldsymbol{y}$ denotes the scalar product between vectors $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^3$. A straightforward generalization of (1) is

$$u(\boldsymbol{x}) = \int_{\mathbb{S}^2} g(\widehat{\boldsymbol{y}}) \, e^{ik\boldsymbol{x}\cdot\widehat{\boldsymbol{y}}} \, ds(\widehat{\boldsymbol{y}}), \tag{2}$$

where $\mathbb{S}^2$ denotes the unit sphere, $s(\widehat{\boldsymbol{y}})$ is a surface element of $\mathbb{S}^2$ with unit normal $\widehat{\boldsymbol{y}}$ and $g \in L^1(\mathbb{S}^2)$ is the kernel representing an angular amplitude distribution of farfield sources. Representation (2) implies any sources which contribute to the field are farfield ones.

When g is in $L^2(\mathbb{S}^2)$, a stronger condition than $g \in L^1(\mathbb{S}^2)$, it is known as a *Herglotz Kernel*, and representation (2) is known as the *Herglotz Wave Function* [1, p.55]. Herglotz wave functions primarily find use in inverse scattering problems where it is natural to find a scattered field satisfying the condition $g \in L^2(\mathbb{S}^2)$.

In our context, we are interested in using representations of the form (2), or generalizations thereof, to model any physically realizable scattering environment. If we use (2) then we exclude fields which have components from nearfield sources, and if, further, $g \in L^2(\mathbb{S}^2)$ then we exclude some farfield sources as well including those of the form (1). Representation (2) has been used to model spatial correlation in wireless communication scattering scenarios [2]. However, there is a need for even more general representations than (2) such as in the case when the sources and scatterers are in the nearfield. This paper develops such a representation.

The logical arena for the Herglotz Kernel is the Hilbert Space $L^2(\mathbb{S}^2)$ with the natural inner product defined on $\mathbb{S}^2$. However, important classes of multipath fields cannot be directly associated with kernels belonging to such spaces. Our objective is to rework (2) to find a integral representation for multipath fields where the kernel associated with practically important scattering fields can be associated with a Hilbert Space corresponding to finite energy signals. Then the full machinery of Hilbert Space theory can be brought to bear on the representation to transparently render its properties, which to a large degree fully emulate the remarkable properties of the classical Herglotz wave function. We will see that this leads to an infinite number of orthonormal representations for multipath fields and associated L^2 Fourier Series. As a by-product our theory subsumes and provides a modest simplification to the results given in [1].

II. Problem Formulation

A. Subspace Interpretation

Fields of the form (1), (2) and Herglotz wave functions satisfy the homogeneous Helmholtz equation in $\mathbb{R}^3$, sometimes referred to as the reduced wave equation:

$$\triangle u(\boldsymbol{x}) + k^2 u(\boldsymbol{x}) = 0, \tag{3}$$

where $\triangle$ is the Laplacian, and k is the wave number given by the real positive constant $k = 2\pi/\lambda$ [1,3]. Equation (3) holds in any region of space, a subset of $\mathbb{R}^3$, that excludes any sources. That all such solutions to (3) for a given source-free region define a linear subspace of functions follows from the linearity and homogeneity of (3). That is, if $u_1(\boldsymbol{x})$ and $u_2(\boldsymbol{x})$ are solutions to (3) in a region then $\alpha_1 u_1(\boldsymbol{x}) + \alpha_2 u_2(\boldsymbol{x})$ is also a solution in the same region.

B. Helmholtz Balls

The kernel in (2) is defined on $\mathbb{S}^2$ and implicitly the sources may be regarded as being defined on an infinite sphere. Many of the special properties that can be attributed to the Herglotz wave function are actually a manifestation of the high degree of spherical symmetry and the implicit choice of spherically symmetric domains (albeit infinite domains). Hence in studying (3) we expect highly structured solutions whenever the region of interest is a ball

$$\mathbb{B}^3_R \triangleq \bigl\{\boldsymbol{x} \in \mathbb{R}^3 \colon \|\boldsymbol{x}\| \leq R\bigr\}, \tag{4}$$

where R is the radius (usually finite but possibly infinite). We write the above problem more compactly as follows.

Spatial Concentration Problem 5. *Determine the complete subspace of solutions $u \equiv u(\boldsymbol{x})$ to*

$$\triangle u + k^2 u = 0, \ \textit{in}\ \mathbb{B}^3_R. \tag{5}$$

Physically this means we wish to understand the complete set of valid wave-fields for arbitrary source locations with the only condition that all the sources are not in $\mathbb{B}^3_R$.

Our task will be to find a Hilbert Space formulation where we can study the geometry of solutions to (5) using complete orthonormal sequences and the like. In what follows we assume the reader is well-familiar with separable Hilbert Spaces [4,5].

Given we are dealing with spherical regions, we often utilize a spherical coordinate system (r, θ, ϕ) representing radius, co-latitude and longitude. In a coordinate system independent form we have $\boldsymbol{x} \equiv (r, \theta, \phi)$, $\|\boldsymbol{x}\|$ representing radius r, and $\widehat{\boldsymbol{x}}$ representing the direction θ and ϕ .

C. Modal Representation

From [1], solutions to (5) can be represented as the following entire series expansion (which converges in the mean)

$$u(\boldsymbol{x}) = \sum_{m,n} \bigl(4\pi i^n \alpha_n^m\bigr)\, j_n\bigl(k\|\boldsymbol{x}\|\bigr) Y_n^m(\widehat{\boldsymbol{x}}), \tag{6}$$

where $j_n\bigl(k\|\boldsymbol{x}\|\bigr)$ is the spherical Bessel function of integer order n, α_n^m are complex coefficients, and $Y_n^m(\widehat{\boldsymbol{x}}) \equiv Y_n^m(\theta, \phi)$ are the spherical harmonics (orthonormal on $\mathbb{S}^2$) given by

$$Y_n^m(\theta, \phi) \triangleq \sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}}\, P_n^{|m|}(\cos\theta)\, e^{im\phi}, \tag{7}$$

$P_n^{|m|}(\cdot)$ are the associated Legendre functions, and we have introduced the shorthand $\sum_{m,n} \triangleq \sum_{n=0}^{\infty}\sum_{m=-n}^{n}$ [1,6].

Example 3 (Point Sources). *The fundamental solution to the Helmholtz equation, which is that of a point source at $\boldsymbol{y}$, is given by*

$$\Phi(\boldsymbol{x}, \boldsymbol{y}) \triangleq \frac{ik}{4\pi} h_0^{(1)}\bigl(k\|\boldsymbol{x} - \boldsymbol{y}\|\bigr) \equiv \frac{e^{ik\|\boldsymbol{x}-\boldsymbol{y}\|}}{4\pi\|\boldsymbol{x} - \boldsymbol{y}\|} \tag{8}$$

where $h_n^{(1)}(\cdot)$ is the order n spherical hankel function [3,7]. In the spherical coordinate system the fundamental solution has an expansion, called the addition theorem for the fundamental solution,

$$\Phi(\boldsymbol{x}, \boldsymbol{y}) = ik \sum_{m,n} j_n\bigl(k\|\boldsymbol{x}\|\bigr) h_n^{(1)}\bigl(k\|\boldsymbol{y}\|\bigr) Y_n^m(\widehat{\boldsymbol{x}})\, \overline{Y_n^m(\widehat{\boldsymbol{y}})}, \tag{9}$$

valid for $\|\boldsymbol{x}\| < \|\boldsymbol{y}\|$, from which we glean

$$\alpha_n^m = \frac{(-i)^{n-1} k}{4\pi} h_n^{(1)}\bigl(k\|\boldsymbol{y}\|\bigr) \overline{Y_n^m(\widehat{\boldsymbol{y}})}. \tag{10}$$

are the coefficients in expansion (6).

We can interpret the countable set

$$\bigl\{A_n^m(\boldsymbol{x})\bigr\} \triangleq \bigl\{i^n j_n\bigl(k\|\boldsymbol{x}\|\bigr) Y_n^m(\widehat{\boldsymbol{x}})\bigr\} \tag{11}$$

as specifying a basis for the subspace of solutions to (5). These basis functions are orthogonal over any spherically symmetric region. The sense in which these can be normalized for any size ball is now determined.

Denote the volume element at $\boldsymbol{x}$ as

$$dv(\boldsymbol{x}) \triangleq r^2 \sin\theta\, d\phi\, d\theta\, dr, \tag{12}$$

and define the inner product[1] and associated induced norm by

$$\langle f, g\rangle_{\mathbb{B}^3_R} \triangleq \int_{\mathbb{B}^3_R} f(\boldsymbol{x})\overline{g(\boldsymbol{x})} h_R\bigl(\|\boldsymbol{x}\|\bigr)\, dv(\boldsymbol{x}), \tag{13a}$$

$$\equiv \int_0^R h_R(r) r^2 \int_{\mathbb{S}^2} f(r, \widehat{\boldsymbol{x}})\, \overline{g(r, \widehat{\boldsymbol{x}})}\, ds(\widehat{\boldsymbol{x}})\, dr, \tag{13b}$$

and

$$\bigl\|f\bigr\|^2_{\mathbb{B}^3_R} \triangleq \bigl\langle f, f\bigr\rangle_{\mathbb{B}^3_R} \equiv \int_{\mathbb{B}^3_R} \bigl|f(\boldsymbol{x})\bigr|^2 h_R\bigl(\|\boldsymbol{x}\|\bigr)\, dv(\boldsymbol{x}), \tag{14}$$

parametrized by a non-negative bounded real weighting function $h_R(r) \geq 0$ that may depend on either R or r.

[1] This is actually a class of inner products given that the radial weighting term $h_R(r)$ can take different forms. We are assuming that $h_R(r)$ is chosen such that we do have an inner product, particularly we require that $\langle f, f\rangle_{\mathbb{B}^3_R} = 0$ implies $f = 0$ which is not obviously satisfied. To avoid clutter, for the inner product we suppress in the notation the explicit dependence on $h_R(r)$.

Then, with a suitable choice of weight function $h_R(r)$, the set of functions satisfying $\|f\|^2_{\mathbb{B}^3_R} < \infty$ is a separable Hilbert Space, and solutions of (5) are a strict subspace. This subspace naturally forms a Hilbert Space.

The $\{A_n^m(\boldsymbol{x})\}$ are orthogonal with respect to the inner product (13). This follows from the orthonormality of the spherical harmonics defined on the unit sphere, $\mathbb{S}^2$, which induces orthogonality of our wave expansion over $\mathbb{B}^3_R$ leading to

$$\langle A_n^m, A_q^p\rangle_{\mathbb{B}^3_R} = \delta_{nq}\,\delta_{mp}\int_0^R h_R(r)\,[j_n(kr)]^2\,r^2\,dr \tag{15}$$

where δ_{nq} and δ_{mp} are delta functions. Equation (15) shows orthogonality holds independently of the choice of the non-negative bounded real function $h_R(r) \geq 0$. However, $h_R(r)$ does influence the normalization, as we explore next.

D. Complete Orthonormal Sequences

Orthonormality is subtly connected with the size of the spherical region. Beginning with the unnormalized set (11), equation (15) indicates how to achieve orthonormality, givening the orthonormal basis functions,

$$\{\varphi^m_{n;R}\}_{m,n} \triangleq \left\{\frac{i^n j_n(k\|\boldsymbol{x}\|)Y_n^m(\widehat{\boldsymbol{x}})}{\left(\int_0^R h_R(r)\,[j_n(kr)]^2\,r^2\,dr\right)^{1/2}}\right\}_{m,n} \tag{16}$$

note that normalizing factor is a function of both n and R. Whence, given completeness, any solution $u(\boldsymbol{x})$ to our problem (5) has representation (in the sense of convergence in the mean of the induced norm)

$$u = \sum_{m,n}\beta^m_{n;R}\,\varphi^m_{n;R}, \tag{17}$$

with Fourier coefficients

$$\beta^m_{n;R} \triangleq \langle u, \varphi^m_{n;R}\rangle_{\mathbb{B}^3_R} \tag{18a}$$

$$= \frac{\int_{\mathbb{B}^3_R} u(\boldsymbol{x})(-i)^n j_n(k\|\boldsymbol{x}\|)\overline{Y_n^m(\widehat{\boldsymbol{x}})}h_R(r)\,dv(\boldsymbol{x})}{\left(\int_0^R h_R(r)\,[j_n(kr)]^2\,r^2\,dr\right)^{1/2}} \tag{18b}$$

E. Field Representations

Direct comparison of (17) and (18) with (6) leads to

$$\alpha_n^m = \frac{\langle u(\boldsymbol{x}), \varphi^m_{n;R}(\boldsymbol{x})\rangle_{\mathbb{B}^3_R}}{4\pi\left(\int_0^R h_R(r)\,[j_n(kr)]^2\,r^2\,dr\right)^{1/2}} \tag{19a}$$

$$= \frac{\int_{\mathbb{B}^3_R} u(\boldsymbol{x})\overline{Y_n^m(\widehat{\boldsymbol{x}})}(-i)^n j_n(k\|\boldsymbol{x}\|)h_R(\|\boldsymbol{x}\|)\,dv(\boldsymbol{x})}{\int_{\mathbb{B}^3_R}[j_n(k\|\boldsymbol{x}\|)]^2 h_R(\|\boldsymbol{x}\|)\,dv(\boldsymbol{x})}. \tag{19b}$$

Hence there are a plethora of ways to compute these coefficients based on different values of R and different choices of $h_R(r)$. Here are some of the more interesting choices for $h_R(r)$ and the resulting expressions for α_n^m (19):

Example 4. *If $h_R(\|\boldsymbol{x}\|) = \delta(\|\boldsymbol{x}\| - r_0)$ where $0 < r_0 \leq R$, and $j_n(kr_0) \neq 0$ then*

$$\alpha_n^m = \frac{1}{4\pi i^n j_n(kr_0)}\int_{\mathbb{S}^2} u_{r_0}(\widehat{\boldsymbol{x}})\overline{Y_n^m(\widehat{\boldsymbol{x}})}\,ds(\widehat{\boldsymbol{x}}), \tag{20}$$

where $u_{r_0}(\widehat{\boldsymbol{x}})$ is $u(\boldsymbol{x})$ restricted to the shell $\|\boldsymbol{x}\| = r_0$. That is, provided k is not a Dirichlet eigenvalue we can use the Spherical Harmonic Transform to determine α_n^m [1,8,9].

Example 5. *If $h_R(\|\boldsymbol{x}\|) = 1$ then we have the most natural case for a finite sphere ($R < \infty$)*

$$\alpha_n^m = \frac{\int_{\mathbb{B}^3_R} u(\boldsymbol{x})(-i)^n j_n(k\|\boldsymbol{x}\|)\overline{Y_n^m(\widehat{\boldsymbol{x}})}\,dv(\boldsymbol{x})}{\int_{\mathbb{B}^3_R}[j_n(k\|\boldsymbol{x}\|)]^2\,dv(\boldsymbol{x})} \tag{21a}$$

$$= \frac{\int_{\mathbb{B}^3_R} u(\boldsymbol{x})(-i)^n j_n(k\|\boldsymbol{x}\|)\overline{Y_n^m(\widehat{\boldsymbol{x}})}\,dv(\boldsymbol{x})}{4\pi\int_0^R[j_n(kr)]^2\,r^2\,dr} \tag{21b}$$

The denominator in (21) *will be written*

$$\mathcal{J}_n(R) \triangleq \int_0^R [j_n(kr)]^2\,r^2\,dr \tag{22}$$

for which there are known closed form expressions.

Example 6. *If $h_R(\|\boldsymbol{x}\|) = 1/R$, and let $R \to \infty$ then we have the most natural case for the infinite sphere. In this case we use the expression*

$$\lim_{R\to\infty}\frac{1}{R}\int_0^R r^2[j_n(kr)]^2\,dr = \frac{1}{2k^2} \tag{23}$$

to simplify denominator of (19) *to glean*

$$\alpha_n^m = \lim_{R\to\infty}\frac{(-i)^n k^2}{2\pi R}\int_{\mathbb{B}^3_R} u(\boldsymbol{x})j_n(k\|\boldsymbol{x}\|)\overline{Y_n^m(\widehat{\boldsymbol{x}})}\,dv(\boldsymbol{x}). \tag{24}$$

III. Orthonormal Expansions in Balls

A. Finite Sphere Case

We now focus on the natural inner product, where $h_R(\|\boldsymbol{x}\|) = 1$. We show that there is a more general representation than (6) when dealing with spherical regions $\mathbb{B}^3_R$ of radius R.

Theorem 1 (Expansion for Finite Source-Free Ball). *Consider the space of finite energy solutions to the homogeneous Helmholtz equation $\triangle u + k^2 u = 0$ in a spherical domain $\mathbb{B}^3_R$ of radius $R < \infty$. Then any bounded solution u can be expressed in terms of an expansion*

$$u(\boldsymbol{x}) = \sum_{m,n}\beta^m_{n;R}\,\frac{i^n j_n(k\|\boldsymbol{x}\|)Y_n^m(\widehat{\boldsymbol{x}})}{[\mathcal{J}_n(R)]^{1/2}} \tag{25}$$

such that

$$\left\{\varphi^m_{n;R}(\boldsymbol{x})\right\}_{m,n} \triangleq \left\{\frac{i^n j_n(k\|\boldsymbol{x}\|)Y_n^m(\widehat{\boldsymbol{x}})}{[\mathcal{J}_n(R)]^{1/2}}\right\}_{m,n} \tag{26}$$

are orthonormal with respect to the inner product

$$\langle f, g\rangle_{\mathbb{B}^3_R} \triangleq \int_{\mathbb{B}^3_R} f(\boldsymbol{x})\overline{g(\boldsymbol{x})}\,dv(\boldsymbol{x}). \tag{27}$$

The Fourier Coefficients β are given by

$$\beta^m_{n;R} = \langle u, \varphi^m_{n;R}\rangle_{\mathbb{B}^3_R} \tag{28a}$$

$$= \int_{\mathbb{B}^3_R} u(\boldsymbol{x})\,\frac{(-i)^n j_n(k\|\boldsymbol{x}\|)\overline{Y_n^m(\widehat{\boldsymbol{x}})}}{[\mathcal{J}_n(R)]^{1/2}}\,dv(\boldsymbol{x}). \tag{28b}$$

and are square summable, that is, $\beta \in \ell^2$.

Proof. The orthonormality of (26) is a special case of (16) with $h_R(\|\boldsymbol{x}\|) = 1$. Using the Parseval Relation we have

$$\sum_{m,n} \left|\beta^m_{n;R}\right|^2 = \langle u, u\rangle_{\mathbb{B}^3_R} = \int_{\mathbb{B}^3_R} \left|u(\boldsymbol{x})\right|^2 dv(\boldsymbol{x}) < \infty \tag{29}$$

by the finite energy of u, that is, $\|\beta_R\|^2_{\ell^2} = \|u\|^2_{\mathbb{B}^3_R} < \infty$. □

B. *Infinite Sphere Case*

We now consider the case where the region is the whole space $\mathbb{R}^3$ which can be regarded as an infinite spherical volume. In this case we use $h_R(\|\boldsymbol{x}\|) = 1/R$ and let $R \to \infty$. As we will see, the results indicate that the expansion in (6) is most naturally associated with the infinite sphere.

Theorem 2 (Expansion for Infinite Source-Free Ball). *Let u be any bounded solution to the homogeneous Helmholtz equation $\triangle u + k^2 u = 0$ in $\mathbb{R}^3$. Then u can be expressed in terms of an expansion*

$$u(\boldsymbol{x}) = \sum_{m,n} \beta^m_{n;\infty}\, k\sqrt{2}\, i^n j_n\big(k\|\boldsymbol{x}\|\big) Y^m_n(\widehat{\boldsymbol{x}}) \tag{30}$$

such that

$$\left\{\varphi^m_{n;\infty}\right\}_{m,n} \triangleq \left\{k\sqrt{2}\, i^n j_n\big(k\|\boldsymbol{x}\|\big) Y^m_n(\widehat{\boldsymbol{x}})\right\}_{m,n} \tag{31}$$

are orthonormal with respect to the natural inner product

$$\langle f, g\rangle_{\mathbb{B}^3_\infty} \triangleq \lim_{R\to\infty} \frac{1}{R} \int_{\mathbb{B}^3_R} f(\boldsymbol{x})\overline{g(\boldsymbol{x})}\, dv(\boldsymbol{x}), \tag{32}$$

and the Fourier coefficients are given by

$$\beta^m_{n;\infty} = \big\langle u(\boldsymbol{x}), \varphi^m_{n;\infty}(\boldsymbol{x})\big\rangle_{\mathbb{B}^3_\infty} \tag{33a}$$

$$= \lim_{R\to\infty} \frac{(-i)^n k\sqrt{2}}{R} \int_{\mathbb{B}^3_R} u(\boldsymbol{x}) j_n\big(k\|\boldsymbol{x}\|\big)\overline{Y^m_n(\widehat{\boldsymbol{x}})}\, dv(\boldsymbol{x}) \tag{33b}$$

and are square summable, that is, $\beta_\infty \in \ell^2$.

Proof. The orthonormality of (31) is a special case of (16) with $h_R(\|\boldsymbol{x}\|) = 1/R$ and letting $R \to \infty$. □

Now we present a key representation result — captured in Theorem 3.22 in [1] — for a class of solutions to the homogeneous Helmholtz equation.

Theorem 3 (Classical Herglotz Wave Function). *Let u be any bounded solution to the homogeneous Helmholtz equation $\triangle u + k^2 u = 0$ in $\mathbb{R}^3$ satisfying the growth condition*

$$\lim_{R\to\infty} \frac{1}{R} \int_{\mathbb{B}^3_R} \left|u(\boldsymbol{x})\right|^2 dv(\boldsymbol{x}) < \infty. \tag{34}$$

Then we have the representation for u

$$u(\boldsymbol{x}) = \frac{k}{2\pi\sqrt{2}} \int_{\mathbb{S}^2} b(\widehat{\boldsymbol{x}}) e^{ik\boldsymbol{x}\cdot\widehat{\boldsymbol{y}}}\, ds(\widehat{\boldsymbol{y}}) \tag{35}$$

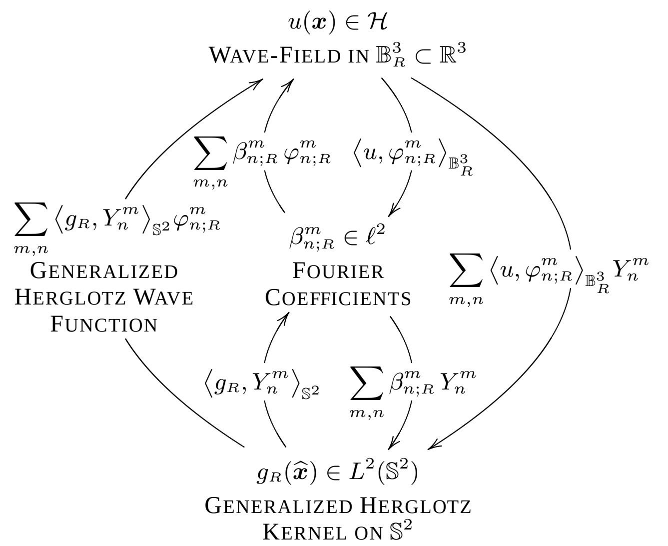

Fig. 1: Isomorphisms between the Wave Field $u(\boldsymbol{x})$ in $\mathcal{H}$, the Fourier Coefficients $\beta^m_{n;R}$ in ℓ^2 and the Herglotz Kernel $g_R(\widehat{\boldsymbol{x}})$ in $L^2(\mathbb{S}^2)$. The mapping between the Generalized Herglotz Kernel and the Wave Field, $\sum_{m,n}\langle g_R, Y^m_n\rangle_{\mathbb{S}^2}\varphi^m_{n;R}$ is the Generalized Herglotz Wave Function.

where $b(\widehat{\boldsymbol{x}})$ is, up to a constant factor, the Herglotz Kernel and can be expressed as the Inverse Spherical Harmonic Transform

$$b(\widehat{\boldsymbol{x}}) \triangleq \sum_{m,n} \beta^m_{n;\infty}\, Y^m_n(\widehat{\boldsymbol{x}}) \in L^2(\mathbb{S}^2) \tag{36}$$

of the Fourier coefficients $\beta_\infty \in \ell^2$ given in Theorem 2.

Proof. By Parseval

$$\sum_{m,n} \left|\beta^m_{n;\infty}\right|^2 = \|u\|^2_{\mathbb{B}^3_\infty} \triangleq \lim_{R\to\infty} \frac{1}{R} \int_{\mathbb{B}^3_R} \left|u(\boldsymbol{x})\right|^2 dv(\boldsymbol{x}) \tag{37}$$

which is finite by the growth condition (34). Hence $\beta_\infty \in \ell^2$ which implies that (36) is well-defined and in $L^2(\mathbb{S}^2)$. Then (36) can be inverted and this implies $\beta^m_n = \langle b, Y^m_n\rangle_{\mathbb{S}^2}$, leading to

$$u = \sum_{m,n} \beta^m_n \varphi^m_{n;\infty} = \sum_{m,n} \big\langle b, Y^m_n\big\rangle_{\mathbb{S}^2} \varphi^m_{n;\infty}, \tag{38}$$

which equals

$$u(\boldsymbol{x}) = \sum_{m,n} \Big(\int_{\mathbb{S}^2} b(\widehat{\boldsymbol{y}})\overline{Y^m_n(\widehat{\boldsymbol{y}})}\, ds(\widehat{\boldsymbol{y}})\Big) \times \sqrt{2}k\, i^n j_n\big(k\|\boldsymbol{x}\|\big) Y^m_n(\widehat{\boldsymbol{x}}) \tag{39a}$$

$$= \Big(\frac{k}{2\pi\sqrt{2}}\Big) \int_{\mathbb{S}^2} b(\widehat{\boldsymbol{y}}) \times \underbrace{\Big\{4\pi \sum_{m,n} i^n j_n\big(k\|\boldsymbol{x}\|\big) Y^m_n(\widehat{\boldsymbol{x}})\overline{Y^m_n(\widehat{\boldsymbol{y}})}\Big\}}_{e^{ik\boldsymbol{x}\cdot\widehat{\boldsymbol{y}}} \text{ by the Jacobi-Anger Expansion [1]}} ds(\widehat{\boldsymbol{y}}) \tag{39b}$$

□

COMMENT. *Comparing* (6), (24) *and* (33) *we can see that*

$$\beta^m_{n;\infty} = \Big(\frac{2\pi\sqrt{2}}{k}\Big)\, \alpha^m_n \equiv \big(\lambda\sqrt{2}\big)\, \alpha^m_n, \quad \forall m, n. \tag{40}$$

Clearly $\|\beta_\infty\|^2_{\ell^2} < \infty$ iff $\|\alpha\|^2_{\ell^2} < \infty$ where $\alpha \triangleq \{\alpha^m_n\}_{m,n}$.

C. Generalized Herglotz Wave Functions

As previously mentioned (35) is a Herglotz Wave Function. We now show how to generalize the classical Herglotz wave function to broaden its applicability and more transparently render its derivation and properties.

Let $\mathcal{H}$ be the separable Hilbert Space of solutions to the homogeneous Helmholtz equation $\triangle u + k^2 u = 0$ in $\mathbb{B}^3_R \triangleq \{\boldsymbol{x} \in \mathbb{R}^3 \colon \|\boldsymbol{x}\| \leq R\}$, with inner product

$$\langle f, g\rangle_{\mathbb{B}^3_R} \triangleq \int_{\mathbb{B}^3_R} f(\boldsymbol{x})\overline{g(\boldsymbol{x})}\, dv(\boldsymbol{x}), \quad f, g \in \mathcal{H}. \tag{41}$$

Definition 1 (Generalized Herglotz Wave Function). *Let $\varphi^m_{n;R}$ be a complete orthonormal sequence in the separable Hilbert Space of solutions to the homogeneous Helmholtz equation given by $\triangle u + k^2 u = 0$ in $\mathbb{B}^3_R \triangleq \{\boldsymbol{x} \in \mathbb{R}^3 \colon \|\boldsymbol{x}\| \leq R\}$ then*

$$v = \langle g_R, Y^m_n\rangle_{\mathbb{S}^2} \varphi^m_{n;R}. \tag{42}$$

is a Generalized Herglotz Wave Function with Generalized Herglotz Kernel $g_R \in L^2(\mathbb{S}^2)$.

That this definition does reduce to the classical Herglotz Wave Function was demonstrated in the proof of Theorem 3, particularly (38). The essence of the Generalized Herglotz Wave Function is that a field can be represented by a finite energy function defined on the unit sphere $\mathbb{S}^2$ and this is captured by (42).

Fig. 1 indicates the relationships between the various representations. It shows that the classical and generalized Herglotz Wave Functions can be viewed as a isomorphism between the space of square integrable functions defined on the unit sphere $\mathbb{S}^2$ and the space of wave-fields generated by sources no closer than distance R from the origin. In the case when we have farfield sources, $R \to \infty$, using the approach in section III-B, we obtain Fig. 2 which shows how the various general Fourier expansions (akin to Fig. 1) "collapse" to the classical Herglotz Wave Function Theory. We can infer that

$$\|g_\infty\|^2_{\mathbb{S}^2} = \|\beta^m_{n;\infty}\|^2_{\ell^2} = \|u\|^2_{\mathbb{B}^3_\infty} < \infty. \tag{43}$$

IV. Applications

A. Spatial Correlation

In [2], expansion (2), with kernel in $L^1(\mathbb{S}^2)$, was used to render in closed form an expression for the spatial correlation. This spatial correlation expression subsumed a number of other explicit models which have appeared in the literature. The kernel can be associated with a farfield angular distribution of power from the scattering environment. With the generalized Herglotz Wave Function there are three further advantages: i) nearfield sources and scatterers can now be incorporated; ii) an expression can be made over any finite ball, parametrized by R, provided it excludes all sources; and iii) the kernel can be chosen in $L^2(\mathbb{S}^2)$ from which the theory of Hilbert Spaces can be applied.

B. Channel Representation

The Generalized Herglotz Wave Function permits one to dispense with a potential complicated source and scatter geometry and replace it with a fully equivalent distribution defined on a spherical region. This decomposition of space and the model it implies will be presented in a future publication.

C. Single Layer Potentials

Finally, we can make a connection with the theory of Single Layer Potentials [1]. Consider the Single Layer Potential defined on a sphere of radius R which can be written

$$u(\boldsymbol{x}) = \int_{\mathbb{S}^2} \psi_R(\widehat{\boldsymbol{y}})\, \Phi(\boldsymbol{x}, R\widehat{\boldsymbol{y}})\, ds(\widehat{\boldsymbol{y}}), \quad \|\boldsymbol{x}\| < R \tag{44}$$

where $\psi_R(\cdot) \in L^2(\mathbb{S}^2)$ is the density defined on $\mathbb{S}^2$, and $\Phi(\cdot,\cdot)$ is the fundamental solution given in (8). Equation (44) can be interpreted as the field in the ball $\mathbb{B}^3_R$ of radius R expressed as the superposition of point sources on its boundary $\partial\mathbb{B}^3_R$ (the sphere of radius R).

Now given the addition theorem (9) we can substitute into (44) and obtain

$$u(\boldsymbol{x}) = ik \sum_{m,n} j_n\big(k\|\boldsymbol{x}\|\big) h^{(1)}_n(kR) Y^m_n(\widehat{\boldsymbol{x}}) \times \underbrace{\int_{\mathbb{S}^2} \psi_R(\widehat{\boldsymbol{y}}) \overline{Y^m_n(\widehat{\boldsymbol{y}})}\, ds(\widehat{\boldsymbol{y}})}_{\triangleq\, \gamma^m_{n;R}} \tag{45a}$$

$$= ik \sum_{m,n} \gamma^m_{n;R}\, h^{(1)}_n(kR) j_n\big(k\|\boldsymbol{x}\|\big) Y^m_n(\widehat{\boldsymbol{x}}) \tag{45b}$$

where $\gamma^m_{n;R}$ is the Spherical Harmonic Transform of density $\psi_R(\cdot)$. By comparison with the modal representation (6) we see that for all valid m and n

$$\gamma^m_{n;R} \equiv \frac{4\pi i^{n-1}}{k h^{(1)}_n(kR)} \alpha^m_n. \tag{46}$$

So the coefficients $\gamma^m_{n;R}$ have the advantage over α^m_n that they are in ℓ^2 whenever $\psi_R(\cdot) \in L^2(\mathbb{S}^2)$, due to the Parseval Relation of the Spherical Harmonics. Numerically, the expansion (45) with coefficients (46) can be much superior to its mathematical equivalent (6) provided one finds a means to compute the term $h^{(1)}_n(kR) j_n\big(k\|\boldsymbol{x}\|\big)$ without directly computing the product. This can be gleaned from the asymptotics of the spherical hankel function

$$h^{(1)}_n(kr) = O\Big(\frac{2n}{ekr}\Big)^n, \quad \text{as } n \to \infty \tag{47}$$

for fixed r.

In summary, (44) can be viewed as another form of Generalized Herglotz Wave Function. The relationships derived above and additional ones are shown in Fig. 3.

V. Conclusions

In general, multipath may be a manifestation of specular or diffuse, farfield or nearfield sources. Modeling of such multipath has been considered in a form which can be interpreted as

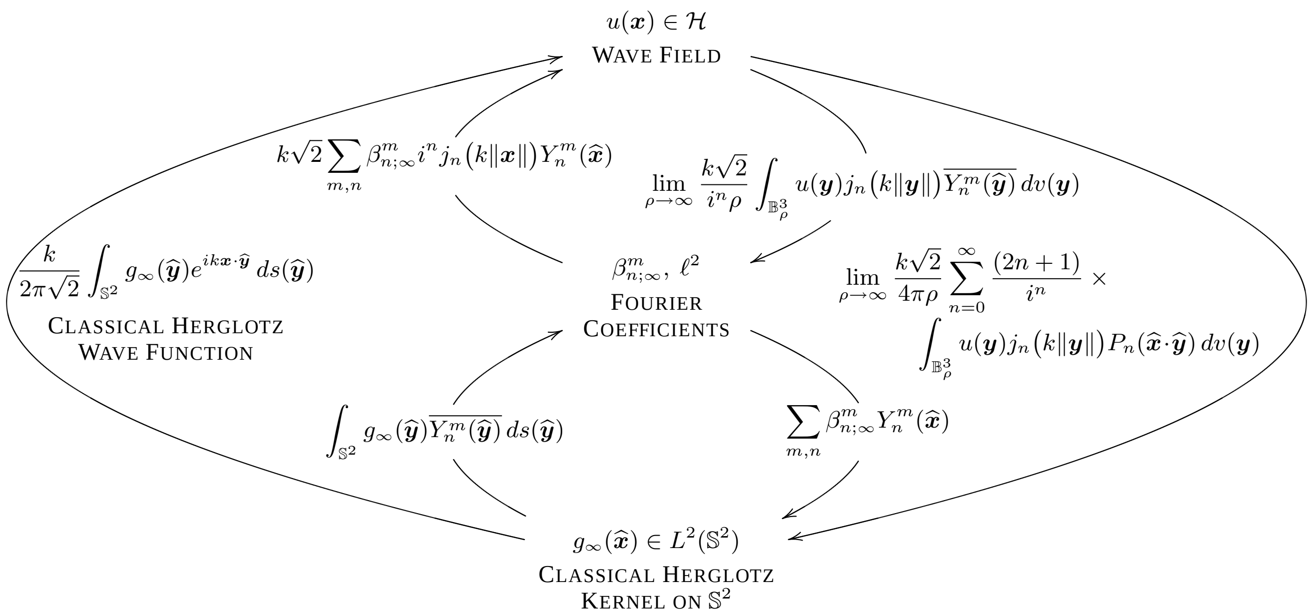

Fig. 2: Mappings between the Wave Field $u(\boldsymbol{x})$, the Fourier Coefficients $\beta_{n;\infty}^m$ and the Classical Herglotz Kernel $g_\infty(\widehat{\boldsymbol{x}})$.

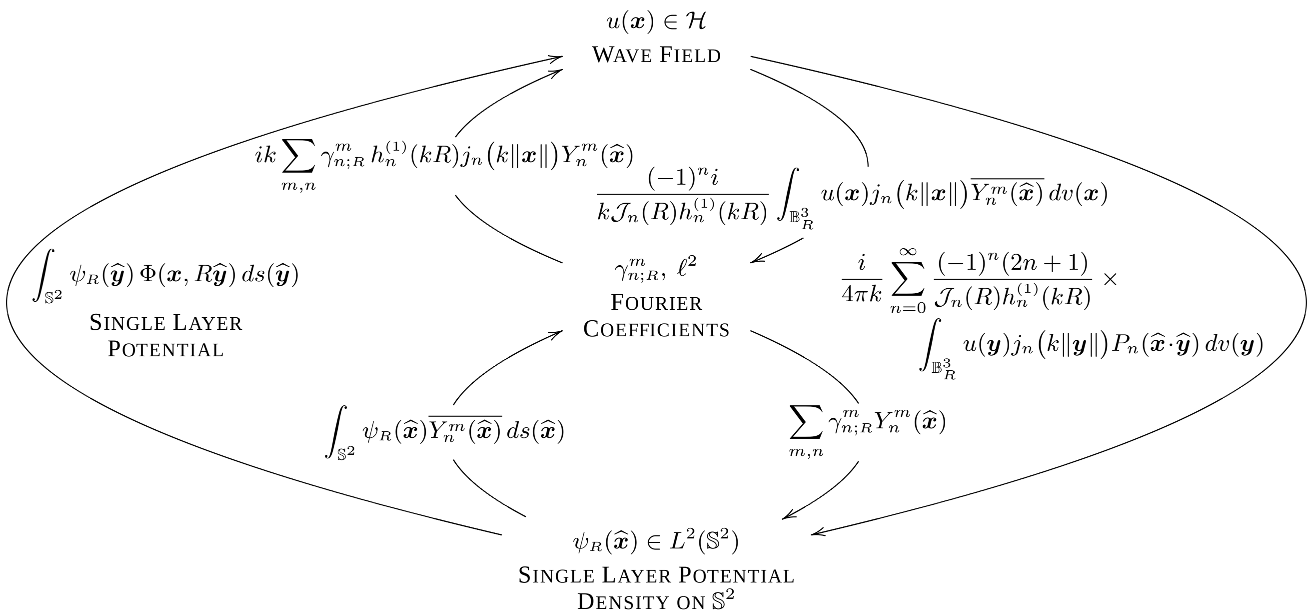

Fig. 3: Mappings between the Wave Field $u(\boldsymbol{x})$, the Fourier Coefficients $\gamma_{n;R}^m$ and the Single Layer Potential Density $\psi(\widehat{\boldsymbol{x}})$.

a generalization of the classical Herglotz Wave Function. We have shown that there exist a number of useful Fourier expansions in spatial variables which are complete in the sense of orthonormal expansions and in the sense of being able to model every physically realizable multipath field. This opens the possibility of developing closed form expansions for quantities of interest such as spatial correlation and bounds on capacity of MIMO systems in a nearfield scattering environment.

REFERENCES

[1] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer-Verlag, Berlin, Germany, second edition, 1998.

[2] P. D. Teal, T. D. Abhayapala, and R. A. Kennedy, "Spatial Correlation for General Distributions of Scatterers," *IEEE Signal Processing Letters*, vol. 9, no. 10, pp. 305–308, October 2002.

[3] C. A. Coulson and A. Jeffrey, *Waves: A Mathematical Approach to the Common Types of Wave Motion*, Longman, London, second edition, 1977.

[4] F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Dover Publications Inc., New York, NY, second edition, 1990.

[5] L. Debnath and P. Mikusiński, *Introduction to Hilbert Spaces with Applications*, Academic Press, San Diego, CA, 1999.

[6] N. N. Lebedev, *Special Functions and Their Applications*, Dover Publications Inc., New York, NY, 1972.

[7] N. W. McLachlan, *Bessel Functions for Engineers*, Oxford University Press, London, second edition, 1961.

[8] I. Stakgold, *Green's Functions and Boundary Value Problems*, John Wiley & Sons, Inc., New York, NY, second edition, 1998.

[9] P. D. Teal and R. A. Kennedy, "Bounds on the Spatial Extrapolation of Multipath in Long-Range Prediction of Mobile Signals," in *Proc. Int. Conf. Telecommunications, ICT'03*, Papeete, Tahiti, February 2003, (to appear).