Spatial Correlation for General Distributions of Scatterers

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***Abstract*—The well-known results of the spatial correlation function for two-dimensional and three-dimensional diffuse fields of narrowband signals are generalized to the case of general distributions of scatterers. A method is presented that allows closed-form expressions for the correlation function to be obtained for arbitrary scattering distribution functions. These closed-form expressions are derived for a variety of commonly used scattering distribution functions.**

***Index Terms*—Scattering distribution, sensors, spatial correlation.**

I. Introduction

There is growing interest in the literature in the use of multiple sensors—particularly, multiple antennas for transmission and/or reception of wireless signals. As well as diversity reception [1], this includes such areas as acoustic systems, fixed and mobile multiple-input multiple-output (MIMO) systems, spatio-temporal equalization, adaptive arrays, and space-time coding. Most of the work assumes that each receiving sensor receives uncorrelated signals and, conversely, that the signal received from each transmitting source is uncorrelated. A widely used "rule" is that half a wavelength separation is required in order to obtain decorrelation. This arises from the first null of the $\mathrm{sinc}(\cdot)$ function, which is the spatial correlation function for a three-dimensional (3-D) diffuse field [2].

Several approaches have been used in the case of signals confined to a limited azimuth and/or elevation [3]. In this letter, a modal analysis approach is presented that can be used to obtain closed-form expressions for the spatial correlation function for narrowband signals for a wide variety of scattering distribution functions.

II. Spatial Correlation Formulation

Consider two sensors located at points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Let $s_1(t)$ and $s_2(t)$ denote the complex envelope of the received signals at two sensors, respectively. Then, the normalized spatial correlation function between the complex envelopes of the two received signals is defined by

$$\rho(\boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{E\{s_1(t)s_2^*(t)\}}{E\{s_1(t)s_1^*(t)\}} \tag{1}$$

where $E\{\cdot\}$ denotes the expectation operator, and $\cdot^*$ denotes complex conjugation. We consider a general scattering environment with the scatterers distributed sufficiently far from the two sensors. If the transmitted signal is a narrowband signal $e^{i\omega t}$ (where ω is the angular frequency, $i = \sqrt{-1}$, and t is the time), then the received signal at the ℓth receiver is

$$s_\ell(t) = e^{i\omega t} \int_\Omega A(\hat{\boldsymbol{y}}) e^{-i(\omega/c)\boldsymbol{x}_\ell \cdot \hat{\boldsymbol{y}}}\, d\hat{\boldsymbol{y}}, \qquad \ell = 1, 2 \tag{2}$$

where c is the speed of wave propagation; $\hat{\boldsymbol{y}}$ is a unit vector pointing in the direction of wave propagation; and $A(\hat{\boldsymbol{y}})$ is the *complex gain* of scatterers as a function of direction that captures both the amplitude and phase distribution. Also note that the integration in (2) is over Ω, the unit sphere in the case of a 3-D multipath environment or the unit circle in the two-dimensional (2-D) case. We substitute (2) in (1), and we assume that scattering from one direction is independent from another direction to get

$$\rho(\boldsymbol{x}_1, \boldsymbol{x}_2) \equiv \rho(\boldsymbol{x}_2 - \boldsymbol{x}_1) = \int_\Omega \mathcal{P}(\hat{\boldsymbol{y}}) e^{ik(\boldsymbol{x}_2 - \boldsymbol{x}_1)\cdot \hat{\boldsymbol{y}}}\, d\hat{\boldsymbol{y}} \tag{3}$$

where $k = \omega/c$ is the wave number, and

$$\mathcal{P}(\hat{\boldsymbol{y}}) = \frac{E\left\{|A(\hat{\boldsymbol{y}})|^2\right\}}{\int_\Omega E\left\{|A(\hat{\boldsymbol{y}})|^2\right\} d\hat{\boldsymbol{y}}} \tag{4}$$

is the normalized average power of a signal received from direction $\hat{\boldsymbol{y}}$, or the *distribution function* of scatterers over all angles.

III. Three-Dimensional Scattering Environment

To gain a better understanding of spatial correlation, we now use a spherical harmonic expansion of plane waves, which is given [4] as

$$e^{ik\boldsymbol{x}\cdot\hat{\boldsymbol{y}}} = 4\pi \sum_{n=0}^{\infty} i^n j_n(k\|\boldsymbol{x}\|) \sum_{m=-n}^{n} Y_{nm}(\hat{\boldsymbol{x}}) Y_{nm}^*(\hat{\boldsymbol{y}}) \tag{5}$$

where $\hat{\boldsymbol{x}} = \boldsymbol{x}/\|\boldsymbol{x}\|$, $j_n(r) \triangleq \sqrt{\pi/2r}\, J_{n+(1/2)}(r)$ are spherical Bessel functions, and

$$\begin{aligned} Y_{nm}(\hat{\boldsymbol{x}}) &\equiv Y_{nm}(\theta_{\boldsymbol{x}}, \varphi_{\boldsymbol{x}}) \\ &\triangleq \sqrt{\frac{2n+1}{4\pi} \frac{(n-|m|)!}{(n+|m|)!}}\, P_n^{|m|}(\cos\theta_{\boldsymbol{x}}) e^{im\varphi_{\boldsymbol{x}}} \end{aligned} \tag{6}$$

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where $\theta_{\boldsymbol{x}}$ and $\varphi_{\boldsymbol{x}}$ are the elevation and azimuth, respectively, of the unit vector $\boldsymbol{x}$ and where $P_n^m(\cdot)$ are the associated Legendre functions of the first kind. Equations (3) and (5) may be combined to obtain

$$\rho(\boldsymbol{x}_2-\boldsymbol{x}_1)=4\pi\sum_{n=0}^{\infty} i^n j_n(k\|\boldsymbol{x}_2-\boldsymbol{x}_1\|) \cdot \sum_{m=-n}^{n}\beta_{nm}Y_{nm}\left(\frac{\boldsymbol{x}_2-\boldsymbol{x}_1}{\|\boldsymbol{x}_2-\boldsymbol{x}_1\|}\right) \quad (7)$$

where

$$\beta_{nm}=\int_{\Omega}\mathcal{P}(\hat{\boldsymbol{y}})Y_{nm}^*(\hat{\boldsymbol{y}})\,d\hat{\boldsymbol{y}}. \quad (8)$$

The fact that the higher order spherical Bessel functions (and in Section IV the higher order Bessel functions) have small values for arguments near zero means that to evaluate the correlation for points near each other in space, only a few terms in the sum need to be evaluated in order to obtain a very good approximation [5].

A. Three-Dimensional Omnidirectional Diffuse Field

If waves are incident on the two points from all directions in 3-D space, then (7) reduces to a single term, and so the correlation coefficient is given by

$$\rho=j_0(k\|\boldsymbol{x}_2-\boldsymbol{x}_1\|)=\mathrm{sinc}(k\|\boldsymbol{x}_2-\boldsymbol{x}_1\|)$$

where $\mathrm{sinc}(x)\triangleq\sin(x)/x$ for $x\neq 0$, and $\mathrm{sinc}(0)=1$. This is the classical result [2]. The first zero crossing is at $\lambda/2$.

B. Uniform Limited Azimuth/Elevation Field

Without loss of generality, the coordinate system may be chosen so that $\theta_{\boldsymbol{x}_2-\boldsymbol{x}_1}=\pi/2$ and $\varphi_{\boldsymbol{x}_2-\boldsymbol{x}_1}=0$, i.e., $\boldsymbol{x}_2-\boldsymbol{x}_1=[r,0,0]'$. If the scatters are uniformly distributed over the sector $\{(\theta,\varphi);\theta\in[\theta_1,\theta_2],\varphi\in[\varphi_1,\varphi_2]\}$, then the correlation can be expressed as

$$\rho([r,0,0]')=\frac{1}{(\cos\theta_1-\cos\theta_2)}\sum_{n=0}^{\infty}(2n+1)i^n j_n(kr) \cdot\left(P_n(0)\int_{\theta_1}^{\theta_2}P_n(\cos\theta)\sin\theta\,d\theta + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}\frac{\sin(m\varphi_2)-\sin(m\varphi_1)}{m(\varphi_2-\varphi_1)} \cdot P_n^m(0)\int_{\theta_1}^{\theta_2}P_n^m(\cos\theta)\sin\theta\,d\theta\right). \quad (9)$$

Note that (9) is actually expressible in closed form (the integrals can be evaluated, for example, by using recursions in m and n). Hence, this result can be used to build up the result for a general scattering situation. An arbitrary scattering can be regarded as the limiting summation of a weighted set of uniformly distributed incremental solid-angle contributions.

Fig. 1 shows the case where energy is arriving from all azimuth directions (uniformly) but the elevation spread is in some range of angles on both sides of zero elevation. The spatial correlation is shown for four sets of elevation spread, each centered on $\theta=\pi/2$. It is clear that given an azimuthal distribution, spatial correlation in the horizontal plane is relatively insensitive to the elevational distribution. Multipath that has some small elevation spread (in many practical situations, we would not expect much) may be modeled as only coming from the horizontal plane.

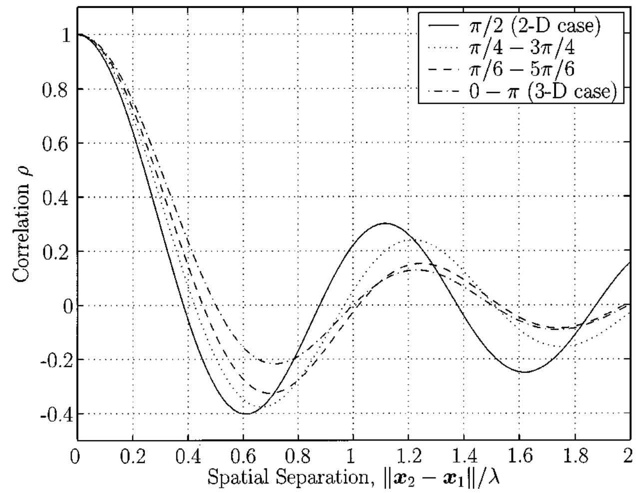

Fig. 1. Spatial correlation versus separation for different elevation ranges. This figure shows that the spread of interference in elevation plays only a secondary role in influencing spatial correlation and, hence, diversity.

C. Spherical Harmonic Model

Given that the distribution function $\mathcal{P}(\hat{\boldsymbol{y}})$ is defined on the unit sphere, then the scattering environment may be specified by giving the coefficients ζ_{nm} in the spherical harmonic expansion

$$\mathcal{P}(\hat{\boldsymbol{y}})=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\zeta_{nm}Y_{nm}(\hat{\boldsymbol{y}}) \quad (10)$$

where the coefficients ζ_{nm} are chosen so that the distribution function is normalized. However, this equation simply expresses the inverse of (8), from which we conclude that $\zeta_{nm}=\beta_{nm}$. That is, the coefficients of the spherical harmonic expansion (10) are the same as required for the coefficients in the spatial correlation expansion (7).

IV. Two-Dimensional Scattering Environment

If the fields may be considered as arriving from the azimuthal plane only, it is more useful to consider the 2-D modal expansion [4]

$$e^{ik\boldsymbol{x}\cdot\hat{\boldsymbol{y}}}=\sum_{m=-\infty}^{\infty}i^m J_m(k\|\boldsymbol{x}\|)e^{im(\varphi_x-\varphi_y)} \quad (11)$$

where φ_x and φ_y are the angles of $\boldsymbol{x}$ and $\hat{\boldsymbol{y}}$. Equations (3) and (11) may be combined to obtain

$$\rho(\boldsymbol{x}_2-\boldsymbol{x}_1)=\sum_{m=-\infty}^{\infty}i^m\gamma_m J_m(k\|\boldsymbol{x}_2-\boldsymbol{x}_1\|)e^{im\varphi_{21}} \quad (12)$$

where

$$\gamma_m = \int_0^{2\pi} \mathcal{P}(\varphi) e^{-im\varphi}\, d\varphi \tag{13}$$

with $\mathcal{P}(\varphi)$ the distribution function equivalent to $\mathcal{P}(\hat{\boldsymbol{y}})$ in (4), and where φ_{21} is the angle of the vector connecting $\boldsymbol{x}_2$ and $\boldsymbol{x}_1$. Without loss of generality, φ_{12} can be considered to be zero.

A. Two-Dimensional Omnidirectional Diffuse Field

For the special case of scattering over *all* angles in the plane containing two points, (12) reduces to a single term, and so the correlation coefficient is given by $\rho = J_0(k\|\boldsymbol{x}_2 - \boldsymbol{x}_1\|)$), another classical result [1]. As can be seen by an examination of Fig. 1, $J_0(\cdot)$ and $\mathrm{sinc}(\cdot)$ are qualitatively similar.

B. Uniform Limited Azimuth Field

In the case of energy arriving uniformly from a restricted range of azimuth ($\varphi_0 - \Delta$, $\varphi_0 + \Delta$), we have

$$\gamma_m = e^{-im\varphi_0}\,\mathrm{sinc}(m\Delta). \tag{14}$$

This result is equivalent to that derived in [6].

C. $\cos^{2p}\varphi$ *Distributed Field*

Another azimuthal distribution used for calculation of correlation is [7]

$$\mathcal{P}(\varphi) = Q\cos^{2p}\left(\frac{\varphi - \varphi_0}{2}\right), \qquad |\varphi - \varphi_0| \le \pi \tag{15}$$

where Q is a normalization constant and $p > 0$. Using [8, (335.19)], we have

$$\gamma_m = e^{-im\varphi_0}\,\frac{\Gamma^2(p+1)}{\Gamma(p-m+1)\Gamma(p+m+1)}. \tag{16}$$

D. von-Mises Distributed Field

Nonisotropic scattering in the azimuthal plane may be modeled by the von-Mises distribution (e.g., [9]), for which the density is given by

$$\mathcal{P}(\varphi) = \frac{1}{2\pi I_0(\kappa)}\, e^{\kappa\cos(\varphi-\varphi_0)}, \qquad |\varphi - \varphi_0| \le \pi \tag{17}$$

where φ_0 represents the mean direction; $\kappa > 0$ represents the degree of nonisotropy, and $I_m(\kappa)$ is the modified Bessel function of the first kind. In this case, using [10, (3.937)] we have

$$\begin{aligned}\gamma_m &= \frac{1}{2\pi I_0(\kappa)}\int_0^{2\pi} e^{\kappa\cos(\varphi-\varphi_0)} e^{im\varphi}\, d\varphi \\ &= e^{-im\varphi_0}\,\frac{I_{-m}(\kappa)}{I_0(\kappa)}.\end{aligned} \tag{18}$$

For the von-Mises distribution, the correlation can be calculated without the need for a summation [9]. There are situations, however, as reported in [11] where other models of the scattering distribution are more appropriate than von-Mises.

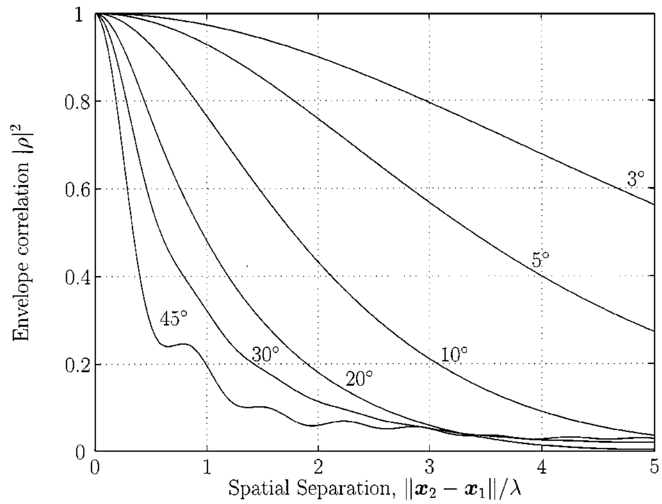

Fig. 2. Correlation for angle of incidence 60° from broadside against separation of sensors with angular spread as parameter, based on a Laplacian power distribution function.

E. Laplacian Distributed Field

In [12], the Laplacian distribution is proposed as a realistic model of the power distribution function in some circumstances. Here

$$\mathcal{P}(\varphi) = \frac{Q}{\sqrt{2}\sigma}\, e^{-\sqrt{2}|\varphi-\varphi_0|/\sigma}, \qquad |\varphi - \varphi_0| \le \frac{\pi}{2} \tag{19}$$

where Q is a normalization constant. It is straightforward to show that

$$\gamma_m = e^{-im\varphi_0}\,\frac{(1-(-1)^{\lceil m/2\rceil}\xi F_m)}{(1+\sigma^2 m^2/2)(1-\xi)} \tag{20}$$

where $\xi = e^{-\pi/(\sqrt{2}\sigma)}$; $F_m = 1$ for m even; and $F_m = m\sigma/\sqrt{2}$ for m odd. The power of the technique is demonstrated in Fig. 2, which is similar to [11, Fig. 5b], except that the distribution used here is the Laplacian, considered in [11] to be realistic but mathematically intractable. The angular spread used to distinguish between the datasets in Fig. 2 is defined as the square root of the variance, which for this distribution is given by

$$S_\sigma^2 = \frac{1}{1-\xi}\left(\sigma^2 - \frac{\xi}{4}\left(\pi^2 + 4\sigma^2 + \sqrt{8}\pi\sigma\right)\right). \tag{21}$$

Approximately 100 terms of the summation (12) were required to obtain results to about ten significant digits at the largest spatial separation of 5λ, or 70 terms for five significant digits. For separations up to 2λ only 40–50 terms are required. Evaluation of the spatial correlation to the same accuracy via adaptive numerical quadrature requires between 100 and 500 function evaluations. The summation in (12) is over terms that are the product of three components, dependent on the center angle φ_0, the spatial separation $\|\boldsymbol{x}_2 - \boldsymbol{x}_1\|$, and the nonisotropy parameter (such as Δ, κ, σ, or p) respectively, so that if any of these parameters are constant, as they are in many situations, then the components need not be recalculated. Fig. 2, for instance, takes approximately 175 times more CPU time to draw using quadrature than using the modal technique.

F. Gaussian Distributed Field

Several researchers [3], [7] have used the Gaussian distribution for modeling the distribution of scatterers, thus

$$\mathcal{P}(\varphi) = \frac{Q}{\sqrt{2\pi}\sigma} e^{-((\varphi-\varphi_0)^2/2\sigma^2)}, \qquad |\varphi - \varphi_0| \leq \frac{\pi}{2} \tag{22}$$

where σ is the standard deviation, and Q is a normalization constant. It can be shown using [13, (313.6)] and the symmetries of the error function for complex arguments discussed in [14] that

$$\gamma_m = e^{-im\varphi_0 - m^2\sigma^2/2} \frac{\mathrm{Re}\left(\mathrm{erf}\left(\frac{\pi/2 + im\sigma^2}{\sqrt{2}\sigma}\right)\right)}{\mathrm{erf}\left(\frac{\pi/2}{\sqrt{2}\sigma}\right)}. \tag{23}$$

It can be seen that contrary to the assertion of [11], a closed-form solution for the correlation function is available for distribution functions other than the Gaussian (such as the $\cos^n \varphi$ distribution), and that the Gaussian case is neither a simple nor natural choice. If, however, the beam is narrow, a good approximation for the Gaussian case can be obtained by performing integration over the domain $(-\infty, \infty)$, since the tails will cause very little error, so using [8, (337.3)], it is straightforward to show that $\gamma_m \approx e^{im\varphi_0 - m^2\sigma^2/2}$.

G. Cylindric Harmonic Model

The distribution function $\mathcal{P}(\varphi)$ may be expressed as the sum of orthogonal basis functions

$$\mathcal{P}(\varphi) = \sum_{m=-\infty}^{\infty} \zeta_m e^{jm\varphi} \tag{24}$$

where the coefficients ζ_k are chosen so that the distribution function is normalized. In many cases, the number of basis functions required to approximate the distribution function may be quite small. The coefficients γ_m in (12) can be simply expressed in terms of the coefficients ζ_m as

$$\gamma_m = 2\pi\zeta_m. \tag{25}$$

V. Mutual Coupling

When the sensors are antennas, there can be mutual coupling effects between the sensors. For a given spacing, mutual coupling of terminated antennas can actually *decrease* the correlation between the sensors from that calculated using the expressions above. One interpretation of this is that the presence of other antennas creates a slow-wave structure that, in effect, decreases the wavelength of the signal in their vicinity and, thus, increases the number of wavelengths separation between the elements. This phenomenon was first reported in [15] and has recently been reported in [16]. If it proves true, in general, that mutual coupling effects decrease correlation, then the closed-form expressions of the previous sections may be considered as upper bounds on the correlation between antennas in an electromagnetic field.

VI. Conclusion

Closed-form solutions have been presented for the correlation between the signal at physically separated points in a field with the signal power distributed in solid or planar angle following a variety of commonly used distribution functions. Unlike some other approaches, the method can be applied to many distribution functions, and for small spatial separations, accurate results are obtained with very few terms.

Application of the method reveals that restriction of the signal elevation does not have a large impact on the signal correlation coefficient, provided there is omnidirectionality in one plane containing the two points. Restriction of the signal in both planes considerably increases signal correlation. In order to exploit diversity for sensors spaced $\lambda/2$ or closer, there must be omnidirectionality in at least one plane. To obtain uncorrelated signals with directional sensors, the spacing must be considerably larger.

References

[1] W. C. Jakes, Ed., *Microwave Mobile Communications*. New York: Wiley, 1974.

[2] R. K. Cook, R. V. Waterhouse, R. D. Berendt, S. Edelman, and M. D. Thompson, Jr., "Measurement of correlation coefficients in reverberant sound fields," *J. Acoust. Soc. Amer.*, vol. 27, no. 6, pp. 1072–1077, Nov. 1955.

[3] M. Kalkan and R. H. Clarke, "Prediction of the space–frequency correlation function for base station diversity reception," *IEEE Trans. Veh. Technol.*, vol. 46, pp. 176–184, Feb. 1997.

[4] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*. Berlin, Germany: Springer-Verlag, 1992.

[5] H. M. Jones, R. A. Kennedy, and T. D. Abhayapala, "On dimensionality of multipath fields: Spatial extent and richness," in *Proc. ICASSP*, vol. 3, Orlando, FL, May 2002, pp. 2837–2840.

[6] J. Salz and J. H. Winters, "Effect of fading correlation on adaptive arrays in digital mobile radio," *IEEE Trans. Veh. Technol.*, vol. 42, pp. 1049–1057, Nov. 1994.

[7] R. G. Vaughan, "Pattern translation and rotation in uncorrelated source distributions," *IEEE Trans. Antennas Propagat.*, vol. 46, pp. 982–990, July 1998.

[8] W. Gröbner and N. Hofreiter, *Integraltafel, Zweiter Teil, Bestimmte Integrale*. Berlin, Germany: Springer-Verlag, 1950.

[9] A. Abdi and M. Kaveh, "A versatile spatio–temporal correlation function for mobile fading channels with non-isotropic scatterers," in *Proc. IEEE Workshop Statistical Signal Processing*, Pocono Manor, PA, 2000, pp. 58–62.

[10] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 5th ed. San Diego, CA: Academic, 1994.

[11] J. Fuhl, A. F. Molisch, and E. Bonek, "Unified channel model for mobile radio systems with smart antennas," *Proc. Inst. Elect. Eng.—Radar, Sonar Navig.*, vol. 4, no. 1, pp. 32–41, Feb. 1998.

[12] K. L. Pedersen, P. E. Mogensen, and B. H. Fleury, "Power azimuth spectrum in outdoor environments," *IEE Electron. Lett.*, vol. 33, no. 18, pp. 1583–1584, 1997.

[13] W. Gröbner and N. Hofreiter, *Integraltafel, Erster Teil, Unbestimmte Integrale*. Berlin, Germany: Springer-Verlag, 1949.

[14] W. Gautschi, "Efficient computation of the complex error function," *SIAM J. Numer. Anal.*, vol. 7, pp. 187–198, 1970.

[15] R. G. Vaughan and N. L. Scott, "Terminated in-line monopoles for vehicular diversity," in *Proc. Int. Union Radio Science (USRI) Trienniel Symp. Electromagnetic Theory*, Sydney, Australia, Aug. 1992.

[16] T. Svantesson and A. Ranheim, "Mutual coupling effects on the capacity of multielement antenna systems," in *Proc. ICASSP*, vol. 4, Salt Lake City, UT, 2001, pp. 2485–2488.