Dual Selection Diversity SNR Performance in Spatially Correlated Scattering Environments

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***Abstract*– A new expression is given for the average signal-to-noise ratio (SNR) of a dual selection-combining receiver where the antenna signals are correlated in a Rayleigh fading multipath environment. This formula is based on a recently developed expression for the bivariate Rayleigh cumulative distribution function by associating the correlation parameter of the distribution with the spatial correlation of the multipath field. By examining various multipath and receiver antenna geometries we can determine the critical parameters which affect the average SNR performance. Our results are shown to generalize and subsume well-known expressions based on an independence assumption of the signals at the antennas. We relate the average SNR diversity gain to critical parameters such as the spatial correlation, beamwidth, angle of arrival and antenna separation. These results show the subtle interplay between these parameters and reveal there is a reduction in performance relative to unrealistic models previously studied.**

I. Introduction

The presence of multipath scatterers in wireless communication channels generally significantly degrades the performance of wireless systems [1]. With the use of multiple antennas, space diversity techniques can combat this degradation [2]. In the most common form of space diversity, signals from multiple receiver antennas are combined to achieve a diversity performance gain relative to a single receiving antenna. The most common combining methods are maximal ratio combining, equal gain combining and selection combining [1, 3]. The first two diversity schemes require cophasing on each received signal before combining the signals, while selection combining does not. This paper focuses on the selection diversity problem but restricts attention to the use of two antennas – dual selection diversity.

The strategy in a dual selection diversity is to select and only use the antenna that has the higher SNR, at any time instant, and discard the signal from the other antenna. Whilst this represents a loss of information, this strategy has many benefits and the performance loss relative to more optimal strategies need not be great. That is, in a multipath fading environment this strategy largely protects the receiver, in a probabilistic sense, from deep fades. Further, selection diversity systems have a simplicity comparable to single antenna systems which makes them attractive from an implementation viewpoint.

One of the meaningful diversity performance metrics is the average SNR diversity gain. An analytical expression for it was determined under the statistical assumption that the received signals from different antennas are independent [4]. However, this assumption generally (but not necessarily) requires that the antennas be sufficiently well separated from each other and this is not always feasible in practice. In fact, the case of greater interest is when the antennas are brought into close proximity because this matches the trend of making communication devices smaller and less intrusive. Hence, the independence assumption needs to be revisited with the objective of determining average SNR diversity gain in less trivial spatially correlated scattering environments.

The broad goal here is to determine the geometric parameters that characterize a multipath scattering environment which are most critical for the best average SNR gain performance of a dual selection diversity receiver. Towards this end, the concept of spatial correlation is crucial as this directly influences the gains that are possible (relative to a single antenna system) and gives insight into how antennas should be placed in space. The critical geometric parameters of the scattering environment in conjunction with the antenna arrays considered in this study are: the beamwidth of the scattering angle impinging the receiver, the angle of arrival (ranging from broadside to endfire), the antenna separation and the various interdependencies.

In this paper, we rely on a recently determined expression for the bivariate Rayleigh cumulative density function (CDF) [5] to develop a novel analytical formula for the average SNR of the correlated dual selection combining system. Then we study the various geometric configurations of scattering using our expression and corroborate the theoretical results with simulations.

II. Problem Formulation

We assume here, as most papers do in the literature, that the superposition of the multipath signals is complex Gaussian distributed by applying (non-rigorously) the central limit theorem [1, 6, 7]. The correlated multiplicative complex Gaussian processes on the two receivers are also assumed to be jointly Gaussian [1, 3]. This implies that the multipath environment is assumed to be sufficiently rich (this is difficult to precisely quantify) and that it is sufficient to only determine the (spatial) correlation and not higher order moments of various joint distributions to fully characterize the system statistically.

Define $z \triangleq \max\{y_1^2, y_2^2\}$, where $y_i^2 = |s_i|^2/(2N_i)$ is the local mean SNR, s_i is the received signal and N_i is additive while gaussian noise (AWGN). The joint PDF of y_1 and y_2 at any instant [3, 7] can be expressed in terms of the modified Bessel function as follows

$$f(y_1, y_2) = \frac{4y_1y_2}{\Omega_1\Omega_2(1-|\rho|^2)}\, \mathrm{I}_0\Big(\frac{2|\rho|\, y_1y_2}{(1-|\rho|^2\sqrt{\Omega_1\Omega_2}}\Big) \times \exp\left(-\frac{1}{(1-|\rho|^2)}\big(y_1^2/\Omega_1 + y_2^2/\Omega_2\big)\right) \tag{1}$$

where $\Omega_i = E\{y_i^2\}$, $\rho = E\{s_1s_2^*\}/E\{s_1s_1^*\}$, $E\{\cdot\}$ denotes the expectation operator, and * denotes complex conjugation. Furthermore, the probability that the output SNR z is below some value γ (which is the same as the CDF) is defined as

$$\Pr(z < \gamma) = \Pr(y_1 < \sqrt{\gamma}, y_2 < \sqrt{\gamma}). \tag{2}$$

With the use of infinite series representations as in [5], the CDF becomes

$$\Pr(z<\gamma) = \big(1-|\rho|^2\big)\sum_{k=0}^{\infty}\frac{|\rho|^{2k}}{(k!)^2}\times G\Big(k+1, \frac{\gamma}{(1-|\rho|^2)\Omega_1}\Big)G\Big(k+1, \frac{\gamma}{(1-|\rho|^2)\Omega_2}\Big), \tag{3}$$

where $G(n,x) = \int_0^x t^{n-1}e^{-t}dt,\ n>0$, is the incomplete Gamma function [8, (8.35)]. This expression contrasts with the classical results which are expressed in terms of the Marcum Q-function [1, 7]. With this new expression, we can derive a novel expression for the average SNR of the correlated dual selection combining system under slow Rayleigh fading, which is shown in the next section.

III. Derivation of Average SNR

Based on the CDF expression (3), we now derive the expression for the average SNR of the correlated dual selection combining system. The expected value of the SNR of the system is defined as

$$E\{z\} = \int_0^{\infty}\gamma\, f_z(\gamma)\, d\gamma, \tag{4}$$

where $f_z(\gamma)$ is the PDF of z. To simplify notation, let

$$\xi_i(\gamma) \triangleq \frac{\gamma}{(1-|\rho|^2)\,\Omega_i}, \quad i\in\{1,2\}. \tag{5}$$

We can calculate the PDF by differentiating the CDF in (3) with respect to γ, as follows

$$\begin{aligned} f_z(\gamma) &\triangleq \frac{d}{d\gamma}\Pr(z<\gamma) \\ &= \sum_{k=0}^{\infty}\frac{(1-|\rho|^2)|\rho|^{2k}}{(k!)^2}\times \\ &\quad \frac{d}{d\gamma}\Big\{\int_0^{\xi_1(\gamma)} t^k e^{-t}\,dt \int_0^{\xi_2(\gamma)} t^k e^{-t}\,dt\Big\}. \end{aligned} \tag{6}$$

After some simplifications, the PDF will be equal to

$$f_z(\gamma) = \sum_{k=0}^{\infty}\Big\{\frac{|\rho|^{2k}\gamma^k e^{-\xi_1(\gamma)}}{(1-|\rho|^2)^k(k!)^2\Omega_1^{k+1}}\, G\big(k+1, \xi_2(\gamma)\big) + \frac{|\rho|^{2k}\gamma^k e^{-\xi_2(\gamma)}}{(1-|\rho|^2)^k(k!)^2\Omega_2^{k+1}}\, G\big(k+1, \xi_1(\gamma)\big)\Big\} \tag{7}$$

Substituting (7) into (4), the expected value of the SNR becomes

$$E\{z\} = \sum_{k=0}^{\infty}\frac{|\rho|^{2k}}{(1-|\rho|^2)^k(k!)^2}\big(B_k + C_k\big) \tag{8}$$

where

$$B_k \triangleq \frac{1}{\Omega_1^{k+1}}\int_0^{\infty}\gamma^{k+1}e^{-\xi_1(\gamma)}\, G\big(k+1, \xi_2(\gamma)\big)\, d\gamma \tag{9}$$

$$C_k \triangleq \frac{1}{\Omega_2^{k+1}}\int_0^{\infty}\gamma^{k+1}e^{-\xi_2(\gamma)}\, G\big(k+1, \xi_1(\gamma)\big)\, d\gamma. \tag{10}$$

We can further simplify (9) and (10) by using [8, (6.455)],

$$\int_0^{\infty} x^{\mu-1}e^{-\beta x}G(v, \alpha x)\, dx = \frac{\alpha^v\Gamma(\mu+v)}{v(\alpha+\beta)^{\mu+v}}\times F\Big(1, \mu+v; v+1; \frac{\alpha}{\alpha+\beta}\Big) \tag{11}$$

where $\mathrm{Re}(\alpha+\beta) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\mu+v) > 0$ and $F(\cdot,\cdot;\cdot;\cdot)$ is the Gauss hypergeometric function. Therefore, substituting (11) into (9) and (10), we obtain

$$B_k = D_k \cdot F\left(1, 2k+3; k+2; \frac{\Omega_1}{\Omega_1+\Omega_2}\right) \tag{12}$$

$$C_k = D_k \cdot F\left(1, 2k+3; k+2; \frac{\Omega_2}{\Omega_1+\Omega_2}\right) \tag{13}$$

where

$$D_k \triangleq \frac{(1-|\rho|^2)^{k+2}(2k+2)!\Omega_1^{k+2}\Omega_2^{k+2}}{(k+1)(\Omega_1+\Omega_2)^{2k+3}}. \tag{14}$$

Furthermore, from [8, (9.14)], we know that

$$F(a,b;c;d) \triangleq \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{d^n}{n!} \tag{15}$$

where $(\cdot)_n$ is the Pochhammer symbol which is defined by $(y)_n \triangleq (y-n-1)!/(y-1)!$. Therefore,

$$B_k = D_k\sum_{n=0}^{\infty}\frac{(2k+2+n)!}{(2k+2)!}\frac{(k+1)!}{(k+1+n)!}\frac{\Omega_1^n}{(\Omega_1+\Omega_2)^n} \tag{16}$$

$$C_k = D_k\sum_{n=0}^{\infty}\frac{(2k+2+n)!}{(2k+2)!}\frac{(k+1)!}{(k+1+n)!}\frac{\Omega_2^n}{(\Omega_1+\Omega_2)^n}. \tag{17}$$

Finally, we substitute (16) and (17) into (8); the expected value of the SNR becomes the more compact expression

$$E\{z\} = \left(1-|\rho|^2\right)^2 \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \left\{ |\rho|^{2k}(2k+n+2) \times \frac{\Omega_1^{k+2}\Omega_2^{k+2}(\Omega_1^n+\Omega_2^n)}{(\Omega_1+\Omega_2)^{2k+n+3}} \binom{2k+n+1}{k} \right\} \tag{18}$$

where

$$\binom{p}{n} \triangleq \frac{p!}{n!(p-n)!}.$$

Now we define

$$m \triangleq \frac{\Omega_2}{\Omega_1}, \tag{19}$$

then the expected value of the SNR becomes

$$E\{z\} = \left(1-|\rho|^2\right)^2 \Omega_1 \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \left\{ \binom{2k+n+1}{k} \times \frac{|\rho|^{2k}(2k+n+2)m^{k+2}(1+m^n)}{(1+m)^{2k+n+3}} \right\}. \tag{20}$$

The average SNR is a function of the absolute value of correlation, $|\rho|$, and the local SNR average in the two receivers, Ω_1 and Ω_2. In the context of this paper the correlation is the spatial correlation which means the signals are obtained by detecting them at physically separated points in space. By explicitly relating the spatial correlation to distance we can then explicitly determine how diversity is affected by antenna separation.

IV. Results

In previous section we described the novel formulation of the expected value of the SNR for the dual selection diversity system. Here we will use (20) to examine the average SNR diversity gain of the system, relate it with the spatial correlation ρ, angle of arrival θ, beamwidth $\triangle$, and antenna spacing D; all analytically.

A. *Two Standard Results*

Here we want to show that (20) will lead to two well known results [1], when two received signals are uncorrelated, $\rho = 0$ or fully correlated, $\rho = 1$. Here we consider the case when each receiver antenna has an identical local average SNR, $m = 1$ ($\Omega_1 = \Omega_2$), then (20) reduces to

$$E\{z\} = \left(1-|\rho|^2\right)^2 \Omega_1 \times \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \binom{2k+n+1}{k} \frac{|\rho|^{2k}(2k+n+2)}{2^{2k+n+2}}. \tag{21}$$

When both received signals are uncorrelated, $\rho = 0$, (21) simplifies to $1.5\,\Omega_1$ ($= 1.761$ dB), which agrees with the result in [1, 4]. When two received signals are fully correlated, $\rho = 1$, we obtain, by taking the limit $\rho \to 1$ in (21), $1\,\Omega_1$ ($= 0$ dB). That is, there is no average SNR gain when the two received signals are identical, as expected.

Let us now consider for all other spatial correlations coefficients, the average SNR diversity gain is defined as

$$\begin{aligned} G_{\mathrm{dB}}(\rho) &\triangleq 10\log_{10}\left\{\frac{E\{z\}}{\Omega_1}\right\} \\ &= 10\log_{10}\left\{ \left(1-|\rho|^2\right)^2 \times \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \binom{2k+n+1}{k} \frac{|\rho|^{2k}(2k+n+2)}{2^{2k+n+2}} \right\}. \end{aligned} \tag{22}$$

For the numerical calculations, we use only the first 1001 terms of k and n as they are sufficient to bound the error (caused by the truncation of the infinite series) to below 10^{-10}. Using the truncated form of (22), in Fig. 1 we show relationship between the diversity gain and the correlation coefficients. Again, from Fig. 1, we can check that the standard cases, $\rho = 0$ and $\rho = 1$, both agree with the result in [1, 4].

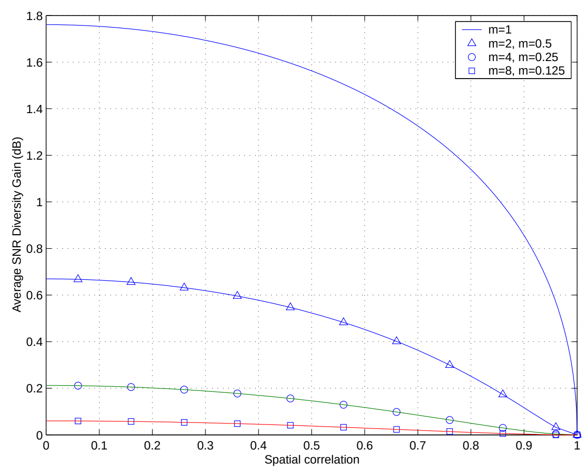

Fig. 1. Average SNR diversity gain in dual selection combining system under slow Rayleigh fading. $m = \Omega_2/\Omega_1$. Our analytical result agreed in both trivial cases, $\rho = 0$ and $\rho = 1$. Notice that once $m \neq 1$, the diversity gain reduces.

B. *Branch Average SNR Imbalance*

Here we want to investigate the important question of how the local average SNR affects the diversity power gain. We consider seven different cases here, where $m = \Omega_2/\Omega_1 = 0.125, 0.25, 0.5, 1, 2, 4, 8$. We use (20) to derive formulae similar to (21) and (22), each with different m. The average SNR diversity gain with different m is plotted in Fig. 1. Notice that both $\Omega_2 = m\Omega_1$ and $m\Omega_2 = \Omega_1$ give the same power gain curves, as expected. We also notice that once the two antennas have different local average SNRs, the diversity gain reduces. If the local SNR average of antennas is different, there will be a greater chance that the antenna with the higher local SNR average will be selected, therefore, it reduces the diversity of the system. From Fig. 1, we observed that when one local average SNR is the twice the other, the available diversity gain drops by 1.28dB (from 1.761dB to 0.6695dB), compared to the balance average SNR.

C. Diversity Gain versus Antenna Separation

In this section we want to relate the average SNR diversity gain to the antenna separation. This is particularly interesting when the separations are close as this corresponds to the case when there is limited real estate on a mobile terminal for example. We will also corroborate our analytical results with simulation results. It is well known that if the scatters form a 2D Omni-directional diffuse field, the spatial correlation is given by [1]

$$\rho(d) = J_0(2\pi d/\lambda), \tag{23}$$

where $J_0(\cdot)$ is the zero order Bessel function of the first kind, and d is the antenna separation. For $m = \Omega_2/\Omega_1 = 1$, we can simply substitute (23) into (22) to relate diversity gain and antenna separation, which is shown in Fig. 2. The simulation curves is also plotted in Fig. 2 to corroborate the correctness of our analytical expression. Our simulation method is reported in [11] and is summarized in four steps here:

Step 1) Generate 40 randomly directed multipaths with Gaussian distributed amplitude $N(0,1)$ and uniformly distributed phase ($0°$ to $360°$).
Step 2) Randomly locate two antennas at a distance d apart to each other.
Step 3) Record the received signal power from the antenna with the higher SNR.
Step 4) Repeat step (1) through (3) $10,000$ times.
Step 5) Estimate the gain by averaging these samples.

From the analytical plot, it is noticed that the diversity gain approximately increases at a rate of 7dB/λ from 0λ to 0.2λ. In order to achieve 90% of the available diversity gain (1.584dB in this example), the requirement of antenna separation is only 0.25λ, under a 2D Omni-directional diffuse field.

D. Diversity Gain versus Beamwidth and Angle of Arrival

The diversity of the system is also depending on the beamwidth Δ and the angle of arrival θ of the multipath scatters. Fig. 3 shows a scenario where all signals arrive at the receivers within $\pm\Delta$ at the angle θ. The diversity of the system reduces when the beamwidth reduces. The idea becomes clear if we consider an extreme case that no multipath and only direct path (line of sight) is received (i.e. $\Delta = 0$); it is obvious that no matter how many receiving antennas are employed, there will be no diversity gain. With the help of the analytical expression (20), we can investigate those relationships in more details. For a 2D uniform limited azimuth field with certain beamwidth $(\theta - \Delta, \theta + \Delta)$, the spatial correlation is given by [9, 10]

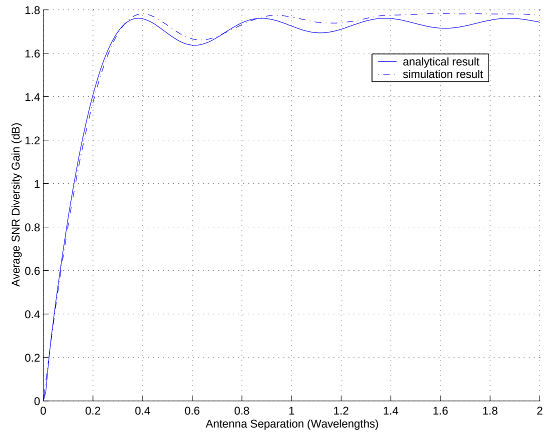

Fig. 2. The relationship between average SNR diversity gain and antenna separation for a 2D diffuse field. The simulation result is plotted to demonstrate the correctness of our analytical expression. Notice that only 0.25λ of antenna separation is required in order to achieve 90% of the available diversity gain.

$$\rho(d) = \sum_{m=-\infty}^{\infty} e^{im(\pi/2-\theta)} \frac{\sin(m\Delta)}{m\Delta} J_m(2\pi\, d/\lambda). \tag{24}$$

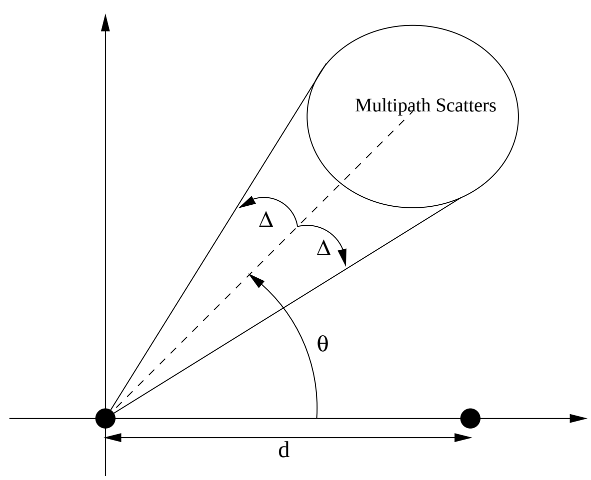

Fig. 3. The typical scenario where all signals arrive at the receivers within $\pm\Delta$ at the angle θ.

Substitute (24) into (20), we can relate the diversity gain to the antenna separation, given a particular combination of Δ and θ.

E. Minimum Antenna Separation

We have shown that the diversity average SNR gain is related to the beamwidth Δ, the angle of arrival θ as well as the antenna separation d. In this section we would like to use our analytical expression (20) to find out the minimum antenna separation, in order to achieve 90% of the available diversity gain, given a particular value of Δ and θ. We choose the local average SNR, such that $\Omega_1 = \Omega_2$ here. A similar result can be plotted by changing the local average SNR ratio m. In

Fig. 4, we keep the angle of arrival θ constant and relate the minimum antenna separation to the beamwidth Δ. Then in Fig. 5, we keep the beamwidth Δ constant and relate the minimum antenna separation to the angle of arrival θ. Notice that both Fig. 4 and Fig. 5 are generated analytically by the use of (20), not by simulation. Those plots are useful to design the antenna separation of a practical system. For example, if we know both the beamwidth Δ of the scatters and the angle of arrival θ are greater than 30^o, then 1.3λ of antenna separation is sufficient for providing 90% of the available diversity gain.

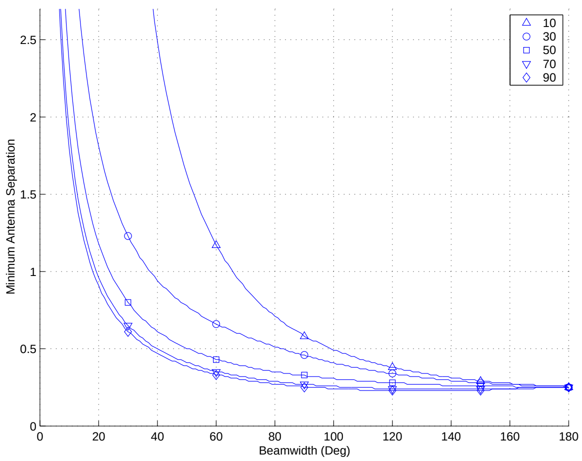

Fig. 4. The figure show the minimum antenna separation required (in wavelength) to achieve 90% of the available average SNR diversity gain. The angle of arrival $\theta = \{10^o, 30^o, 50^o, 70^o, 90^o\}$, and the beamwidth Δ is from 0^o to 180^o.

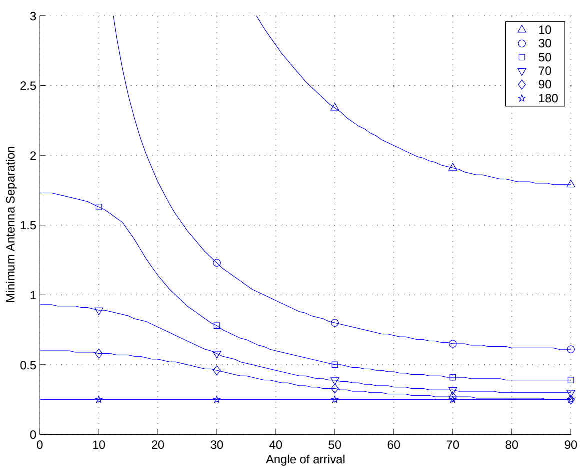

Fig. 5. The figure show the minimum antenna separation required (in wavelength) to achieve 90% of the available average SNR diversity gain. The beamwidth $\Delta = \{10^o, 30^o, 50^o, 70^o, 90^o, 190^o\}$, and the angle of arrival θ is from 0^o to 90^o. Notice that if both the beamwidth Δ of the scatters and the angle of arrival θ are greater than 30^o, then 1.3λ of antenna separation is sufficient for providing 90% of the available diversity gain.

V. Conclusion

We derived an analytical expression of the average SNR diversity gain in a dual selection combining system under slow Rayleigh fading and related this to the spatial correlation of the multipath field. This expression enabled us to analytically relate the diversity gain to critical system parameters such as beamwidth, angle of arrival and antenna separation. The new analytical approach should generalize well to correlated Nakagami-m fading channels. This is the subject of current investigations and we will report it shortly.

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