SPATIAL CORRELATION IN NON-ISOTROPIC SCATTERING SCENARIOS

Paul D. Teal *

Industrial Research Limited
PO Box 31-310, Lower Hutt
NEW ZEALAND
p.teal@irl.cri.nz

Thushara D. Abhayapala†, *Rodney A. Kennedy*

Department of Telecommunications
RSISE, Institute of Advanced Studies
The Australian National University
ACT 200, AUSTRALIA

ABSTRACT

The well known results of the spatial correlation function for 2-dimensional and 3-dimensional diffuse fields of narrowband signals are generalised to the case of general distributions of scatterers. A method is presented which allows closed form expressions for the correlation function to be obtained for arbitrary scattering distribution functions. These closed form expressions are derived for a variety of commonly used scattering distribution functions.

1. INTRODUCTION

THERE is growing interest in the literature in the use of multiple sensors - particularly multiple antennas for transmission and/or reception of wireless signals. As well as diversity reception [1], this includes such areas as Multiple-Input Multiple Output (MIMO) systems, spatio-temporal equalisation, adaptive arrays and space-time coding [2]. Most of the work assumes that each receiving sensor receives uncorrelated signals, and conversely that the signal received from each transmitting source is uncorrelated. A widely used "rule" is that half a wavelength separation is required in order to obtain de-correlation. This arises from the first null of the sinc$(\cdot)$ function, which is the spatial correlation function for a three dimensional diffuse field [3].

Several approaches have been used in the case of signals confined to a limited azimuth and/or elevation [4]. In this letter a modal analysis approach is presented which can be used to obtain closed form expressions for the spatial correlation function for narrowband signals for a wide variety of scattering distribution functions.

2. SPATIAL CORRELATION FORMULATION

Consider two sensors located at points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Let $s_1(t)$ and $s_2(t)$ denote the complex envelope of the received signal at two sensors, respectively. Then the normalized spatial correlation function between the complex envelopes of the two received signals, is defined by

$$\rho(\boldsymbol{x}_2 - \boldsymbol{x}_1) = \frac{E\{s_1(t)s_2^*(t)\}}{E\{s_1(t)s_1^*(t)\}} \tag{1}$$

where $E\{\cdot\}$ denotes the expectation operator and $\cdot^*$ denotes complex conjugation. We consider a general scattering environment with a large number of scatters distributed sufficiently far from the two sensors. If the transmitted signal is a narrowband signal $e^{i\omega t}$,

*This work was performed while the author was at The Australian National University.

†T. D. Abhayapala has a joint appointment with the Faculty of Engineering and Information Technology, ANU

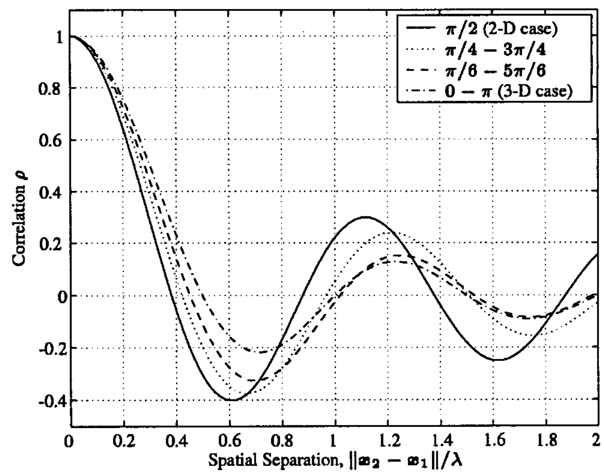

Fig. 1. Spatial Correlation versus Separation for different elevation ranges. This figure shows that the spread of interference in elevation plays only a secondary role in influencing spatial correlation and hence diversity within the horizontal plane.

where ω is the angular frequency, $i = \sqrt{-1}$ and t is the time, then the received signal at the lth receiver is

$$s_l(t) = e^{i\omega t} \int A(\hat{\boldsymbol{y}}) e^{-i\frac{\omega}{c}\boldsymbol{x}_n \cdot \hat{\boldsymbol{y}}} d\hat{\boldsymbol{y}}, \; l = 1, 2 \tag{2}$$

where c is the speed of wave propagation, $\hat{\boldsymbol{y}}$ is a unit vector pointing in the direction of wave propagation, and $A(\hat{\boldsymbol{y}})$ is the *complex gain* of scatterers as a function of direction which captures both the amplitude and phase distribution. Also note that the integration in (2) is over a unit sphere in the case of a 3-dimensional multipath environment or unit circle in the 2-dimensional case. We substitute (2) in (1) and assume that scattering from one direction is independent from another direction, to get

$$\rho(\boldsymbol{x}_2 - \boldsymbol{x}_1) = \int \mathcal{P}(\hat{\boldsymbol{y}}) e^{ik(\boldsymbol{x}_2 - \boldsymbol{x}_1)\cdot\hat{\boldsymbol{y}}} d\hat{\boldsymbol{y}}, \tag{3}$$

where $k = \omega/c$ is the wave number, and

$$\mathcal{P}(\hat{\boldsymbol{y}}) = \frac{E\{|A(\hat{\boldsymbol{y}})|^2\}}{\int E\{|A(\hat{\boldsymbol{y}})|^2\} d\hat{\boldsymbol{y}}}, \tag{4}$$

is the normalized average power of a signal received from direction $\hat{\boldsymbol{y}}$, or the *distribution function* of scatterers over all angles.

0-7803-7402-9/02/$17.00 ©2002 IEEE

3. 3 DIMENSIONAL SCATTERING ENVIRONMENT

To gain a better understanding of spatial correlation, we now use a spherical harmonic expansion of plane waves, which is given [5] as

$$e^{ik\boldsymbol{x}\cdot\hat{\boldsymbol{y}}} = 4\pi\sum_{n=0}^{\infty}(-i)^n j_n(k\|\boldsymbol{x}\|)\sum_{m=-n}^{n} Y_{nm}(\hat{\boldsymbol{x}})Y_{nm}^*(\hat{\boldsymbol{y}}) \quad (5)$$

where $\hat{\boldsymbol{x}} = \boldsymbol{x}/\|\boldsymbol{x}\|$, $j_n(r) \triangleq \sqrt{\pi/2r}J_{n+\frac{1}{2}}(r)$ are spherical Bessel functions, and

$$\begin{aligned} Y_{nm}(\hat{\boldsymbol{x}}) &\equiv Y_{nm}(\theta_x,\phi_x) \\ &\triangleq \sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta_x)e^{im\phi_x}, \end{aligned} \quad (6)$$

where θ_x and ϕ_x are the elevation and azimuth respectively of the unit vector $\boldsymbol{x}$, and where $P_n^m(\cdot)$ are the associated Legendre functions of the first kind. Equations (3) and (5) may be combined to obtain

$$\begin{aligned} \rho(\boldsymbol{x}_2-\boldsymbol{x}_1) = 4\pi\sum_{n=0}^{\infty}(-i)^n j_n(k\|\boldsymbol{x}_2-\boldsymbol{x}_1\|)\sum_{m=-n}^{n}\beta_{nm} \\ Y_{nm}\left(\frac{\boldsymbol{x}_2-\boldsymbol{x}_1}{\|\boldsymbol{x}_2-\boldsymbol{x}_1\|}\right) \end{aligned} \quad (7)$$

where

$$\beta_{nm} = \int \mathcal{P}(\hat{\boldsymbol{y}})Y_{nm}^*(\hat{\boldsymbol{y}})d\hat{\boldsymbol{y}}. \quad (8)$$

The fact that the higher order spherical Bessel functions, (and in section 4 the higher order Bessel functions) have small values for arguments near zero, means that to evaluate the correlation for points near each other in space, only a few terms in the sum need to be evaluated in order to obtain a very good approximation.

3.1. 3D Omni-directional Diffuse Field

If waves are incident on the two points from all directions in 3-dimensional space, then (7) reduces to a single term, and so the correlation coefficient is given by $\rho = j_0(k\|\boldsymbol{x}_2-\boldsymbol{x}_1\|) = \mathrm{sinc}(k\|\boldsymbol{x}_1-\boldsymbol{x}_2\|)$, where $\mathrm{sinc}(x) \triangleq \sin(x)/x$ for $x \neq 0$, and $\mathrm{sinc}(0) \triangleq 1$. This is the classical result [3]. The first zero crossing is at $\lambda/2$.

3.2. Uniform Limited Azimuth/Elevation Field

Without loss of generality, the coordinate system may be chosen so that $\theta_{x_2-x_1} = \pi/2$ and $\phi_{x_2-x_1} = 0$. If the scatters are uniformly distributed over the sector $\Omega \in \{(\theta,\phi); \theta \in [\theta_1,\theta_2], \phi \in [\phi_1,\phi_2]\}$, then the correlation can be expressed as

$$\begin{aligned} \rho(\boldsymbol{x}) = &\frac{1}{(\cos\theta_1-\cos\theta_2)(\phi_2-\phi_1)}\sum_{n=0}^{\infty}(-i)^n j_n(k\|\boldsymbol{x}\|) \\ &\Bigg[(\phi_2-\phi_1)P_n(0)\int_{\theta_1}^{\theta_2}P_n(\cos\theta)\sin\theta\,d\theta \\ &+2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}\frac{\sin(m\phi_2)-\sin(m\phi_1)}{m}P_n^m(0) \\ &\int_{\theta_1}^{\theta_2}P_n^m(\cos\theta)\sin\theta\,d\theta\Bigg] \end{aligned} \quad (9)$$

Note that (9) is actually expressible in closed form (the integrals can be evaluated for example by using recursions in m and n). Hence, this result can be used to build up the result for a general scattering situation. An arbitrary scattering can be regarded as the limiting summation of a weighted set of uniformly distributed incremental solid angle contributions.

The case where energy is arriving from all azimuth directions (uniformly) but the elevation spread is in some range of angles either side of zero elevation is shown in Fig. 1. The spatial correlation is shown for four sets of elevation spread: π, $\pi/2$, $\pi/3$ and 0, each centred on $\theta = \pi/2$. Generally we can conclude that multipath spread in elevation does not have a great effect on spatial correlation in the horizontal plane. Multipath which may have some small elevation spread (in many practical situations we would not expect much) may be modeled as only coming from the horizontal plane.

3.3. Spherical Harmonic Model

The distribution function $\mathcal{P}$ in (4) may be expressed as a weighted sum of orthonormal spherical basis functions

$$\mathcal{P}(\hat{\boldsymbol{y}}) = \sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'}\gamma_{n'm'}Y_{n'm'}(\hat{\boldsymbol{y}}), \quad (10)$$

where the coefficients $\gamma_{n'm'}$ are chosen so that the distribution function is normalised. In many cases the number of basis functions required to approximate the distribution function may be quite small. By substitution into (8) β_{nm} can be simply expressed in terms of the coefficients $\gamma_{n'm'}$ as

$$\begin{aligned} \beta_{nm} &= \int\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'}\gamma_{n'm'}Y_{n'm'}(\hat{\boldsymbol{y}})Y_{nm}^*(\hat{\boldsymbol{y}})d\hat{\boldsymbol{y}} \\ &= \gamma_{nm}. \end{aligned} \quad (11)$$

4. 2 DIMENSIONAL SCATTERING ENVIRONMENT

If the fields may be considered as arriving from the azimuthal plane only, it is more useful to consider the 2 dimensional modal expansion [5]

$$e^{ik\boldsymbol{x}\cdot\hat{\boldsymbol{y}}} = \sum_{m=-\infty}^{\infty} i^m J_m(k\|\boldsymbol{x}\|)e^{jm(\phi_x-\phi_y)} \quad (12)$$

where ϕ_x and ϕ_y are the angles of $\boldsymbol{x}$ and $\hat{\boldsymbol{y}}$. Equations (3) and (12) may be combined to obtain

$$\rho(\boldsymbol{x}_2-\boldsymbol{x}_1) = \sum_{m=-\infty}^{\infty}\alpha_m J_m(k\|\boldsymbol{x}_2-\boldsymbol{x}_1\|)e^{im\phi_{12}} \quad (13)$$

where

$$\alpha_m = i^m\int_0^{2\pi}\mathcal{P}(\phi)e^{-im\phi}d\phi, \quad (14)$$

$\mathcal{P}(\phi)$ is the distribution function equivalent to $\mathcal{P}(\hat{\boldsymbol{y}})$ in (4), and ϕ_{12} is the angle of the vector connecting $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Without loss of generality, this can be considered to be zero.

4.1. 2D Omni-directional Diffuse Field

For the special case of scattering over *all* angles in the plane containing two points, (13) reduces to a single term, and so the correlation coefficient is given by $\rho = J_0(k\|\boldsymbol{x}_2 - \boldsymbol{x}_1\|)$), another classical result [1]. As can be seen by an examination of Fig. 1, $J_0(\cdot)$ and $\text{sinc}(\cdot)$ are qualitatively similar.

4.2. Uniform Limited Azimuth Field

In the case of energy arriving uniformly from a restricted range of azimuth $(\phi_0 - \Delta, \phi_0 + \Delta)$,

$$\alpha_m = e^{im(\pi/2-\phi_0)}\,\text{sinc}(m\Delta). \tag{15}$$

This result is equivalent to that derived in [6].

4.3. von-Mises distributed field

Non-isotropic scattering in the azimuthal plane may be modeled by the von-Mises distribution [e.g., 7], for which the density is given by

$$\mathcal{P}(\phi) = \frac{1}{2\pi I_0(\kappa)} e^{\kappa\cos(\phi-\phi_0)}, \quad |\phi - \phi_0| \le \pi, \tag{16}$$

where ϕ_0 represents the mean direction, $\kappa > 0$ represents the degree of non-isotropy, and $I_m(\kappa)$ is the modified Bessel function of the first kind. In this case, using (3.937) of [8]

$$\begin{aligned}\alpha_m &= \frac{i^m}{2\pi I_0(\kappa)}\int_0^{2\pi} e^{\kappa\cos(\phi-\phi_0)} e^{im\phi}\,d\phi \\ &= e^{im(\pi/2-\phi_0)}\frac{I_{-m}(\kappa)}{I_0(\kappa)}.\end{aligned} \tag{17}$$

4.4. $\cos^{2p}\phi$ distributed field

Another azimuthal distribution used for calculation of correlation is [9, 10]

$$\mathcal{P}(\phi) = Q\cos^{2p}\left(\frac{\phi-\phi_0}{2}\right), \quad |\phi - \phi_0| \le \pi, \tag{18}$$

where Q is a normalisation constant. Using (335.19) of [11],

$$\begin{aligned}\alpha_m &= \frac{i^m 2^{2p-1}\Gamma^2(p+1)}{\pi\Gamma(2p+1)}\int_{\phi_0-\pi}^{\phi_0+\pi}\cos^{2p}\left(\frac{\phi-\phi_0}{2}\right)e^{-im\phi}\,d\phi \\ &= e^{im(\pi/2-\phi_0)}\frac{\Gamma^2(p+1)}{\Gamma(p-m+1)\Gamma(p+m+1)}.\end{aligned} \tag{19}$$

4.5. Laplacian distributed field

In [12] the Laplacian distribution is proposed as a realistic model of the power distribution function in some circumstances. Here

$$\mathcal{P}(\phi) = \frac{Q}{\sqrt{2}\sigma}e^{-\sqrt{2}|\phi-\phi_0|/\sigma}, \quad |\phi - \phi_0| \le \frac{\pi}{2}, \tag{20}$$

where Q is a normalisation constant. It is straightforward to show that

$$\alpha_m = e^{im(\pi/2-\phi_0)}\frac{(1-(-1)^{\lceil m/2\rceil}\xi F_m)}{(1+\sigma^2 m^2/2)(1-\xi)} \tag{21}$$

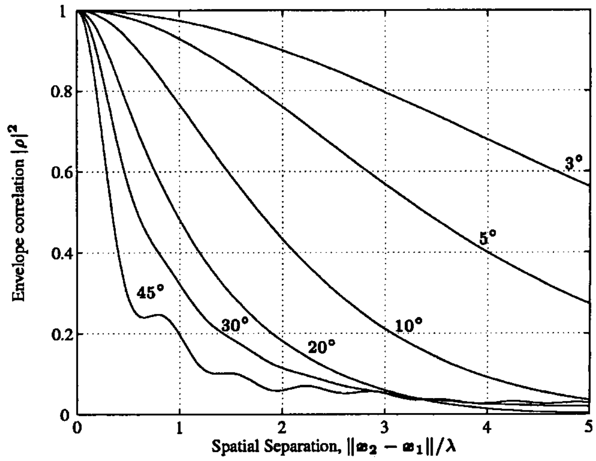

Fig. 2. Correlation for angle of incidence 60° from broadside against separation of sensors with angular spread as parameter, based on a Laplacian power distribution function

where $\xi = e^{\frac{-\pi}{\sqrt{2}\sigma}}$, and $F_m = 1$ for m even, and $F_m = m\sigma/\sqrt{2}$ for m odd.

To demonstrate the power of the technique, Fig. 5b from [13] is reproduced in Fig. 2, except that the distribution used here is the Laplacian, considered in [13] to be realistic but mathematically intractable. Approximately 100 terms of the summation (13) were required to obtain results to about 10 significant figures at the largest spatial separation on the figure (5λ), or 70 terms for 5 significant figures. For separations up to 2λ only 40–50 terms are required. The angular spread used to distinguish between the data sets in the figure is defined as the square root of the variance, which for this distribution is given by

$$S_\sigma^2 = \frac{1}{1-\xi}\left(\sigma^2 - \frac{\xi}{4}(\pi^2 + 4\sigma^2 + \sqrt{8}\pi\sigma)\right). \tag{22}$$

4.6. Gaussian distributed field

Several researchers [4, 10] have used the Gaussian distribution for modelling the distribution of scatterers, thus

$$\mathcal{P}(\phi) = \frac{Q}{\sqrt{2\pi}\sigma}e^{-\frac{(\phi-\phi_0)^2}{2\sigma^2}}, \quad |\phi - \phi_0| \le \frac{\pi}{2}, \tag{23}$$

where σ is the standard deviation, and Q is a normalisation constant. It can be shown using (313.6) of [14], and the symmetries of the error function for complex arguments discussed in [15], that

$$\alpha_m = e^{im(\pi/2-\phi_0)-m^2\sigma^2/2}\frac{\text{Re}\left(\text{erf}\left(\frac{\pi/2+im\sigma^2}{\sqrt{2}\sigma}\right)\right)}{\text{erf}\left(\frac{\pi/2}{\sqrt{2}\sigma}\right)}. \tag{24}$$

It can be seen that contrary to the assertion of [13], a closed form solution for the correlation function is available for distribution functions other than the Gaussian (such as the $\cos^n\phi$ distribution), and that the Gaussian case is neither a simple nor natural choice. If however, the beam is narrow, a good approximation for the Gaussian case can be obtained by performing integration over the domain $(-\infty, \infty)$, since the tails will cause very little error. In this case using (337.3) of [11], it is straightforward to show that

$$\alpha_m \approx e^{im(\pi/2-\phi_0)-m^2\sigma^2/2}. \tag{25}$$

4.7. Cylindric Harmonic Model

The distribution function $\mathcal{P}$ may be expressed as the sum of orthogonal basis functions

$$\mathcal{P}(\phi) = \sum_{k=-\infty}^{\infty} \gamma_k e^{jk\phi}, \tag{26}$$

where the coefficients γ_k are chosen to that the distribution function is normalised. In many cases the number of basis functions required to approximate the distribution function may be quite small. The coefficients α_m in (13) can be simply expressed in terms of the coefficients γ_m as

$$\alpha_m = i^m 2\pi\gamma_m. \tag{27}$$

5. MUTUAL COUPLING

When the sensors are antennas, there can be mutual coupling effects between the sensors. For a given spacing, mutual coupling of terminated antennas can actually *decrease* the correlation between the sensors from that calculated using the expressions above. One interpretation of this is that the presence of other antennas creates a slow-wave structure which in effect decreases the wavelength of the signal in their vicinity, and thus increases the number of wavelengths separation between the elements. This phenomenon was first reported in [16], and has recently been reported in [17]. If it proves true in general that mutual coupling effects decrease correlation, then the closed form expressions of the previous sections may be considered as upper bounds on the correlation between antennas in an electromagnetic field.

6. CONCLUSION

Closed form solutions have been presented for the correlation between the signal at physically separated points in a field with the signal power distributed in solid or planar angle following a variety of commonly used distribution functions. Unlike some other approaches the method can be applied to many distribution functions, and for small spatial separations, accurate results are obtained with very few terms.

Application of the method reveals that restriction of the signal elevation does not have a large impact on the signal correlation coefficient, provided there is omni-directionality in one plane containing the two points. Restriction of the signal in both planes considerably increases signal correlation. In order to exploit diversity for sensors spaced $\lambda/2$ or closer, there must be omni-directionality in at least one plane. To obtain un-correlated signals with directional sensors the spacing must be considerably larger.

7. REFERENCES

[1] W. C. Jakes, Ed., *Microwave Mobile Communications*, John Wiley, New York, 1974.

[2] G. J. Foschini and M. J. Gans, "On limits of wireless communications in a fading environment when using multiple antennas," *Wireless Personal Communications*, vol. 6, pp. 311–335, 1998.

[3] R. K. Cook, R. V. Waterhouse, R. D. Berendt, S. Edelman, and M. D. Thompson, Jr., "Measurement of correlation coefficients in reverberant sound fields," *J. Acoustical Soc. America*, vol. 27, no. 6, pp. 1072–1077, Nov 1955.

[4] M. Kalkan and R. H. Clarke, "Prediction of the space-frequency correlation function for base station diversity reception," *IEEE Trans. Vehic. Tech.*, vol. VT-46, no. 1, pp. 176–184, Feb 1997.

[5] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer-Verlag, Berlin, 1992.

[6] J. Salz and J. H. Winters, "Effect of fading correlation on adaptive arrays in digital mobile radio," *IEEE Trans. Vehic. Tech.*, vol. 42, no. 4, pp. 1049–1057, Nov 1994.

[7] A. Abdi and M. Kaveh, "A versatile spatio-temporal correlation function for mobile fading channels with non-isotropic scatterers," in *IEEE Workshop Stat. Signal Processing*, Pocono Manor, PA, 2000, pp. 58–62.

[8] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, San Diego, Fifth edition, 1994.

[9] R. G. Vaughan, "Spaced Directive Antennas for Mobile Communications by the Fourier Transform Method," *IEEE Trans. Ant. and Prop.*, vol. AP-48, no. 7, pp. 1025–1032, Jul 2000.

[10] R. G. Vaughan, "Pattern translation and rotation in uncorrelated source distributions," *IEEE Trans. Antennas and Propagation*, vol. 46, no. 7, pp. 982–990, Jul 1998.

[11] W. Gröbner and N. Hofreiter, *Integraltafel, Zweiter Teil, Bestimmte Integrale*, Springer-Verlag, Wein and Innsbruck, 1950.

[12] K. L. Pedersen, P. E. Mogensen, and B. H. Fleury, "Power azimuth spectrum in outdoor environments," *IEE Electronics Letters*, vol. 33, no. 18, pp. 1583–1584, 1997.

[13] J. Fuhl, A. F. Molisch, and E. Bonek, "Unified channel model for mobile radio systems with smart antennas," *IEE Proc. - Radar, Sonar Navig.*, vol. 145, no. 1, pp. 32–41, Feb 1998.

[14] W. Gröbner and N. Hofreiter, *Integraltafel, Erster Teil, Unbestimmte Integrale*, Springer-Verlag, Wein and Innsbruck, 1949.

[15] W. Gautschi, "Efficient computation of the complex error function," *SIAM J. Numerical Analysis*, vol. 7, pp. 187–198, 1970.

[16] R. G. Vaughan and N. L. Scott, "Terminated in-line monopoles for vehicular diversity," in *Proc. Int. Union of Radio Science (USRI) Trienniel Symposium on Electromagnetic Theory*, Sydney, Australia, August 1992.

[17] T. Svantesson and A. Ranheim, "Mutual coupling effects on the capacity of multi-element antenna systems," in *IEEE Int. Conf. on Acoustics, Speech and Signal Processing*, Salt Lake City, 2001.