Bounds on the Spatial Richness of Multipath

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Abstract — **In this paper wireless multipath fields are modelled using classes of orthogonal functional expansions based on the solutions to the Helmholtz (wave) equation. These expansions permit a multipath field—generated by any number of nearfield, farfield, specular and diffuse multipath reflections—to be modelled to any precision in a region of interest. Two expansions are provided, one suitable for multipath fields which show no variation with height, and the other suitable for general 3D fields. We establish that the dimensionality of the functional expansion, and thereby the multipath field, scales with the size of the boundary of the region in space. When the region of interest is small in wavelengths, typical of antenna arrays apertures in practice, then the multipath field can be modelled by only a small number, given by the dimensionality, of arbitrarily located sources. Multipath field synthesis can be done using combinations of arbitrarily located distinct farfield, nearfield, point and spatially distributed sources. These results establish rigorous bounds on the spatial richness of multipath.**

I. Introduction

Using the spatial aspects of multipath is an increasingly active thread of research in wireless communications [1]. This has lead to the general notion of space-time receivers which exploit the spatial variations of the signal by using multiple antennas with suitable processing augmented to conventional temporal signal processing techniques. This improves the performance of wireless receivers because of spatial diversity gains. However, the degree of benefit is rarely quantified apart from asymptotic gains obtained when antenna elements are spaced so far apart that one can assume independence and hence full diversity gains. This spatially asymptotic case often bears little relevance in practice. For example, wireless handsets are tending to become so small that they could well pose a choking hazard to small children. For multiple antennas, the case of interest that needs to be studied is when the antenna elements are very close (fraction of a wavelength) which corresponds to a regime where the diversity and capacity gains can be expected to be much less in comparison with the independent antennas case.

A wireless multipath environment can be modelled in a number of ways. It is useful to briefly review these as they contrast with the model that we introduce. The most direct model is to use multiple discrete farfield sources [2,3] where the gains associated with each path are obtained by assuming some physical configuration of scatterers. Other models target the statistical characterization of a multipath environment and have been guided by measurements and experiments. The investigated parameter characteristics include angle of arrival [4], spatial signal correlation [5] and changes in multipath profile [6]. In yet other approaches, geometrical models are used to characterize diffuse multipath fields [7], and in [8] a theoretical model is introduced to derive fading statistics.

II. Multipath Field Expansions

Let $\boldsymbol{x}$ represent a vector in 3D space and let $\|\boldsymbol{x}\|$ denote the euclidean distance of $\boldsymbol{x}$ from the origin which is the centre of some region of interest. Let $\widehat{\boldsymbol{x}} \triangleq \boldsymbol{x}/\|\boldsymbol{x}\|$. Further, let $\phi \in [0, 2\pi)$ represent the azimuth angle and $\theta \in [-\pi/2, \pi/2]$ the elevation. Consider two cases: height invariant multipath (2D) and general 3D multipath.

II.A. Height Invariant Multipath (2D)

First, consider the situation where the multipath is restricted to the horizontal plane, having no components arriving at significant elevations. Without loss of generality, this may be considered as a 2D multipath environment (since the multipath field is height invariant). In this case the use of polar coordinates[2] is natural and we write $\boldsymbol{x} \equiv (\|\boldsymbol{x}\|, \phi)$.

The narrowband (low fractional bandwidth) multipath field, $F(\boldsymbol{x}; k)$, is a complex function of the position, $\boldsymbol{x}$, and the wave number, $k = 2\pi/\lambda$, where λ is the wavelength. It is a solution to the 2D Helmholtz wave equation in polar coordinates [9], and can be written

$$F(\boldsymbol{x}; k) = \sum_{n=-\infty}^{\infty} \alpha_n \, J_n\big(k\|\boldsymbol{x}\|\big) \, e^{in\phi} \qquad (1)$$

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[2] This can be regarded as a cylindrical coordinate representation where the third coordinate, the height, plays a minor role since the field does not change with height. There is no confusion in using $\boldsymbol{x}$ to represent either a 2D space or this special case in 3D space.

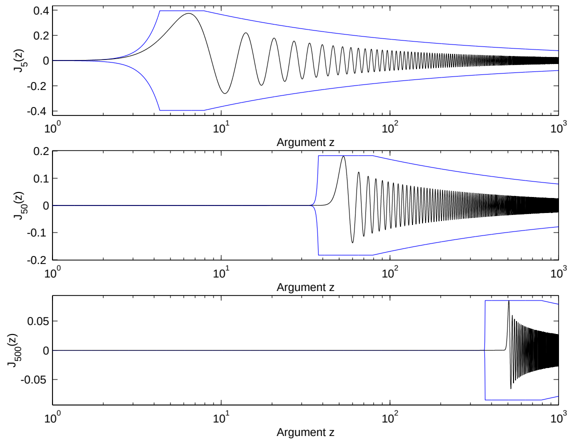

Fig. 1: High pass character of the Bessel functions $J_5(z)$, $J_{50}(z)$ and $J_{500}(z)$ versus argument z (log scale).

where $\alpha_n \in \mathbb{C}$ are complex constants independent of position and $J_n(\cdot)$ is the order n Bessel function [10]. That is, in (1) the field strength at a point $\boldsymbol{x}$ can be represented as a weighted sum of orthogonal basis functions. This is our preferred model for height invariant multipath fields.

Bessel Function Properties and Bounds

Bessel functions $J_n(\cdot)$ for $n \geq 1$ in (1) have a spatial high pass character ($J_0(\cdot)$ is spatially low pass). That is, as illustrated in Fig. 1, for $n \in \{5, 50, 500\}$, $J_n(z)$ starts small increasing monotonically to its maximum at arguments around $O(n)$ before decaying slowly asymptotically to zero as $z \to \infty$. Also shown overlaid in Fig. 1 are limits imposed by three upper bounds on $|J_n(z)|$. The uniform in argument z bounds, $0.6748851/n^{1/3}$, and the uniform n bounds, $0.7857468704/z^{1/3}$, come from the recent work of Landau [11]. The third bound

$$\left|J_n(z)\right| \leq \frac{z^n}{2^n \Gamma(n+1)}, \quad z \geq 0 \tag{2}$$

was proven in [12] is asymptotically tight as $z \to 0$ and is a central inequality for this work.[3]

Superpositions of 2D Sources

A classical multipath model is to model every distinct path explicitly as a plane wave, *viz.*,

$$F(\boldsymbol{x}; k) = \sum_p a_p \, e^{-ik\boldsymbol{x}\cdot\widehat{\boldsymbol{\eta}}_p}, \tag{3}$$

where the plane wave of index p has complex amplitude $a_p \in \mathbb{C}$ and propagation direction $\phi_p \in [0, 2\pi)$ with normalized (zero elevation) direction $\widehat{\boldsymbol{\eta}}_p \equiv (\cos\phi_p, \sin\phi_p)'$. Cartesian form (3) can be brought into the polar form (1) by noting the component plane waves have expansion coefficients [13, p.66]

$$\alpha_n = \alpha_n(a_p, \phi_p) \triangleq a_p \, i^n \, e^{-in\phi_p}. \tag{4}$$

[3]Note $\Gamma(n+1) = n!$ for integer n.

Hence (3) can be written

$$F(\boldsymbol{x}; k) = \sum_p a_p \sum_{n=-\infty}^{\infty} i^n \, J_n\big(k\|\boldsymbol{x}\|\big) e^{in(\phi-\phi_p)} \tag{5}$$

which is in form (1) with

$$\alpha_n \triangleq \sum_p a_p \, i^n \, e^{-in\phi_p} = \sum_p a_p \, e^{-in(\phi_p - \pi/2)}. \tag{6}$$

Similarly, a nearfield source of complex amplitude $a \in \mathbb{C}$ at location $\boldsymbol{y} \equiv (\|\boldsymbol{y}\|, \phi)$ yielding circular waves[4] has expansion coefficients [13, p.66]

$$\alpha_n = \alpha_n(a, \boldsymbol{y}) \triangleq a \, e^{-in\phi} H_n^{(1)}\big(k\|\boldsymbol{y}\|\big), \tag{7}$$

where $H_n^{(1)}(\cdot)$ is the order n Hankel function of the first kind. The superposition of multiple such sources will also be of the form (1).[5]

II.B. General 3D Multipath

To model the case when multipath can come from any range of elevations and azimuths we adopt an expansion in spherical coordinates

$$F(\boldsymbol{x}; k) = \sum_{n=0}^{\infty} j_n\big(k\|\boldsymbol{x}\|\big) \sum_{m=-n}^{n} \alpha_{nm} \, Y_{nm}(\widehat{\boldsymbol{x}}) \tag{8}$$

where $\alpha_{nm} \in \mathbb{C}$ are complex constants independent of position, and

$$j_n(z) \triangleq \sqrt{\frac{\pi}{2r}} J_{n+1/2}(z), \tag{9}$$

are the spherical Bessel functions which can be bounded, using (2), as follows

$$\left|j_n(z)\right| \leq \frac{z^n}{(2n+1)!!} \equiv \frac{\sqrt{\pi}}{2} \frac{1}{\Gamma(n+\frac{3}{2})} \left(\frac{z}{2}\right)^n, \; z \geq 0 \tag{10}$$

where $(2n+1)!! = (2n+1) \cdots 5 \cdot 3 \cdot 1$, and

$$Y_{nm}(\widehat{\boldsymbol{x}}) \triangleq \sqrt{\frac{2n+1}{4\pi} \frac{(n-|m|)!}{(n+|m|)!}} \, P_n^{|m|}(\cos\theta) \, e^{im\phi} \tag{11}$$

are the spherical harmonic functions, which are expressed in terms of the associated Legendre functions $P_n^m(\cos\theta)$. The spherical harmonics are the most natural basis for functions defined on a sphere. A property which is useful for bounding the spherical harmonic functions is the non-trivial identity

$$\sum_{m=-n}^{n} Y_{nm}(\widehat{\boldsymbol{x}}) \, \overline{Y_{nm}(\widehat{\boldsymbol{y}})} = \frac{2n+1}{2\pi} P_n(\widehat{\boldsymbol{x}} \cdot \widehat{\boldsymbol{y}}) \tag{12}$$

$$\leq \frac{2n+1}{2\pi} \tag{13}$$

[4]The sources will need to be line sources in 3D to ensure the field is height invariant and the corresponding waves will be cylindrical.

[5]By linearity of the wave equation, superpositions of arbitrary numbers of fields which are in the form of (1) trivially will also be of the form (1). That all solutions can be put in the form (1) in the appropriate sense is not treated here.

where $\overline{z}$ means the complex conjugate of z, and $P_n(\cos\theta)$ is the Legendre function of order n which is bounded by unity, and θ is the angle between $\widehat{\boldsymbol{x}}$ and $\widehat{\boldsymbol{y}}$.

The spherical harmonics are orthonormal with respect to the inner product

$$\langle f, h\rangle \triangleq \int_0^{\pi}\!\!\int_0^{2\pi} f(\theta,\phi)\,\overline{h(\theta,\phi)}\,\sin\theta\,d\phi\,d\theta. \tag{14}$$

It can be readily verified that $\langle Y_{nm}, Y_{nm}\rangle = \|Y_{nm}\|^2 = 1$ (the L_2-norm), $\langle Y_{nm}, Y_{pq}\rangle = 0$ when $n \neq p$ or $m \neq q$ and $\overline{Y_{nm}} = Y_{n,-m}$. Using this inner product any square-integrable function on the sphere may be expanded in the spherical harmonic basis. The expansion is given by

$$f = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \widetilde{f}(n,m)\,Y_{nm} \tag{15}$$

where $\widetilde{f}(n,m) \triangleq \langle f, Y_{nm}\rangle$ is the Spherical Harmonic Transform of f. Let $F_\rho(\widehat{\boldsymbol{x}}) \equiv F_\rho(\theta,\phi)$ be the field $F(\boldsymbol{x};k)$ on the surface of a sphere of radius ρ, $\|\boldsymbol{x}\| = \rho$, where ρ satisfies $j_n(k\rho) \neq 0$. Then the α_{nm} can be obtained by the following scaled spherical harmonic transform expression:

$$\alpha_{nm} \triangleq \frac{1}{j_n(k\rho)}\langle F_\rho, Y_{nm}\rangle. \tag{16}$$

III. Dimensionality of Multipath

We wish to quantify the complexity of an arbitrary multipath field $F(\boldsymbol{x};k)$ in a region of maximum radius R by defining the effective *dimensionality* of the field. We do this by truncating the series expansion such that $|n| \leq N$ in (1) and $0 \leq n \leq N$ in (8), calling this $F_N(\boldsymbol{x};k)$, and dividing it by a bound on the maximum field intensity to ensure normalization. A bound on the maximum field intensity follows from an alternate decomposition, which generalizes (3). The field $F(\boldsymbol{x};k)$ can be written

$$F(\boldsymbol{x};k) = \int_\Omega a_F(\widehat{\boldsymbol{y}})\,e^{ik\boldsymbol{x}\cdot\widehat{\boldsymbol{y}}}d\widehat{\boldsymbol{y}} \tag{17}$$

where $a_F(\widehat{\boldsymbol{y}})$ is the angular distribution of multipath energy and the integral is over Ω meaning all azimuth directions (in the 2D case).[6] Whence we have the bound

$$\gamma_F \triangleq \int_\Omega \big|a_F(\widehat{\boldsymbol{y}})\big|\,d\widehat{\boldsymbol{y}} \geq \big|F(\boldsymbol{x};k)\big|. \tag{18}$$

The *relative error* is defined by normalizing with respect to γ_F in (18), *viz.*,

$$\varepsilon_N(\boldsymbol{x}) \triangleq \frac{\big|F(\boldsymbol{x};k) - F_N(\boldsymbol{x};k)\big|}{\gamma_F}. \tag{19}$$

The two multipath cases will now be considered; the 3D case in less depth due to space limitations.

[6] This expression works equally well in the 2D and 3D case. In the 3D, Ω represents the unit sphere.

III.A. Height Invariant Multipath Dimensionality (2D)

Here the region of interest is a circular region of radius R/λ wavelengths or $\|\boldsymbol{x}\| \leq R$. Truncating the series in (1) to $2N+1$ terms gives

$$F(\boldsymbol{x};k) \triangleq \sum_{n=-N}^{N} \alpha_n\,J_n\big(k\|\boldsymbol{x}\|\big)\,e^{in\phi}, \tag{20}$$

where using (17) we have the alternate representation

$$\alpha_n = \int_0^{2\pi} a_F(\phi)\,i^n\,e^{-in\phi}\,d\phi. \tag{21}$$

Now from (21) it is clear that $|\alpha_n| \leq \gamma_F$ using (18), and then combining with (19) and (20) we can write

$$\begin{aligned}\gamma_F\cdot\varepsilon_N(\boldsymbol{x}) &= \Big|\sum_{|n|>N} \alpha_n\,J_n\big(k\|\boldsymbol{x}\|\big)\,e^{in\phi}\Big| \\ &\leq \sum_{|n|>N} |\alpha_n|\,\big|J_n\big(k\|\boldsymbol{x}\|\big)\big|. \end{aligned}\tag{22}$$

Hence, the normalized error, (19), can be bounded

$$\varepsilon_N(\boldsymbol{x}) \leq 2\sum_{n>N}\big|J_n\big(k\|\boldsymbol{x}\|\big)\big| \tag{23}$$

$$\leq 2\sum_{n>N}\frac{\big(\pi\|\boldsymbol{x}\|/\lambda\big)^n}{n!}. \tag{24}$$

by applying (2) in (23) with $k = 2\pi/\lambda$ and using the symmetry $\big|J_n(\cdot)\big| = \big|J_{-n}(\cdot)\big|$.

In (24) it is clear that, for a fixed $\boldsymbol{x}$, by taking the truncation depth N large enough we can make $\varepsilon_N(\boldsymbol{x})$ as small as we like. It is also clear that such a choice of N needs to be an increasing function of $\|\boldsymbol{x}\|$. The next result makes this statement more precise.

Theorem 1 (Relative Truncation Error). *Given the relative truncation error, $\varepsilon_N(\boldsymbol{x})$ (19), on a multipath field $F(\boldsymbol{x};k)$ define the critical threshold function*

$$\mathcal{N}(\boldsymbol{x}) \triangleq \lceil e\,\pi\|\boldsymbol{x}\|/\lambda\rceil. \tag{25}$$

Then the following hold:

$$\varepsilon_{\mathcal{N}(\boldsymbol{x})}(\boldsymbol{x}) \leq 0.16127 \tag{26}$$

$$\varepsilon_{\mathcal{N}(\boldsymbol{x})+\Delta}(\boldsymbol{x}) \leq \exp(-\Delta)\,\varepsilon_{\mathcal{N}(\boldsymbol{x})}(\boldsymbol{x}) \tag{27}$$

for any integer $\Delta > 0$. That is, the relative truncation error is small once N equals the critical threshold $\mathcal{N}(\boldsymbol{x})$, and decreases at least exponentially to zero as N increases.

Note that when truncating $F(\boldsymbol{x};k)$ to (20), the number of terms is $2N+1$, hence, we can assert:

Definition 1 (Dimensionality of 2D Multipath). *For a circular region in space given by $\|\boldsymbol{x}\| \leq R$ the maximum* dimensionality of 2D multipath *is given by*

$$\mathcal{D}_R \triangleq 2\lceil e\,\pi\,R/\lambda\rceil + 1 \approx 17.079\,R/\lambda. \tag{28}$$

We make a few comments on the interpretation of dimensionality before giving a proof of Theorem 1:

1) The dimensionality, $\mathcal{D}_R$, *bounds* the spatial richness of multipath and increases *only linearly* with the size of the region R/λ (in wavelengths). Hence the dimension scales only with the size of the region's boundary (the circumference) and not the area.

2) The dimension increases if greater precision is desired. What (28) is really specifying is a *threshold* when the truncation error is small and becoming exponentially smaller with the truncation depth.

3) A vector of α's of dimension $\mathcal{D}_R$ specifies the given field $F(\boldsymbol{x}; k)$. Hence, any given field can be *synthesized using linear algebra* by superimposing $\mathcal{D}_R$ distinct sources.

Proof. Defining the non-negative scalar $z \triangleq \pi\|\boldsymbol{x}\|/\lambda$, bound (24) can be written as

$$\varepsilon_N(\boldsymbol{x}) \le 2\,\mathcal{R}_N(z) \tag{29}$$

where

$$\mathcal{R}_N(z) \triangleq \sum_{n>N} \frac{z^n}{n!} = \exp(z) - \sum_{n=0}^{N} \frac{z^n}{n!} \tag{30}$$

$$= \frac{z^{N+1}}{(N+1)!}\left(\sum_{n=0}^{\infty} \frac{(N+1)!}{(N+1+n)!} z^n\right). \tag{31}$$

By Taylor's theorem, for some $\eta_N \in [0,1]$,

$$\sum_{n=0}^{\infty} \frac{(N+1)!}{(N+1+n)!} z^n = \exp(\eta_N\, z) \tag{32}$$

which is upper bounded by $\exp(z)$. A tighter bound than $\exp(z)$ when z is sufficiently large is derived next.

Note that for integer $n \ge 0$

$$\frac{(N+1)!}{(N+1+n)!} z^n \le \left(\frac{z}{N+2}\right)^n.$$

Hence, for integer $N \ge 0$ satisfying $N > z-2$ we can upper bound (32) by a the sum of a geometric series, *viz.*,

$$\exp(\eta_N\, z) \le \left(\frac{N+2}{N+2-z}\right) \tag{33}$$

Combining (31), (32) and (33) we have for any integer $N \ge 0$ satisfying $N > z-2$

$$\mathcal{R}_N(z) \le \frac{z^{N+1}}{(N+1)!}\left(\frac{N+2}{N+2-z}\right) \tag{34}$$

$$\le \frac{1}{\sqrt{2\pi(N+1)}}\left(\frac{e\,z}{N+1}\right)^{N+1}\left(\frac{N+2}{N+2-z}\right) \tag{35}$$

$$\le \frac{\exp(e\,z - N - 1)}{\sqrt{2\pi(N+1)}}\left(\frac{N+2}{N+2-z}\right) \tag{36}$$

where the second inequality, (35), follows from the Stirling lower bound, $n! > \sqrt{2\pi n}\, n^n\, e^{-n}$, and the third inequality, (36) follows from $(1+x/n)^n \le e^x$, $n \ne 0$ (with equality in the limit $n \to \infty$).

To contain the exponential in (35) we require $N \sim e\,z$ or greater and this motivates the choice of critical threshold (25) given $z \equiv \pi\|\boldsymbol{x}\|/\lambda$. That is, for $z > 0$, we can define a variation of *critical threshold* (25), by

$$\mathcal{N}_z \triangleq \lceil e\,z \rceil \tag{37}$$

from which it can be shown

$$\left(\frac{\mathcal{N}_z + 2}{\mathcal{N}_z + 2 - z}\right) \le \frac{e}{e-1} = 1.58197... \tag{38}$$

and

$$\exp(e\,z - \mathcal{N}_z - 1) \le 1/e = 0.36787.... \tag{39}$$

Then using (37), (38) and (39) in bound (36)

$$\mathcal{R}_{\mathcal{N}_z}(z) \le \frac{1}{(e-1)\sqrt{2\pi(\mathcal{N}_z+1)}} = \frac{0.23217...}{\sqrt{\mathcal{N}_z+1}} \tag{40}$$

The piecewise nature of $\mathcal{R}_{\mathcal{N}_z}(z)$, as a function of z, implies there are local maxima at $z \le N/e$. By searching over these local maxima we can use the exact expression (31) to obtain a uniform tight bound:

$$\mathcal{R}_{\mathcal{N}_z}(z) \le \max_N \mathcal{R}_N(N/e) = \mathcal{R}_2(2/e)$$

$$= \exp\left(\frac{2}{e}\right) - \left(1 + \frac{2}{e} + \frac{2}{e^2}\right) = 0.080635... \tag{41}$$

which improves on (40) when $\mathcal{N}_z \le 7$. Fig. 2 displays the truncation error bounds for (31), (37) and (41).

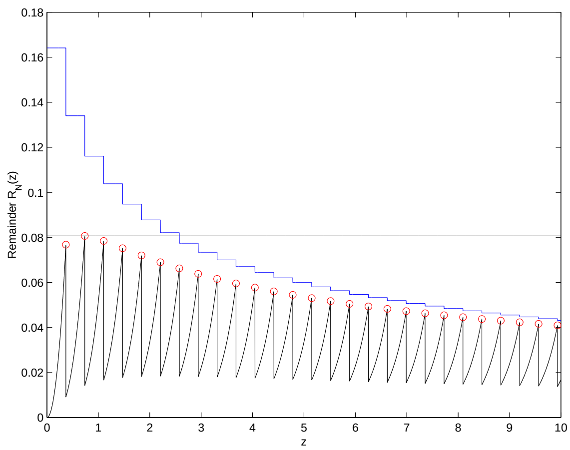

Fig. 2: Remainder $\mathcal{R}_N(z)$, (31), with $N = \mathcal{N}_z \equiv \lceil e\,z \rceil$, (37). The stepped curve is the bound is given by (40) and the uniform bound corresponds to (41).

When $N \ge \mathcal{N}_z$, which we refer to as the *critical regime*, we infer from (41) that $\mathcal{R}_N \le 0.080635...$. Therefore, using (29), $\varepsilon_N(\boldsymbol{x}) \le 0.16127$ which establishes (26), and from (37) and (40) we can show $\mathcal{R}_{\mathcal{N}_z}(z) \le O(z^{-\frac{1}{2}})$ as $z \to \infty$. Therefore, the critical threshold defines the point where N is sufficiently large such that the remainder $\mathcal{R}_N(z)$ is small.

Now we show that $\mathcal{R}_N(z)$, for fixed z, exponentially decreases as N increases provided $N > z-2$. From the definition of $\exp(\eta_N\, z)$ in (32) we observe

$$\exp(\eta_{N+\Delta}\, z) \le \exp(\eta_N\, z) \tag{42}$$

for any integer $\Delta \geq 0$, and hence from (32) the ratio

$$\frac{\mathcal{R}_{N+\Delta}(z)}{\mathcal{R}_N(z)} = \frac{z^{N+\Delta+1}}{(N+\Delta+1)!} \cdot \frac{(N+1)!}{z^{N+1}} \cdot \frac{\exp(\eta_{N+\Delta}\, z)}{\exp(\eta_N\, z)}$$
$$\leq \frac{z^\Delta}{(N+2)(N+3)\cdots(N+\Delta+1)} \qquad (43)$$
$$\leq \Big(\frac{z}{N+2}\Big)^\Delta = \frac{1}{\alpha^\Delta}\Big|_{\alpha=(N+2)/z}. \qquad (44)$$

Therefore, whenever $N > z-2$, we have $\alpha > 1$ and the remainder $\mathcal{R}_{N+\Delta}(z)$ decreases exponentially as Δ increases.

In the critical regime, $N \geq \mathcal{N}_z$, where $\alpha > e$, the exponential decrease is at least as fast as $\exp(-\Delta)$ by (44). That is, at worst, we have $\log_{10} e \approx 0.43429$ orders of magnitude reduction per increment in Δ. For small z the decrease is considerably faster. For example, $z = 3.3$ implies the critical threshold is $\mathcal{N}_z = 9$ and $\Delta = 1, 2, 3, 4$, and 5 give, using (43), the upper bounds on the ratios are $0.27, 0.068, 0.016, 0.0034$, and 0.00067, respectively. □

III.B. General 3D Multipath Dimensionality

Here the region of interest is a spherical region of radius R/λ wavelengths or $\|\boldsymbol{x}\| \leq R$. Conceptually the results for the 3D case, where the truncation takes the form

$$F_N(\boldsymbol{x};k) = \sum_{n=0}^{N} j_n\big(k\|\boldsymbol{x}\|\big) \sum_{m=-n}^{n} \alpha_{nm}\, Y_{nm}(\theta,\phi), \qquad (45)$$

are more difficult but qualitatively similar to the 2D case; and will only be sketched here. From (19)

$$\gamma_F \cdot \varepsilon_N(\boldsymbol{x}) = \Big| \sum_{n>N} j_n\big(k\|\boldsymbol{x}\|\big) \sum_{m=-n}^{n} \alpha_{nm}\, Y_{nm}(\widehat{\boldsymbol{x}}) \Big|$$
$$\leq \sum_{n>N} \big|j_n\big(k\|\boldsymbol{x}\|\big)\big| \, \Big| \sum_{m=-n}^{n} \alpha_{nm}\, Y_{nm}(\widehat{\boldsymbol{x}}) \Big|. \qquad (46)$$

Again we need to implement some form of normalization, and we will rely on a generalization of (21) given by

$$\alpha_{nm} \equiv \int_\Omega a(\widehat{\boldsymbol{y}})\, i^n\, \overline{Y_{nm}(\widehat{\boldsymbol{y}})}\, d\widehat{\boldsymbol{y}}. \qquad (47)$$

Then, using (12) and (13) we see

$$\Big| \sum_{m=-n}^{n} \alpha_{nm}\, Y_{nm}(\widehat{\boldsymbol{x}}) \Big| = \Big| \frac{2n+1}{2\pi} \int_\Omega a(\widehat{\boldsymbol{y}})\, i^n\, P_n\big(\widehat{\boldsymbol{x}} \cdot \widehat{\boldsymbol{y}}\big)\, d\widehat{\boldsymbol{y}} \Big|$$
$$\leq \frac{2n+1}{2\pi} \cdot \gamma_F \qquad (48)$$

and this can be used in (46). Hence, the normalized error, (19), can be bounded

$$\varepsilon_N(\boldsymbol{x}) \leq \frac{1}{2\pi} \sum_{n>N} (2n+1) \big|j_n\big(k\|\boldsymbol{x}\|\big)\big| \qquad (49)$$
$$\leq \frac{1}{2\pi} \sum_{n>N} \frac{\big(2\pi\|\boldsymbol{x}\|/\lambda\big)^n}{(2n-1)!!} \qquad (50)$$

by applying (10) with $k = 2\pi/\lambda$. This is analogous to (24) for the 2D case. Again the same style of proof techniques can be applied as used for Theorem 1. Note that when truncating $F(\boldsymbol{x};k)$ to the form given by (45), the number of terms is $(N-1)^2$, hence, we can assert:

Definition 2 (Dimensionality of 3D Multipath). *For a spherical region in space given by $\|\boldsymbol{x}\| \leq R$ the maximum* dimensionality of 3D multipath *is given by*

$$\mathcal{D}_R \triangleq \big(\lceil e\,\pi\, R/\lambda \rceil - 1\big)^2 \approx 72.923\,(R/\lambda)^2. \qquad (51)$$

IV. Conclusions

A multipath model was developed that can quantify spatial richness. The spatial variation of multipath was modelled by a set of orthogonal functions in space which correspond to elementary solutions to the Helmholtz (wave) equation. We show that only a bounded number of terms are needed to model *any* multipath field to a given precision. As such the spatial variation of multipath cannot be arbitrarily complex and this has important ramifications for a number of problems such as: spatially extrapolating multipath signals; limiting the number of parameters in an adaptive receiver; determining the well-posedness of direction of arrival estimation, etc.

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