

# Analytical Expression for Average SNR of Correlated Dual Selection Diversity System

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**Abstract** — We use a recently developed expression for the bivariate Rayleigh cumulative distribution function (CDF) to derive a novel analytical formula for the average signal-to-noise ratio (SNR) of the correlated dual selection combining (SC) system under slow Rayleigh fading. We relate the average SNR diversity gain to spatial correlation as well as to antenna separation. Furthermore, we also develop a more practical channel model in which multipath scatters are from more than one incident angle, each with a certain beamwidth. For any scattering field from our model, the average SNR diversity gain can be easily related to the antenna separation. Finally, we corroborate our theory with a simulation study.

## I. INTRODUCTION

The presence of multipath scatters in wireless communication channels generally degrades the performance of wireless systems significantly [1]. With the use of multiple antennas, space diversity techniques can combat this degradation caused by multipath scatters [2].

The general idea of space diversity is simple. On the receiving side, antennas are placed in different locations so that the received signals at each location will generally have different multipath signatures. Since more than one version of the transmitted signal are received with differing levels and degrees of distortions, we can in principle combine these signals to achieve some diversity performance gain over a single receiving antenna. The most common combining methods are maximal ratio combining (MRC), equal gain combining (EGC) and selection combining (SC) [1, 3]. The first two diversity schemes require co-phasing on each received signal before combining the signals, while SC does not.

One of the meaningful diversity performance metrics is the average SNR diversity gain. An analytical expression for it was determined under the statistical assumption that the received signals from different antennas are independent [4]. However, this assumption generally (but not necessarily) requires that the antennas be sufficiently

well separated from each other and this is not always feasible in practice. Therefore, it is important to understand how the correlation between received signals affects the diversity gain. That is, the objective is to better quantitatively understand how the degree of correlation relates to the degree of diversity gain. Practically, it is of great interest to relate the separation of antennas to diversity gain. Both questions are considered in this paper, particularly for the correlated dual selection combining diversity (SC) system under slow Rayleigh fading.

In this paper, we first report the recently determined expression for the bivariate Rayleigh CDF [5]. Then, with a new CDF expression developed in this paper, we derive a novel analytical formula for the average SNR of the correlated dual SC system. The average SNR diversity gain is then related to spatial correlation. Moreover, we develop a more practical channel model in which multipath scatters are from more than one incident angle, each with a certain beamwidth. For the diffuse scattering field and a particular scattering field from our model, the average SNR diversity gain is plotted against the antenna separation. At the end of the paper, we corroborate the analytical results with simulations.

## II. PROBLEM FORMULATION

In a dual (two channel or two antenna) selection combining system, the antenna that has the higher SNR, at any time instant, is selected and the signal from other antenna is not used. As in most papers in the literature, the sum of the multipath signals is assumed to be complex Gaussian distributed due to the use of the central limit theorem [1, 6, 7]. The correlated multiplicative complex Gaussian processes on the two receivers are also assumed to be jointly Gaussian [1, 3].

Define  $z \triangleq \max\{y_1^2, y_2^2\}$ , where  $y_i^2$  is the local mean SNR,  $y_i^2 = |s_i|^2/(2N_i)$ ,  $s_i$  is the signal and  $N_i$  is the two sided noise power spectral density, of the  $i$ th antenna. The joint PDF of  $y_1$  and  $y_2$  at any instant [3, 7] can be expressed in terms of the modified Bessel function as fol-

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lows

$$f(y_1, y_2) = \frac{4y_1y_2}{\Omega_1\Omega_2(1-|\rho|^2)} I_0\left(\frac{2|\rho|y_1y_2}{(1-|\rho|^2)\sqrt{\Omega_1\Omega_2}}\right) \times \exp\left(-\frac{1}{(1-|\rho|^2)}(y_1^2/\Omega_1 + y_2^2/\Omega_2)\right) \quad (1)$$

where  $\Omega_i = E\{y_i^2\}$ ,  $\rho = E\{s_1s_2^*\}/E\{s_1s_1^*\}$ ,  $E\{\cdot\}$  denotes the expectation operator, and  $*$  denotes complex conjugation. Furthermore, the probability that the output SNR  $z$  is below some value (which is the same as the CDF) is defined as

$$\Pr(z < \gamma) = \Pr(y_1 < \sqrt{\gamma}, y_2 < \sqrt{\gamma}). \quad (2)$$

With the use of infinite series representations as in [5], the CDF becomes

$$\Pr(z < \gamma) = (1-|\rho|^2) \sum_{k=0}^{\infty} \frac{|\rho|^{2k}}{(k!)^2} \times G\left(k+1, \frac{\gamma}{(1-|\rho|^2)\Omega_1}\right) G\left(k+1, \frac{\gamma}{(1-|\rho|^2)\Omega_2}\right), \quad (3)$$

where  $G(n, x) = \int_0^x t^{n-1}e^{-t}dt$ ,  $n > 0$ , is the incomplete Gamma function [8, (8.35)]. This expression contrasts the classical results which are expressed in terms of the Marcum  $Q$ -function [1, 7]. With this new expression, we can derive a novel expression for the average SNR of the correlated dual SC system under slow Rayleigh fading, which is shown in the next section.

### III. DERIVATION OF AVERAGE SNR

Based on the expression of CDF (3), we now derive the expression for the average SNR of the correlated dual SC system. The expected value of the SNR is defined as

$$E_z\{\gamma\} = \int_0^{\infty} \gamma f_z(\gamma) d\gamma, \quad (4)$$

where  $f_z(\gamma)$  is the PDF of  $z$ . To simplify notation, let

$$\xi_i(\gamma) \triangleq \frac{\gamma}{(1-|\rho|^2)\Omega_i}, \quad i \in \{1, 2\}. \quad (5)$$

We can calculate the PDF by differentiating the CDF in (3) with respect to  $\gamma$ , as follows

$$\begin{aligned} f_z(\gamma) &\triangleq \frac{d}{d\gamma} \Pr(z < \gamma) \\ &= \sum_{k=0}^{\infty} \frac{(1-|\rho|^2)|\rho|^{2k}}{(k!)^2} \times \\ &\quad \frac{d}{d\gamma} \left\{ \int_0^{\xi_1(\gamma)} t^k e^{-t} dt \int_0^{\xi_2(\gamma)} t^k e^{-t} dt \right\}. \end{aligned} \quad (6)$$

After some simplifications, the PDF will be equal to

$$\begin{aligned} f_z(\gamma) &= \sum_{k=0}^{\infty} \left\{ \frac{|\rho|^{2k} \gamma^k e^{-\xi_1(\gamma)}}{(1-|\rho|^2)^k (k!)^2 \Omega_1^{k+1}} G(k+1, \xi_2(\gamma)) \right. \\ &\quad \left. + \frac{|\rho|^{2k} \gamma^k e^{-\xi_2(\gamma)}}{(1-|\rho|^2)^k (k!)^2 \Omega_2^{k+1}} G(k+1, \xi_1(\gamma)) \right\} \end{aligned} \quad (7)$$

The expected value of the SNR becomes

$$E_z\{\gamma\} = \sum_{k=0}^{\infty} \frac{|\rho|^{2k}}{(1-|\rho|^2)^k (k!)^2} (B_k + C_k) \quad (8)$$

where

$$B_k \triangleq \frac{1}{\Omega_1^{k+1}} \int_0^{\infty} \gamma^{k+1} e^{-\xi_1(\gamma)} G(k+1, \xi_2(\gamma)) d\gamma \quad (9)$$

$$C_k \triangleq \frac{1}{\Omega_2^{k+1}} \int_0^{\infty} \gamma^{k+1} e^{-\xi_2(\gamma)} G(k+1, \xi_1(\gamma)) d\gamma. \quad (10)$$

From [8, (6.455)],

$$\begin{aligned} \int_0^{\infty} x^{\mu-1} e^{-\beta x} G(v, \alpha x) dx &= \frac{\alpha^v \Gamma(\mu+v)}{v(\alpha+\beta)^{\mu+v}} \times \\ &\quad F\left(1, \mu+v; v+1; \frac{\alpha}{\alpha+\beta}\right) \end{aligned} \quad (11)$$

where  $\text{Re}(\alpha+\beta) > 0$ ,  $\text{Re}(\beta) > 0$ ,  $\text{Re}(\mu+v) > 0$  and  $F(\cdot, \cdot; \cdot; \cdot)$  is the Gauss hypergeometric function. Therefore, substituting (11) into (9) and (10), we obtain

$$B_k = D_k \cdot F\left(1, 2k+3; k+2; \frac{\Omega_1}{\Omega_1+\Omega_2}\right) \quad (12)$$

$$C_k = D_k \cdot F\left(1, 2k+3; k+2; \frac{\Omega_2}{\Omega_1+\Omega_2}\right) \quad (13)$$

where

$$D_k \triangleq \frac{(1-|\rho|^2)^{k+2} (2k+2)! \Omega_1^{k+2} \Omega_2^{k+2}}{(k+1)(\Omega_1+\Omega_2)^{2k+3}}. \quad (14)$$

Furthermore, from [8, (9.14)], we know that

$$F(a, b; c; d) \triangleq \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{d^n}{n!} \quad (15)$$

where  $(\cdot)_n$  is the Pochhammer symbol which is defined by  $(y)_n \triangleq (y-n-1)!/(y-1)!$ . Therefore,

$$B_k = D_k \sum_{n=0}^{\infty} \frac{(2k+2+n)!}{(2k+2)!} \frac{(k+1)!}{(k+1+n)!} \frac{\Omega_1^n}{(\Omega_1+\Omega_2)^n} \quad (16)$$

$$C_k = D_k \sum_{n=0}^{\infty} \frac{(2k+2+n)!}{(2k+2)!} \frac{(k+1)!}{(k+1+n)!} \frac{\Omega_2^n}{(\Omega_1+\Omega_2)^n} \quad (17)$$

Now substitute (16) and (17) into (8); the expected value of the SNR becomes

$$\begin{aligned} E_z\{\gamma\} &= (1-|\rho|^2)^2 \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \left\{ |\rho|^{2k} (2k+n+2) \times \right. \\ &\quad \left. \frac{\Omega_1^{k+2} \Omega_2^{k+2} (\Omega_1^n + \Omega_2^n)}{(\Omega_1+\Omega_2)^{2k+n+3}} \binom{2k+n+1}{k} \right\} \end{aligned} \quad (18)$$

where

$$\binom{p}{n} \triangleq \frac{p!}{n!(p-n)!}$$

Finally, define

$$m \triangleq \frac{\Omega_1}{\Omega_2}. \quad (19)$$

Then the expected value of the SNR becomes

$$E_z\{\gamma\} = (1 - |\rho|^2)^2 \Omega_1 \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \binom{2k+n+1}{k} \times \frac{|\rho|^{2k} (2k+n+2) m^{k+2} (1+m^n)}{(1+m)^{2k+n+3}} \right\}. \quad (20)$$

The average SNR is a function of the absolute value of correlation,  $|\rho|$ , and the local SNR average in the two receivers,  $\Omega_1$  and  $\Omega_2$ . In the context of this paper the correlation is the spatial correlation which means the signals are obtained by detecting them at physically separated points in space. By explicitly relating the spatial correlation to distance we can then explicitly determine how diversity is affected by antenna separation.

#### IV. AVERAGE SNR DIVERSITY GAIN

Now consider the case when each receiver has an identical local SNR average,  $m = 1$ , i.e.,  $\Omega_1 = \Omega_2$ , then (20) becomes

$$E_z\{\gamma\} = (1 - |\rho|^2)^2 \Omega_1 \times \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \binom{2k+n+1}{k} \frac{|\rho|^{2k} (2k+n+2)}{2^{2k+n+2}}. \quad (21)$$

When both received signals are uncorrelated,  $\rho = 0$ , we can immediately see that (21) becomes  $1.5 \Omega_1$  (1.761 dB), which agrees with the result in [1, 4]. When two received signals are fully correlated,  $\rho = 1$ , we obtain, by taking the limit  $\rho \rightarrow 1$  in (21), that  $1 \Omega_1$  (0 dB). That is, there is no average SNR gain when the two received signals are identical, as expected.

Let us now consider for all other spatial correlations coefficients, the average SNR diversity gain is defined as

$$\begin{aligned} G_{\text{dB}} &\triangleq 10 \log_{10} \left\{ \frac{E_z\{\gamma\}}{\Omega_1} \right\} \\ &= 10 \log_{10} \left\{ (1 - |\rho|^2)^2 \times \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \binom{2k+n+1}{k} \frac{|\rho|^{2k} (2k+n+2)}{2^{2k+n+2}} \right\}. \end{aligned} \quad (22)$$

For the numerical calculations, we use only the first 1001 terms of  $k$  and  $n$  as they are sufficient to bound the error (caused by the truncation of the infinite series) to below  $10^{-10}$ . The truncated version of the average SNR diversity gain becomes

$$G_{\text{dB}} \approx 10 \log_{10} \left\{ (1 - |\rho|^2)^2 \times \sum_{k=0}^{1000} \sum_{n=0}^{1000} \binom{2k+n+1}{k} \frac{|\rho|^{2k} (2k+n+2)}{2^{2k+n+2}} \right\}. \quad (23)$$

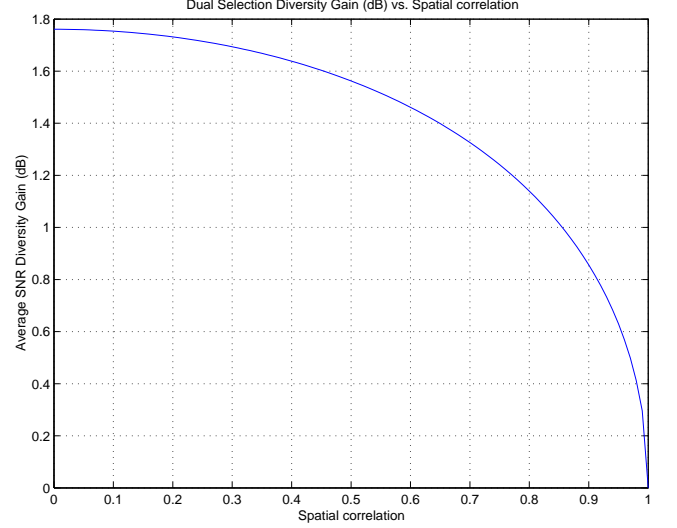


Fig. 1: Average SNR diversity gain in dual SC system under slow Rayleigh fading.

We use (23) to generate Fig. 1. It shows the relationship between the diversity gain and the correlation coefficients.

Again, from Fig. 1, we can check that both trivial cases,  $\rho = 0$  and  $\rho = 1$ , agree with the result in [1, 4]. Now let us relate the average SNR Diversity Gain to the antenna separation. It is well known that if the scatters form a 2D Omni-directional diffuse field, the spatial correlation is given by

$$\rho(d) = J_0(2\pi d/\lambda), \quad (24)$$

where  $J_0(\cdot)$  is the zero order Bessel function of the first kind, and  $d$  is the antenna separation.

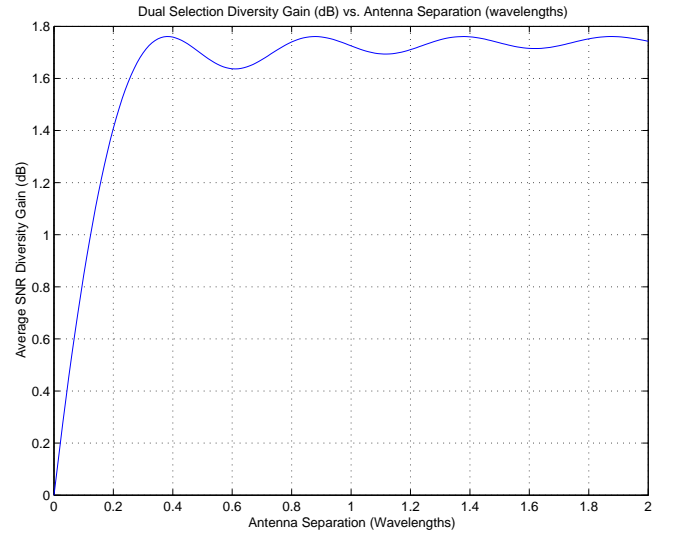


Fig. 2: The relationship between average SNR diversity gain and antenna separation for a 2D diffuse field.

Fig. 2 shows the relationship between diversity gain and antenna separation. Since we know the relationship between diversity and spatial correlation (as shown in Fig. 1, the plot of the diversity gain in Fig. 2 relates to the plot of spatial correlation in Fig. 3, as expected.

It is noticed that slightly under 0.4 wavelength separation is sufficient to have a full diversity gain for a 2D Omni-directional diffuse field. A more practical scattering pattern is discussed in the next section.

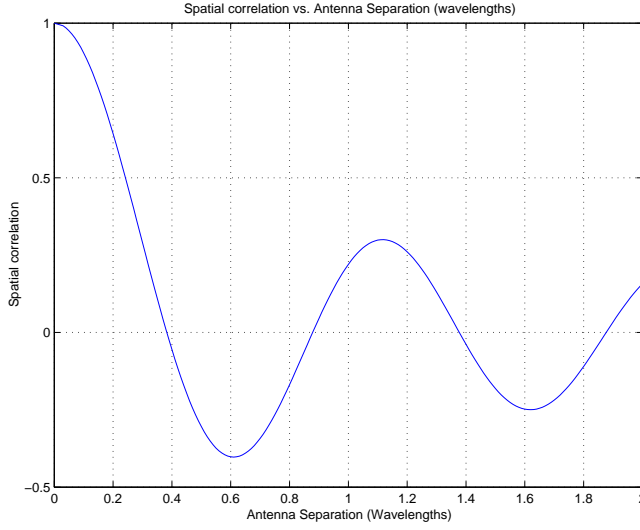


Fig. 3: Spatial correlation between the complex envelopes of the two received signals from two antennas, with  $x$  wavelengths separation to each other.

## V. AVERAGE SNR GAIN IN A PRACTICAL CHANNEL MODEL

In Section IV, we considered scatters, which form a 2D Omni-directional diffuse field. We now generalize the scatters distribution by extending the channel model in [9]. The channel model in [9] assumes multipaths are from a particular incident angle with a certain beamwidth. However, in real environments, multipath scatters are usually reflected from more than one object in different directions. Therefore, our model assumes multipath scatters are uniformly distributed from certain incident angles with certain beamwidths. This model is shown in Fig. 4. We would like, for this more general and practical multipath field channel model, to relate the diversity gain to antenna separation. The received signals from antennas one and two are

$$s_1(t) = e^{i\omega t} \sum_{r=1}^R \int_{\theta_r - \Delta_r/2}^{\theta_r + \Delta_r/2} A_r(\theta) e^{-i\phi_r(\theta)} d\theta \quad (25)$$

$$s_2(t) = e^{i\omega t} \sum_{r=1}^R \int_{\theta_r - \Delta_r/2}^{\theta_r + \Delta_r/2} A_r(\theta) e^{-i\phi_r(\theta)} e^{-i2\pi d \cos \theta / \lambda} d\theta \quad (26)$$

where  $R$  is the number of scattering sources,  $\theta_r$  is the incident angle of the scattering source  $r$ ,  $\Delta_r$  is the angle spread (beamwidth) of the scattering source  $r$ ,  $d$  is the antenna separation in metres,  $A_r(\theta)$  is the signal amplitude of a particular scatter at angle  $\theta$  for source  $r$ , and  $\phi_r(\theta)$  is the signal phase of a particular scatter at angle  $\theta$  for source  $r$ . If there is no overlapping between scattering

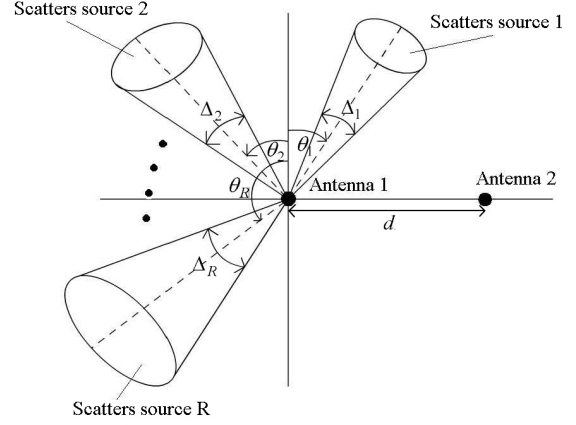


Fig. 4: The 2D multipath scattering model. This model assumes multipath scatters are uniformly distributed from certain incident angles with certain beamwidths.

sources, (25) and (26) can be modified to

$$s_1(t) = e^{i\omega t} \int_0^{2\pi} A(\theta) e^{-i\phi(\theta)} d\theta \quad (27)$$

$$s_2(t) = e^{i\omega t} \int_0^{2\pi} A(\theta) e^{-i\phi(\theta)} e^{-i2\pi d \cos \theta / \lambda} d\theta, \quad (28)$$

where  $A(\theta)$  is the aggregate signal amplitude in analogy to the definition of  $A_r(\theta)$  above. The spatial correlation between antennas is

$$\rho(d) = \frac{\int_0^{2\pi} E\{|A(\theta)|^2\} e^{-i2\pi d \cos \theta / \lambda} d\theta}{\int_0^{2\pi} E\{|A(\theta)|^2\} d\theta} \quad (29)$$

The expected value of  $|A(\theta)|^2$  is equal to 1 when it is from a scattering source and is 0 when it is not from a scattering source. Now we use the same expansion as in paper [10], namely

$$e^{i2\pi d \cos \theta / \lambda} = \sum_{m=-\infty}^{\infty} i^m J_m(2\pi d / \lambda) e^{-im\theta} \quad (30)$$

The correlation coefficient  $\rho$  can be simplified to

$$\rho(d) = \sum_{m=-\infty}^{\infty} \sum_{r=1}^R \frac{\Delta_r}{\Delta_T} e^{im(\pi/2 - \theta_r)} \times \frac{\sin(m\Delta_r/2)}{m\Delta_r/2} J_m(2\pi d / \lambda) \quad (31)$$

where  $\Delta_T = \sum_{r=1}^R \Delta_r$ . If there is only one scattering source,  $R = 1$ , the spatial correlation (31) is equivalent to the one derived in [9] and [10]. By substituting (31) into (22), we can relate the diversity gain to the antenna separation.

## VI. EXAMPLE

Let us consider an example with three scattering sources. The incident angles are  $150^\circ$ ,  $173^\circ$  and  $280^\circ$ ; beamwidths

are  $20^\circ$ ,  $6^\circ$  and  $20^\circ$ , respectively. The analytical result is plotted with the simulating result in Fig. 5. The simulating result is used to indicate the correctness of the analytical formula. Our simulation method is reported in [11] and is summarized in four steps here:

- 1) Replace integrals in  $s_1(t)$  and  $s_2(t)$  as shown in (25) and (26) with the sum of 30 randomly directed multipaths with Gaussian distributed amplitude  $N(0,1)$  and uniformly distributed phase ( $0^\circ$  to  $360^\circ$ ).
- 2) Record the received signal power from the antenna with the higher SNR.
- 3) Repeat step one and two 10,000 times.
- 4) Estimate the gain by averaging these samples.

As we expected, the analytical results and simulation gain plots have a similar shape (as shown in Fig. 5).

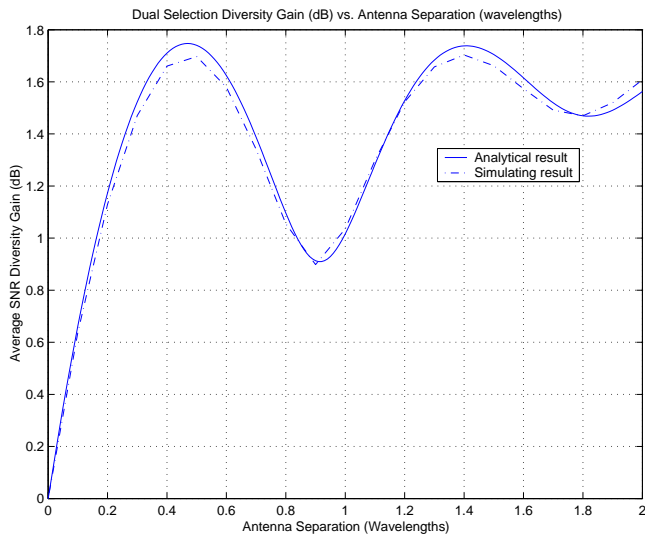


Fig. 5: The analytical gain and simulation gain plots for a particular multipath scattering environment.

## VII. CONCLUSION

We derived an analytical expression of the average SNR diversity gain in a dual selection combining system under slow Rayleigh fading. This expression relates the diversity gain to spatial correlation and antenna separation. Moreover, we developed a more practical channel model in which multipath scatters are from more than one incident angle, each with a certain beamwidth. We showed that the analytical gain agreed were corroborated by the simulations. We have also extended our analytical expression to correlated Nakagami- $m$  fading channels and will report these results shortly.

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