On capacity of multi-antenna wireless channels: Effects of antenna separation and spatial correlation

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Abstract — **Channel capacity of multi-element communication systems in independent Rayleigh channels has been shown to scale linearly with the number of antennas. In reality, the signals received by different receiver antennas can be correlated with each other due to the non-uniform scattering environment and limited aperture of the antenna system. In this paper, the effect of spatial correlation between receiver antennas on capacity is investigated for various scattering environments. The physics of signal propagation is combined with statistics of the scattering environment to derive a capacity expression in terms of spatial correlation, antenna spacings/placement, aperture size, and power distribution of scatters. This result is used to show that for a given aperture size, one can increase the capacity approximately linearly up to a certain value by increasing the number of antennas but further increase will not give any significant capacity gain.**

I. Introduction

Multiple element antenna communication systems have been shown theoretically to drastically improve the capacity over traditional single antenna element systems [1, 2]. A common assumption in the study of such systems is that the channels between different pairs of transmitter-receiver antennas are independent of each other. However, in reality the signals received by different receiver antennas are correlated with each other which will reduce the capacity gain. In this paper, we investigate the effects of spatial correlation between receiver antennas on capacity for not only uniform scattering environment but also for other realistic scattering scenarios.

There are few other works reported in the literature on the effect of spatial correlation on capacity of multi antenna systems [3, 4, 5, 6, 7]. However, most of the previous studies have been either confined to Rayleigh fading channels, where an isotropic scattering environment is implicitly assumed, or they use a one-ring scattering model. In [4], an exponential correlation matrix model has used to investigate channel capacity. However, these approaches fall short of fully describing the effect of scattering environment, aperture size, number of antenna elements, and antenna separation on capacity.

In this paper, we combine the understanding of the physical signal propagation with the statistics of the scattering environment to study these effects.

II. Spatial Correlation

Signals received by two points in space are, in general, correlated, with the correlation coefficient depending on separation of two points and the angular spread of the incoming waves. Consider two antennas located at points $\mathbf{x}_p$ and $\mathbf{x}_q$. Let $z_p(t)$ and $z_q(t)$ denote the complex envelop of the received signal at two antennas respectively. Then the normalized spatial correlation function between the complex envelopes of the two received signals by points p and q, is defined by

$$\rho_{pq} = \frac{E\{z_p(t)z_q^*(t)\}}{E\{z_0(t)z_0^*(t)\}} \tag{1}$$

where $E\{\cdot\}$ denotes the expectation operator, $\cdot^*$ denotes the complex conjugation and $z_0(t)$ is the received signal at a suitably chosen origin. It was shown [8] for a 2 dimensional propagation environment that

$$\rho_{pq} = \sum_{m=-\infty}^{\infty} \alpha_m J_m(k\|\mathbf{x}_p - \mathbf{x}_q\|)e^{im\phi_{pq}} \tag{2}$$

where $k = 2\pi/\lambda$ is the wavenumber, λ is the wave length,

$$\alpha_m = i^m \int_0^{2\pi} \mathcal{P}(\phi)e^{-im\phi}d\phi, \tag{3}$$

where $\mathcal{P}(\phi)$ is the normalized average power received from direction ϕ and ϕ_{pq} is the angle of the vector connecting $\mathbf{x}_p$ and $\mathbf{x}_q$.

The Bessel functions $J_m(\cdot)$ for $m \geq 1$ in (2) have a spatial high pass character ($J_0(\cdot)$ is spatially low pass). The fact that the higher order Bessel functions have small values for arguments near zero, means that to evaluate the correlation for points near each other in space, only a few terms in the sum (2) need to be evaluated in order to obtain a very good approximation. It was shown in

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[9] that the above summation can be truncated for $m = -M : M$ where

$$M \sim \frac{\pi e}{\lambda} \max_{p,q}\{\|\mathbf{x}_p - \mathbf{x}_q\|\}. \tag{4}$$

Thus, we treat that the correlation expression (2) has only a finite number of terms $2M+1$ where M is given by (4).

In this paper, we have only considered the 2D propagation environment, since the practical wireless channels tend to have multipaths mainly in the azimuth plane. The equivalent 3D results can be found in [8].

Note that the spatial correlation expression (2) captures both physical propagation characteristics as well as statistical properties of the scattering environment. We now give explicit expressions for the coefficients α_m for commonly used scattering scenarios.

II.A 2D Omni-Directional (Isotropic) Diffuse Field

For the special case of scattering uniformly over *all* angles in the plane containing two points, (2) reduces to a single term, and so the correlation coefficient is given by

$$\rho = J_0(k\|\mathbf{x}_p - \mathbf{x}_q\|). \tag{5}$$

This is the most simplest form of a multipath field and has been used extensively in modelling space-time channels in the literature. For example, if the received signal at a single antenna is Raleigh distributed, then the scattering environment is omni-directional and the correlation between signals received at two points are given by (5).

II.B Uniform Limited Azimuth Field

In the case of energy arriving uniformly from a restricted range of azimuth $(\phi_0 - \Delta\phi, \phi_0 + \Delta\phi)$, it was shown [8] that α_m in (3)

$$\alpha_m = e^{im(\pi/2-\phi_0)}\,\mathrm{sinc}(m\Delta\phi). \tag{6}$$

II.C Von-Mises Distributed Field

Non-isotropic scattering in the azimuthal plane may be modeled by the von-Mises distribution [10], for which the density is given by

$$\mathcal{P}(\phi) = \frac{1}{2\pi I_0(\kappa)} e^{\kappa\cos(\phi-\phi_0)}, \quad |\phi-\phi_0| \le \pi, \tag{7}$$

where ϕ_0 represents the mean direction, $\kappa > 0$ represents the degree of non-isotropy, and $I_m(\kappa)$ is the modified Bessel function of the first kind. In this case, using (3.937) of [11] α_m in (3) is given by

$$\begin{aligned}\alpha_m &= \frac{i^m}{2\pi I_0(\kappa)} \int_0^{2\pi} e^{\kappa\cos(\phi-\phi_0)} e^{im\phi} d\phi \\ &= e^{im(\pi/2-\phi_0)} \frac{I_{-m}(\kappa)}{I_0(\kappa)}.\end{aligned} \tag{8}$$

III. Capacity of Space-Time Channels

Consider Q receiving antennas and P transmitting antennas, and let $\mathbf{r} = [r_1, r_2, \ldots, r_Q]^T$ be the vector of received signals, $\mathbf{n} = [n_1, n_2, \ldots, n_Q]^T$ be the zero mean additive white gaussian noise vector of the signal received by Q sensors and $\mathbf{s} = [s_1, s_2, \ldots, s_P]$ be the vector of symbols sent by the P transmitting antennas. Also, let

$$\mathbf{r} = \mathbf{z} + \mathbf{n} \tag{9}$$

where $\mathbf{z} = [z_1, \cdots, z_Q]$ is the signal component of the observed vector $\mathbf{r}$ due to the transmitted signal vector $\mathbf{s}$.

The received information, $I(\mathbf{s}, \mathbf{r})$ about $\mathbf{s}$ when $\mathbf{r}$ is observed is defined [12] as

$$I(\mathbf{s}, \mathbf{r}) = H(\mathbf{s}) - H(\mathbf{s}|\mathbf{r}) = H(\mathbf{r}) - H(\mathbf{r}|\mathbf{s}), \tag{10}$$

where $H(\cdot)$ is the entropy of the argument. If $\mathbf{s}$ is a multidimensional gaussian random variable, then its entropy [13] $H(\mathbf{s}) = \log\left((2\pi e)^{\frac{P}{2}}|\mathbf{V}_s|\right)$ where $|\cdot|$ denotes matrix determinant, $\mathbf{V}_s = E\{\mathbf{s}\mathbf{s}^*\}$. Similarly, we can write, $H(\mathbf{r}) = \log\left((2\pi e)^{\frac{Q}{2}}|\mathbf{V}_r|\right)$ and $H(\mathbf{n}) = \log\left((2\pi e)^{\frac{Q}{2}}|\mathbf{V}_n|\right)$ where $\mathbf{V}_r = E\{\mathbf{r}\mathbf{r}^*\}$ and $\mathbf{V}_n = E\{\mathbf{n}\mathbf{n}^*\}$. By observing $H(\mathbf{r}|\mathbf{s}) = H(\mathbf{n})$, we rewrite (10) as

$$I(\mathbf{s}, \mathbf{r}) = \frac{1}{2}\log\left|\frac{\mathbf{V}_r}{\mathbf{V}_n}\right|. \tag{11}$$

Assuming, the noise $\mathbf{n}$ and the signal components $\mathbf{z}$ of the observed signal vector are independent of each other, we use (9) to write

$$\mathbf{V}_r = E\{\mathbf{z}\mathbf{z}^*\} + \mathbf{V}_n. \tag{12}$$

Similar to [14, 12], we remove the factor $\frac{1}{2}$ from (11) by assuming complex data, use base 2 logarithm, and assume the noise at each receiver antenna is gaussian with zero mean and σ^2 variance, to express capacity C in bits/Hz as,

$$C = \log_2\left|\mathbf{I}_Q + \frac{E\{\mathbf{z}\mathbf{z}^*\}}{\sigma^2}\right|, \tag{13}$$

where $\mathbf{I}_Q$ is the $Q \times Q$ identity matrix. Note that $E\{\mathbf{z}\mathbf{z}^*\}$ is the $Q \times Q$ covariance matrix of the signal part of the received signal and can be written as

$$E\{\mathbf{z}\mathbf{z}^*\} = \begin{bmatrix} E\{z_1 z_1^*\} & \ldots & E\{z_1 z_Q^*\} \\ \vdots & \ddots & \vdots \\ E\{z_Q z_1^*\} & \ldots & E\{z_Q z_Q^*\} \end{bmatrix}. \tag{14}$$

Each entry of (14) is proportional to the spatial correlation between the corresponding two points, i.e.,

$$E\{z_p z_q^*\} = \sigma_s^2 \rho_{pq}, \tag{15}$$

where σ_s^2 is the average signal power received at a point and ρ_{pq} is given by (2). Now, we can write

$$\frac{E\{\mathbf{z}\mathbf{z}^*\}}{\sigma^2} = \eta\mathbf{R}, \tag{16}$$

where

$$\mathbf{R} \triangleq \begin{bmatrix} \rho_{11} & \cdots & \rho_{1Q} \\ \vdots & \ddots & \vdots \\ \rho_{Q1} & \cdots & \rho_{QQ} \end{bmatrix}, \tag{17}$$

$\eta = \sigma_s^2/\sigma^2$ is the average signal to noise ratio at any one of the receiving antennas and σ^2 is the noise variance. By writing (16), we also assumed that P transmitted signals are independent of each other, total transmitted signal power is constant irrespective of number transmitting antennas and the scattering environment is same for all transmitted signals.[1]

We call $\mathbf{R}$ as the spatial correlation matrix. We substitute (16) in (13) to derive a new capacity formula for space-time channels which depend on spatial correlation between receiver antenna elements as

$$C = \log_2 \left|\mathbf{I}_Q + \eta\mathbf{R})\right|. \tag{18}$$

Equation (18) together with (17) and (2) form a set of tools which can be used to categorize the capacity of space-time channels. Specifically, the relationship between the capacity and sensor placement, separation, and scattering environment can be explained from these two equations.

IV. Capacity Limits

The capacity of space-time channels will be maximum, when there is no correlation between receiver antenna elements. For this special case, $\mathbf{R} = \mathbf{I}_Q$ and

$$C_{\max} = Q \log_2(1 + \eta). \tag{19}$$

In practice this can only happen if the antenna elements are sufficiently away from one another.

When there is perfect correlation between each pair of antenna elements, the capacity of MIMO channels will be minimum and $\mathbf{R} = \mathbf{1}_Q$, where $\mathbf{1}_Q$ is a $Q \times Q$ matrix of ones. In practice, this can only be achieved if all antennas are located in a single point in space. Thus, the minimum capacity of MIMO channel is

$$C_{\min} = \log_2(1 + \eta Q). \tag{20}$$

That is the capacity will increase logarithematically with the number of receiver antennas. However, this capacity improvement is due to the assumption that the noise at each antenna element is independent of each other. This is an unrealistic assumption in the case of all antennas located in a single point but is valid in other cases. However, (20) gives the capacity improvement due to only noise averaging of the received signal and, the capacity expressions (18) and (19) are due to both the signal and noise components of the received signal.

We can further rearrange (18) to write

$$C = C_{\max} + \log_2 \Delta_Q, \tag{21}$$

where

$$\Delta_Q = \det \begin{bmatrix} 1 & \frac{\eta}{1+\eta}\rho_{12} & \cdots & \frac{\eta}{1+\eta}\rho_{1N_r} \\ \frac{\eta}{1+\eta}\rho_{21} & \ddots & \vdots & \\ \vdots & & \ddots & \vdots \\ \frac{\eta}{1+\eta}\rho_{N_r 1} & \cdots & & 1 \end{bmatrix}. \tag{22}$$

Note that $0 \leq \Delta Q \leq 1$ due to the positive semi-definite property of the matrix involved. Thus, $\log_2 \Delta_Q \leq 0$, that is the capacity gain will be reduce if there is any signal correlation between antenna elements.

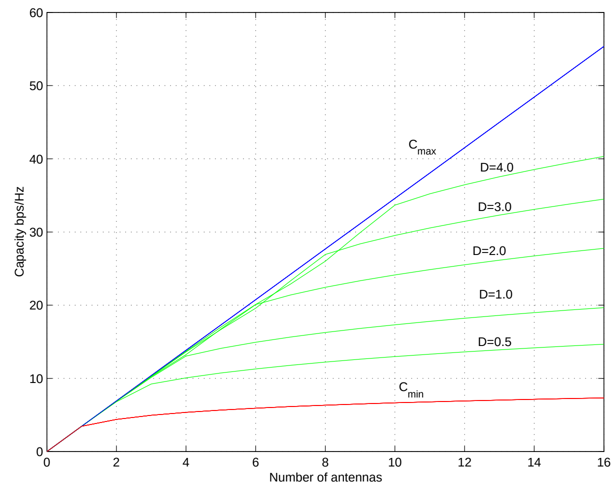

Figure 1: Capacity of a space time channel with 2D omni-directional diffuse scattering field versus number of equally spaced antennas on a constant aperture lengths $D = 0.5, 1, 2, 3$ and 4. Also shown is the theoretical maximum and minimum capacities.

V. Simulations

We use (21) and (2) to calculate the capacity of space time channels for different scattering environments given in Section II. Specifically, we like to answer the questions such as: does the capacity increase linearly with the number of antennas for all scattering scenarios? (as suggested by [2]), How many antennas needed to achieve maximum capacity in a fixed aperture? What will happen to the capacity if the aperture is increased?. For all our simulations, we use equally spaced linear antenna arrays, and set the signal to noise ratio $\eta = 20$dB.

First we consider the 2D omni-directional diffuse scattering environment given in Section II.A. We calculate the capacity by increasing the number of antennas within a constant aperture length and the results are depicted in Fig. 1. It can be observed from Fig.1 that the capacity increases approximately linearly with the number of antennas only up to a certain saturation point and then increase at a decreasing rate. Thus, by over crowding a given aperture by large number of antennas, one cannot achieve large capacities as predicted by [2]. Note that the capacity increase evident in the second part of the curves is mainly due to the assumption of independent noise at each individual antenna.

[1]These assumptions enable us to derive capacity expressions which are independent of the properties of the transmitter antennas but dependent on scattering environment and receiver antenna properties.

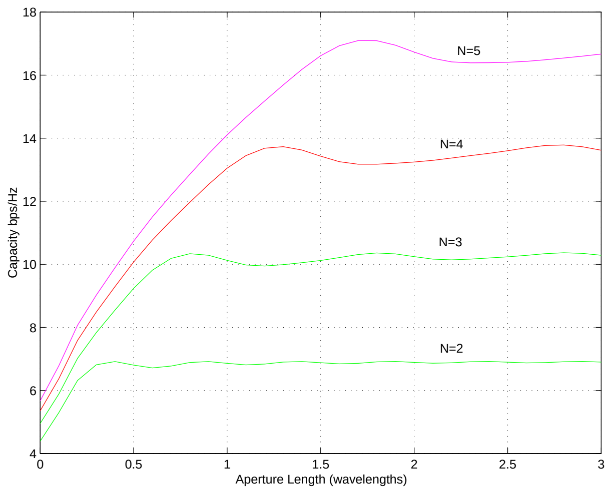

Figure 2: Capacity of a space time channel with 2D diffuse scattering field versus aperture length of equally spaced antennas with constant number of antennas $N = 2, 3, 4$ and 5.

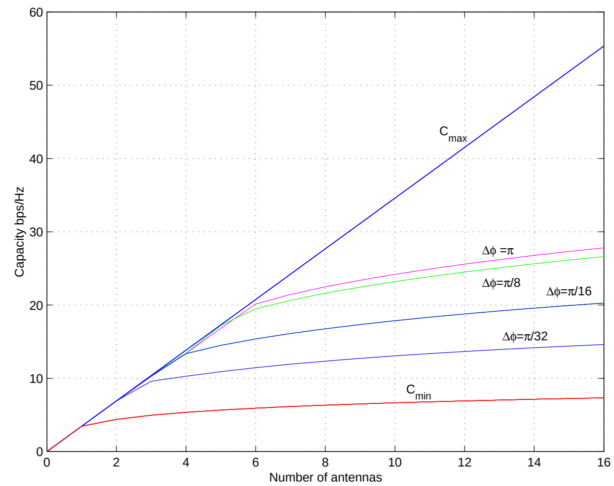

Figure 3: Capacity of a space time channel with uniform limited azimuth scattering field versus number of equally spaced antennas for different range of angular spreads $\Delta\phi = \pi, \pi/8, \pi/16$ and $\pi/32$, for aperture length $D = 2$ wavelengths. Also shown is the theoretical maximum and minimum capacities.

By studying Fig.1, we can propose a "rule of thumb" for choosing the number of antennas to maximize the capacity for a given aperture D for the 2D omni-directional scattering environment. Specifically,

$$N \sim 2(\frac{D}{\lambda} + 1), \qquad (23)$$

will give a guide for smart antenna receiver designers.

Figure 2 shows the capacity for the same environment against the aperture length for a set of fixed number of antenna elements. For zero aperture size, the curves do not start from the same point due to the assumption of independent noise at each antenna. However, it can be seen that for a given number of receiver antenna elements, there is a minimum aperture size which maximize the capacity. Further increase in aperture size will not increase capacity significantly. Observe that the results in Fig.2 can also be used to derive the rule of thumb given by (23).

We have also calculated the capacity versus number of antennas for two other non-isotropic scattering environments. Figure 3 depicts the capacity against the number of antennas for a constant aperture length of $D = 2$ wavelengths for uniform but limited azimuth fields. Angular spread of azimuth is given by $(\phi_0 - \Delta\phi, \phi_0 + \Delta\phi)$ where $\phi_0 = \pi/2$ from the linear antenna array axis and $\Delta\phi = \pi, \pi/8, \pi/16$ and $\pi/32$. Note that $\Delta\phi = \pi$ represents an isotropic field. One can see that there is a reduction of capacity corresponding to a reduction in angular spread of scatterers. However, overall behaviour of capacity with the number of antennas within a given aperture size is qualitatively similar to that of the isotropic scattering case.

Finally, we show the capacity of channels with von-Mises cattering fields with $\kappa = 0, 5, 10$ and 20 in Fig.4. Note that $\kappa = 0$ represents the isotropic case and the width of the angular spread grows as we increase κ. Again, the characteristics of capacity of these channels are qualitatively similar to the isotropic case.

The main message from the simulation results is that for a given aperture size, one can increase the capacity of a communication system approximately linearly up to a certain value by increasing the number of antennas but further increase will not give any significant capacity gain irrespective of the scattering environment. Further, we have suggested a "rule of thumb" to select the optimum number of antennas needed to achieve maximum capacity for given aperture area for the isotropic scattering environment. We are currently investigating to extend this "rule of thumb" to other scattering scenarios and to support them with theoretical results.

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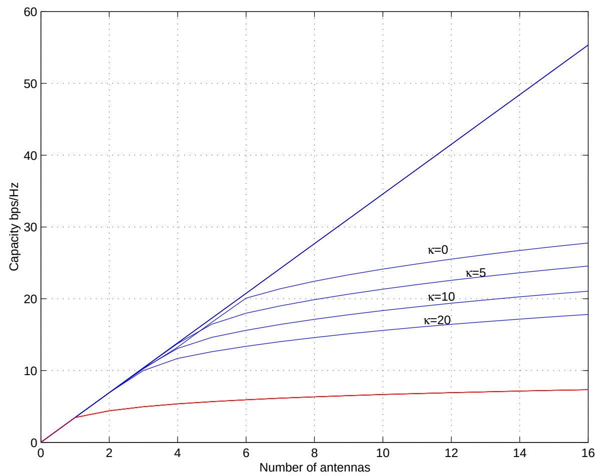

Figure 4: Capacity of a space time channel with von-mises distributed scattering field versus number of equally spaced antennas for angular spreads determined by $\kappa = 0, 5, 10$ and 20, and for aperture length $D = 2$ wavelengths. Also shown is the theoretical maximum and minimum capacities.

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