Tableau metatheory for propositional and syllogistic logics

Part V: Theorems on relationships between tableau systems and generalized semantics

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Tableau lemmas and definitions

To prove that relations $\models_{\mathbf{M}}$ and $\rhd_{\mathbf{TR}}$ are identical, we must define conditions that **M** and **TR** should satisfy, but before we need to state some auxiliary facts that we will use in a proof of Tableau Metatheorem.

Lemma on open tableaus

Lemma (On open tableaux)

Let **TR** be a set of tableau rules, X be a finite subset of For, and $A \in$ For. If there is a maximal and open **TR**–branch starting from $\{\langle B, w_1(i)\rangle : B \in X \cup \{\langle A, w_1(i)\rangle\}\}$, for some $i \in \mathbb{N}$, then all complete **TR**-tableaux $\langle X, A, \Psi\rangle$ are open.

Class of rules **TR** sound to models **M**

The first definition we employ in the context of tableau metatheorem concerns so called sound tableau rules **TR** for some class of models **M**. The word “sound” means that applying the rules leads “from truth to truth".

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The first definition we employ in the context of tableau metatheorem concerns so called sound tableau rules **TR** for some class of models **M**. The word "sound" means that applying the rules leads "from truth to truth".

Definition (Rules sound to models)

For any set of tableau rules **TR** and any class of models **M**, we say that the set of rules **TR** is *sound* to **M** iff for all sets $X_1, \ldots, X_i \subseteq \mathrm{Ex}$ (where $1 < i$), all models $\mathfrak{M} \in \mathbf{M}$ and all rules $R \in \mathbf{TR}$, if:

- $\langle X_1, \ldots, X_i \rangle \in R$
- and $\mathfrak{M}$ suitable to X_1,

then $\mathfrak{M}$ suitable to X_j, for some $1 < j \leq i$.

Lemma on maximal and open branch

Lemma

Let:

- $\mathbf{M}$ be a class of models
- $\mathbf{TR}$ be any set of tableau rules that is sound to $\mathbf{M}$
- $\mathfrak{M} \in \mathbf{M}$
- X be any finite set of formulas and $i \in \mathbb{N}$.

If $\mathfrak{M} \models X$, then there exists a maximal and open **TR**-branch starting from $X^{w_1(i)} = \{\langle A, w_1(i)\rangle : A \in X\}$.

Generating of model

Definition (Generating of model)

Let **TR** be a set of tableau rules. Let ϕ be a **TR**-branch.
The set $AT(\phi)$ is defined as follows: $x \in AT(\phi)$ iff one of the conditions holds

- $x \in \bigcup \phi \cap (\{r_k^l(m_1, \ldots, m_l) : m_1, \ldots, m_l \in \mathbb{N}\} \cup \{i \equiv j : i, j \in \mathbb{N}\})$
- $x \in \bigcup \phi \cap \{\langle x, t_1, \ldots, t_n \rangle : x \in \mathrm{Var}, t_1, \ldots, t_n \in \mathrm{Te}, n \in \mathbb{N}\}$.

Definition (Generating of model cont.)

Branch ϕ *generates* a model $\mathfrak{M} = \langle \{W_i\}_{i \in M}, \{R_j\}_{j \in N}, V \rangle$ iff:

(0) W_i is such a minimal set that $\langle l_1, \ldots, l_n \rangle \in W_i$ iff:

- for no $l_1 \leq k, o \leq l_n$: $k \equiv o \in AT(\phi)$ and $\langle l_1, \ldots, l_n \rangle \in \{\langle j_1, \ldots, j_n \rangle : w_i^n(j_1, \ldots, j_n) \in AT(\phi)\}$
- for some $l_1 \leq k, o \leq l_n$: $k \equiv o \in AT(\phi)$ and $\langle l_1, \ldots, l_n \rangle$ is the result of replacement of all occurrences of k by o in some $\langle g_1, \ldots, g_n \rangle \in \{\langle j_1, \ldots, j_n \rangle : w_i^n(j_1, \ldots, j_n) \in AT(\phi)\}$, for all $k \equiv o \in AT(\phi)$, where $k < o$

(1) $\langle m_1, \ldots, m_l \rangle \in R_k \subseteq W_1 \times \cdots \times W_l$ iff:

- for no $m_1 \leq k, o \leq m_l$: $k \equiv o \in AT(\phi)$ and $r_k(m_1, \ldots, m_l) \in AT(\phi)$
- for some $m_1 \leq k, o \leq m_l$: $k \equiv o \in AT(\phi)$ and $\langle m_1, \ldots, m_l \rangle$ is the result of replacement of all occurrences of k by o in some $r_k(g_1, \ldots, g_l) \in AT(\phi)$, for all $k \equiv o \in AT(\phi)$, where $k < o$

Generating of model

Definition (Generating of model cont.)

(2) $V(x, t_1, \ldots, t_n) = 1$ iff $\langle x, t_1', \ldots, t_n' \rangle \in AT(\phi)$ and $\langle t_1, \ldots, t_n \rangle$ is the result of replacement in $\langle t_1', \ldots, t_n' \rangle$ defined in (0), for all $x \in \mathrm{Var}$.

Collorary

Let **TR** be a set of tableau rules. Let ϕ be an open and maximal **TR**-branch. Then there exists such a model $\mathfrak{M}$ that ϕ generates $\mathfrak{M}$.

Class of models **M** sound to rules **TR**

Definition (Models sound to rules)

Let

- **TR** be a set of tableau rules
- ϕ be an open and maximal branch
- **M** be a class of models.

M is *sound* to rules **TR** iff branch ϕ generates such a model $\mathfrak{M}$ that:

- $\mathfrak{M} \in \mathbf{M}$
- $\mathfrak{M} \models X$.

Closure under rules

Definition (Closure under rules)

Let $X \subseteq \mathrm{Ex}$. We say that $Y \subseteq \mathrm{Ex}$ is a *closure of* X under **TR** iff Y is a set that satisfies conditions:

- $X \subseteq Y$
- for any rule R of **TR** and any n-tuple $\langle Z_1, Z_2, \ldots, Z_n \rangle \in R$, where $n \in \mathbb{N}$, if $X \subseteq Z_1 \subseteq Y$, then $Z_j \subseteq Y$, for some $2 \leq j \leq n$.

Lemma on existence of open and maximal branch

Lemma

Let $X \subseteq$ For and $i \in \mathbb{N}$. If for all finite $Y \subseteq X$ exists a maximal and open branch starting with $Y^{w_1(i)} = \{\langle A, w_1(i)\rangle : A \in Y\}$, then exists some closure of $X^{w_1(i)} = \{\langle A, w_1(i)\rangle : A \in X\}$ under **TR** that is an open and maximal branch.

Theorem (Tableau Metatheorem)

For any set of tableau rules **TR** and any class of models **M**, if:

1. set of rules **TR** is sound to class of models **M**
2. class of models **M** is sound to rules of **TR**,

then for all $X \subseteq \mathrm{For}$, $A \in \mathrm{For}$ the following statements are equivalent:

- $X \models_{\mathbf{M}} A$
- $X \rhd_{\mathbf{TR}} A$
- there is a finite $Y \subseteq X$ and a closed tableau $\langle Y, A, \Phi \rangle$.

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Local entailment

Let us take consequence relation $\models_{\mathbf{M}}$, for some class of models $\mathbf{M}$, and branch consequence relation $\rhd_{\mathbf{TR}}$, for some set of tableau rules $\mathbf{TR}$.

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Now, we assume some $X \subseteq \mathrm{For}$, $A \in \mathrm{For}$, which we want to know about, whether true is:

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Now, we assume some $X \subseteq$ For, $A \in$ For, which we want to know about, whether true is:

$$\text{if } \rhd_{\mathbf{TR}} X, \text{then } \rhd_{\mathbf{TR}} A.$$

To decide this we check in the way presented here, whether $X \rhd_{\mathbf{TR}'} A$, where $\mathbf{TR}'$ is set of rules $\mathbf{TR}$ plus an additional rule:

Local entailment

$R\colon \dfrac{Y}{Y \cup \{\langle C, i \rangle\}}$, where:

1. $C \in X$ and $\langle C, j \rangle \in Y$, for some j
2. $i \in *(Y)$.

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Clause 1 is about a finitary character of tableau proof of $X \rhd_{\mathbf{TR}'} A$, for some finite $Z \subseteq X$, we start a tableau with $\{\langle B, w_1(k)\rangle : B \in Z\} \cup \{\langle \sim A, w_1(k)\rangle\} \subseteq Y$, for some k.

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Clause 2 says that to any C in Z, we can add any index that appears during the proof.

Future problems

Our metatheory can be considered as a methodology of construction of adequate tableau systems, so it results in some other practical consequences:

- a natural problem of estimations of tableau proofs
- problems of economy of formulating and proofs in deductively equivalent tableau systems
- a problem of reduction of labelled tableau systems to non–labelled ones — a problem of a theoretical borderline of such reduction
- a comparison of different tableau systems in one theoretical framework
- a comparison of our approach to other approaches/paradigms — in respect of a level of generality
- a problem of automatization of producing adequate tableau systems for semantically determined logics
- a problem of scope of automatic transformations of our tableau systems into other kind of deductive systems.

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