Tableau metatheory for propositional and syllogistic logics

Part IV: Abstract tableau notions: rules, branches, tableaux

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Program of lecture

We describe the main part of tableau metatheory: general tableau notions:

- all notions are presented as set–theoretical ones (for example: branches are sequences of sets and tableaux are sets of those sequences)
- the rest of tableau notions are defined in a similar, formal way:
  1. tableau rules
  2. branches: open, closed, maximal (aka complete)
  3. tableaux: open, closed, complete
  4. new notions are also presented — branch consequence relation (as a very special set of branches) and useless variant of branch.

Tableau language – set of expressions

We need some language of tableau proofs: set of expressions Ex. Firstly, we list symbols:

1. indexes/labels — set of natural numbers $\mathbb{N}$
2. n–ary functional constants (where $n \geq 1$): w_1^1, w_2^1, w_3^1, ..., w_1^2, w_2^2, w_3^2, ..., w_1^3, w_2^3, w_3^3, ...
3. n–ary predicates (where $n \geq 2$): r_1^2, r_2^2, r_3^2, ..., r_1^3, r_2^3, r_3^3, ..., r_1^4, r_2^4, r_3^4, ...
4. identity symbol: $\equiv$
5. semantic negation: $\sim$.

Tableau language – set of terms

Set of all terms TERM is the least that consists of:

$w_k^l(m_1, \dots, m_l)$, where:

- $k, l, m_1, \dots, m_k \in \mathbb{N}$
- $l \geq 1$
- w_k^l is a functional constant.

The members of TERM we denote by $t_1, t_2, t_3, \dots$

Tableau language – set of expressions

Definition (Expressions)

Ex is the least set that consists of the expressions:

- $r_k^l(m_1, \ldots, m_l)$ $\qquad\sim r_k^l(m_1, \ldots, m_l)$
- $i \equiv j$ $\qquad\sim i \equiv j$
- $\langle A, t_1, \ldots, t_n \rangle$ $\qquad\langle \sim A, t_1, \ldots, t_n \rangle$

for all:

a. $A \in$ For
b. $i, j, k, l, n, m_1, \ldots, m_l \in \mathbb{N}$
c. $t_1, \ldots, t_n \in$ TERM, where $n \geq 1$.

When the context is clear, we write:

- $A, t_1, \ldots, t_n$
- $\sim A, t_1, \ldots, t_n$,

removing brackets: $\langle\ \rangle$.

Fundamental tableau notions: function choosing indexes

Definition (Function choosing indexes)

Function choosing indexes we call a function
$\circ\colon \mathrm{Ex} \cup \mathrm{TERM} \cup \mathrm{P}(\mathrm{Ex} \cup \mathrm{TERM}) \longrightarrow \mathrm{P}(\mathbb{N})$ defined by conditions:

- $\circ(w^l_k(m_1, \ldots, m_l)) = \{m_1, \ldots, m_l\}$
- $\circ(r^l_k(m_1, \ldots, m_l)) = \circ(\sim r^l_k(m_1, \ldots, m_l)) = \{m_1, \ldots, m_l\}$
- $\circ(i \equiv j) = \circ(\sim i \equiv j) = \{i, j\}$
- $\circ(\langle A, w^{l_1}_{k_1}(x^{k_1}_1, \ldots, x^{k_1}_{l_1}), \ldots, w^{l_n}_{h_o}(x^{h_o}_1, \ldots, x^{h_o}_{l_n})\rangle) =$
  $\circ(\langle \sim A, w^{l_1}_{k_1}(x^{k_1}_1, \ldots, x^{k_1}_{l_1}), \ldots, w^{l_n}_{h_o}(x^{h_o}_1, \ldots, x^{h_o}_{l_n})\rangle) =$
  $\{x^{k_1}_1, \ldots, x^{k_1}_{l_1}, \ldots, x^{h_o}_1, \ldots, x^{h_o}_{l_n}\}$
- $\circ(X) = \bigcup\{\circ(y) : y \in X\}$, if $X \subseteq \mathrm{Ex} \cup \mathrm{TERM}$,

for all $A \in \mathrm{For}$ and $h, i, j, k, l, o, m_1, \ldots, m_l, x^{k_1}_1, \ldots, x^{k_1}_{l_1}, \ldots,$
$x^{h_o}_1, \ldots, x^{h_o}_{l_n} \in \mathbb{N}$.

Fundamental tableau notions: similar sets of expressions

Definition (Similar sets of expressions)

Let X, $Y \subseteq \mathrm{Ex}$ be sets of expressions. Let $Z \subseteq \mathbb{N}$.

Set X is *similar to* Y *in respect* of Z iff there is a bijection $\ddagger : \circ(X) \longrightarrow \circ(Y)$ (where $\circ(X)$, $\circ(Y)$ are sets of indexes occurring in expressions of X and Y) such that:

(a) for all $x \in Z$, if $x \in \circ(X)$, then $\ddagger(x) = x$

(b) for all kinds of expressions in Ex:

Fundamental tableau notions: similar sets of expressions

(a) $r^l_k(m_1, \ldots, m_l) \in X$ iff $r^l_k(\ddagger(m_1), \ldots, \ddagger(m_l)) \in Y$

(b) $i \equiv j \in X$ iff $\ddagger(i) \equiv \ddagger(j) \in Y$

(c) $\langle A, w^{l_1}_{k_1}(x^{k_1}_1, \ldots, x^{k_1}_{l_1}), \ldots, w^{l_n}_{h_o}(x^{h_o}_1, \ldots, x^{h_o}_{l_n})\rangle \in X$ iff
$\langle A, w^{l_1}_{k_1}(\ddagger(x^{k_1}_1), \ldots, \ddagger(x^{k_1}_{l_1})), \ldots, w^{l_n}_{h_o}(\ddagger(x^{h_o}_1), \ldots, \ddagger(x^{h_o}_{l_n}))\rangle \in Y$

(d) $\sim r^l_k(m_1, \ldots, m_l) \in X$ iff $\sim r^l_k(\ddagger(m_1), \ldots, \ddagger(m_l)) \in Y$

(e) $\sim i \equiv j \in X$ iff $\sim \ddagger(i) \equiv \ddagger(j) \in Y$

(f) $\langle \sim A, w^{l_1}_{k_1}(x^{k_1}_1, \ldots, x^{k_1}_{l_1}), \ldots, w^{l_n}_{h_o}(x^{h_o}_1, \ldots, x^{h_o}_{l_n})\rangle \in X$ iff
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$\in Y$,

for all $A \in$ For and $h, i, j, k, l, o, m_1, \ldots, m_l, x^{k_1}_1, \ldots, x^{k_1}_{l_1}, \ldots,$
$x^{h_o}_1, \ldots, x^{h_o}_{l_n} \in \mathbb{N}$.

Fundamental tableau notions: tableau inconsistency

Definition (Tableau inconsistent sets of expressions)

Let $X \subseteq \text{Ex}$. We say that X is *tableau inconsistent* iff it consists one of pairs of the expressions:

(a) $r^l_k(m_1, \ldots, m_l)$, $\sim r^l_k(m_1, \ldots, m_l)$

(b) $i \equiv j$, $\sim i \equiv j$

(c) $\langle A, t_1, \ldots, t_n\rangle$, $\langle \sim A, t_1, \ldots, t_n)\rangle$

for all:

- $A \in \text{For}$ and $i, j, k, l, m_1, \ldots, m_l, n \in \mathbb{N}$
- $t_1, \ldots, t_n \in \text{TERM}$.

Otherwise, we call set X *tableau consistent*. We shortly say that X is *t-consistent* or respectively *t-inconsistent*.

Model suitable to a set of expressions

Definition (Model suitable to a set of expressions)

Let $X \in \mathrm{Ex}$. Let $\mathfrak{M} = \langle \{W_i\}_{i \in M}, \{R_j\}_{j \in N}, V \rangle \in \mathbf{M}^i$ and $X \subseteq \mathrm{Ex}$. Model $\mathfrak{M}$ is *suitable* to X iff there are functions:

(a) $f' : \mathbb{N} \longrightarrow M \cup N$

(b) $f'' : \mathbb{N} \longrightarrow \bigcup_{i \in M} W_i$

such that following conditions are fulfilled:

Model suitable to a set of expressions

(a) if $\langle A, w_{k_1}^{l_1}(x_1^{k_1}, \ldots, x_{l_1}^{k_1}), \ldots, w_{h_o}^{l_n}(x_1^{h_o}, \ldots, x_{l_n}^{h_o})\rangle \in X$, then:

- $(\langle f''(x_1^{k_1}), \ldots, f''(x_{l_1}^{k_1})\rangle \in W_{f'(k)_1}^{l_1}, \ldots, \langle f''(x_1^{h_o}), \ldots, f''(x_{l_n}^{h_o})\rangle \in W_{f'(h)_o}^{l_n}$
- $W_{f'(k)_1}^{l_1} \times \cdots \times W_{f'(h)_o}^{l_n} \in \{W_i\}_{i \in M}$
- $\mathfrak{M}, \langle f'(x_1^{k_1}), \ldots, f'(x_{l_1}^{k_1})\rangle, \ldots, \langle f'(x_1^{h_o}), \ldots, f'(x_{l_n}^{h_o})\rangle \models A$

(b) if $\langle \sim A, w_{k_1}^{l_1}(x_1^{k_1}, \ldots, x_{l_1}^{k_1}), \ldots, w_{h_o}^{l_n}(x_1^{h_o}, \ldots, x_{l_n}^{h_o})\rangle \in X$, then:

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Model suitable to a set of expressions

(c) if $r^l_k(m_1, \ldots, m_l) \in X$, then $\langle f''(m_1), \ldots, f''(m_l)\rangle \in R^l_{f'(k)}$

(d) if $\sim r^l_k(m_1, \ldots, m_l) \in X$, then $\langle f''(m_1), \ldots, f''(m_l)\rangle \notin R^l_{f'(k)}$

(e) if $i \equiv j \in X$, then $f''(i)$ is equal to $f''(j)$

(f) if $\sim i \equiv j \in X$, then $f''(i)$ is not equal to $f''(j)$

for all $A \in$ For and $h, i, j, k, l, o, m_1, \ldots, m_l, x^{k_1}_1, \ldots, x^{k_1}_{l_1}, \ldots,$
$x^{h_o}_1, \ldots, x^{h_o}_{l_n} \in \mathbb{N}$.

Complex tableau notions

Having a set of expressions Ex we can put very general conditions defining rules.

Our rules extend properly a set of expressions and they have also an internal mechanism that blocks extending of t-inconsistent sets.

Let us distinguish a set of indexes that plays a role of signs of logical values in our language: $\mathsf{LV} \subseteq \mathbb{N}$ (for some domain in models W_j).

Let $Z \subseteq \mathsf{Ex}$. Z is *co–infinite* iff $\mathbb{N} \setminus \circ(Z)$ is infinite.

Complex tableau notions

Definition (Rule)

Assume that $\mathrm{P(Ex)}$ is the set of all subsets of the set Ex. Let $\mathrm{P(Ex)}^n$ be n–ary Cartesian product $\underbrace{\mathrm{P(Ex)} \times \cdots \times \mathrm{P(Ex)}}_{n}$, for some $n \in \mathbb{N}$, and let $\bigcup_{n \in \mathbb{N}} \mathrm{P(Ex)}^n$ be the union of all such n–ary Cartesian products that $n \geq 2$.

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- *Rule* is such a subset $R \subseteq \bigcup_{n \in \mathbb{N}} \mathrm{P}(\mathrm{Ex})^n$ that if $\langle X_1, \ldots, X_n \rangle \in R$, then the following conditions are satisfied:
  - $X_1 \subset X_i$, for all $1 < i \leq n$
  - X_1 is t-consistent
  - if $k \neq l$, then $X_k \neq X_l$, for all $1 < k, l \leq n$

Rule

Definition (Rule cont.)

- (Closure under similarity) for any such subset of expression Y_1 that Y_1 is similar to X_1 in respect of LV, there exist such sets of expressions $Y_2, \ldots, Y_n$, that $\langle Y_1, \ldots, Y_n \rangle \in R$ and for all $1 < i \leq n$, Y_i is similar to X_i in respect of LV

Rule

Definition (Rule cont.)

- (Closure under similarity) for any such subset of expression Y_1 that Y_1 is similar to X_1 in respect of LV, there exist such sets of expressions $Y_2, \ldots, Y_n$, that $\langle Y_1, \ldots, Y_n \rangle \in R$ and for all $1 < i \leq n$, Y_i is similar to X_i in respect of LV

- (Existence of a core of rule) for some finite set $Y \subseteq X_1$ there exists exactly one such n–tuple $\langle Z_1, \ldots, Z_n \rangle \in R$ that:
  1. $Z_1 = Y$
  2. for any $1 < i \leq n$, $Z_i = Z_1 \cup (X_i \setminus X_1)$
  3. there does not exist a proper subset $U_1 \subset Y$ and such n–tuple $\langle U_1, \ldots, U_n \rangle \in R$ that for $1 < i \leq n$, $U_i = U_1 \cup (Z_i \setminus Z_1)$

  Any such n–tuple $\langle Z_1, \ldots, Z_n \rangle$ is called *a core of rule R in* $\langle X_1, \ldots, X_n \rangle$

Rule

Definition (Rule cont.)

- (Closure under Expansion) for any t-consistent set of expressions Z_1 such that:
  1. $X_1 \subset Z_1$
  2. Z_1 is co–infinite
  3. for all $1 < i \leq n$, X_i is not similar in respect of LV to any subset of Z_1:

  - if n–tuple $\langle W_1, \ldots, W_n \rangle$ is a core of rule R in X_1, then:
    1. there are $n-1$ such sets of expressions $Z_2, \ldots, Z_n$ that $\langle Z_1, \ldots, Z_n \rangle \in R$
    2. and for all $1 < i \leq n$, W_i is similar in respect of LV to $W_1 \cup (Z_i \setminus Z_1)$

- (Closure under Finite Sets) if X_1 is a finite set, then for all $1 < i \leq n$, X_i is a finite set

Rule

Definition (Rule cont.)

- By saying that a rule R *was applied to* X_1, we mean that for $1 < i \leq n$, exactly one X_i of some $\langle X_1, \ldots, X_n \rangle \in R$ was chosen, where $1 < i \leq n$.
- By **R** we denote a set of rules.

Examples of rules

Since we give examples of rules for two–valued modal logics below, we omit indexes for logical values.

If we have classical conjunction $\wedge$ in our language, we can take the following rule:

$$R_{\wedge}: \frac{X \cup \{\langle (A \wedge B), i \rangle\}}{X \cup \{\langle (A \wedge B), i \rangle, \langle A, i \rangle, \langle B, i \rangle\}}.$$

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If we have diamond $\Diamond$ in our language, we can take the following rule.

$$R_{\Diamond}: \frac{X \cup \{\langle \Diamond A, i \rangle\}}{X \cup \{\langle \Diamond A, i \rangle, irj, \langle A, j \rangle\}}, \text{ where:}$$

1. $j \notin \circ(X \cup \{\langle \Diamond A, i \rangle\})$
2. for all $k \in \mathbb{N}$, $\{irk, \langle A, k \rangle\} \not\subseteq X$.

Core of a rule

Definition (Core of a rule)

Let R be a rule and $n \in \mathbb{N}$. Let $\langle X_1, \ldots, X_n \rangle \in R$ and $\langle Z_1, \ldots, Z_n \rangle \in R$.
n–tuple $\langle Z_1, \ldots, Z_n \rangle$ is *a core of rule R in* $\langle X_1, \ldots, X_n \rangle$ iff

1. $Z_1 \subseteq X_1$
2. for all $1 < i \leq n$, $Z_i = Z_1 \cup (X_i \setminus X_1)$
3. there does not exist a proper subset $U_1 \subset Z_1$ and such n–tuple $\langle U_1, \ldots, U_n \rangle \in R$ that $U_i = U_1 \cup (Z_i \setminus Z_1)$, for all $1 < i \leq n$.

Core of a rule

In the case of structurally defined rules:

$$R_\wedge : \frac{X \cup \{\langle (A \wedge B), i\rangle\}}{X \cup \{\langle (A \wedge B), i\rangle, \langle A, i\rangle, \langle B, i\rangle\}}$$

we can easily distinguish a core of the rule. Here it is $\langle (A \wedge B), i\rangle\}$, for all A, B, i.

Non-introducing/introducing rules

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- in case of non–introducing rule R, when we obtain from X an extended by R set Y, then in no expression in set Y there appears any new symbol of alphabet that is not a part of an expression in set X
- in case of introducing rule R, when we obtain from X an extended by R set Y, then at least in one expression in set Y there appears a new symbol of alphabet than does not occur in any expression in set X.

R_X

Definition

Let $X \subseteq \text{Ex}$. Let R be a rule.
By R_X we denote a maximal subset of all n-tuples in R, such that their first member is X and if other members of two n-tuples in R_X differ, then the rule can be applied more times to the extended set.

Formally, R_X is such a maximal set of n–tuples that for all $n \in \mathbb{N}$, $\langle Y_1, \ldots, Y_n \rangle \in R_X$ iff:

- $\langle Y_1, \ldots, Y_n \rangle \in R$ and $Y_1 = X$
- for all $Z_1, \ldots, Z_n \subseteq \text{Ex}$, if $\langle Z_1, \ldots, Z_n \rangle \in R_X$, then:
  - for some core $\langle Z_1', \ldots, Z_n' \rangle$ of R in $\langle Z_1, \ldots, Z_n \rangle$, for some $1 < i \leq n$, $\langle Z_1' \cup Y_i, \ldots Z_n' \cup Y_i \rangle \in R$.

R_X

Having a rule R and a set of expressions X we define a set of n-tuples belonging to R such that after application of R the other n-tuples in R cannot extend the extended set. For example, we have a set

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We apply the rule for $\Diamond$, using the pair in the rule: $\langle\{\langle \Diamond A, 1\rangle\}, \{\langle \Diamond A, 1\rangle, 1r3, \langle \Diamond A, 3\rangle\}\rangle$, and we get:

$$(*) \quad \{\langle \Diamond A, 1\rangle, 1r3, \langle \Diamond A, 3\rangle\},$$

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but from $(*)$ and the rule for $\Diamond$ we cannot get:

$\{\langle \Diamond A, 1\rangle, 1r3, \langle \Diamond A, 3\rangle, 1rk, \langle \Diamond A, k\rangle, \}$, for any $k \in \mathbb{N}$, where $k \neq 3$.

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So in the R_X, where R is the rule for $\Diamond$, we have only one pair $\langle\{\langle \Diamond A, 1\rangle\}, \{\langle \Diamond A, 1\rangle, 1rj, \langle \Diamond A, j\rangle\}\rangle$, for some $j \in \mathbb{N}$.

Definition (Tableau rules)

Let **R** be a set of rules. **R** is *a set of tableau rules* iff

1. **R** is finite
2. for any $X \subseteq \mathrm{Ex}$, if X is finite, then for any rule $R \in \mathbf{R}$, each set R_X is finite.

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By **TR** we denote some fixed set of tableau rules.

Branch

A branch is a sequence of sets of expressions:

$X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n$ (possibly infinite),

where for any $1 \leq i \leq n$, X_{i+1} is a result of application of some rule to set X_i.

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Definition (Branch)

Let $K = \mathbb{N}$ or $K = \{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$. Let X be a set of expressions. *Branch* (or *branch starting from* X) is any string $\phi : K \longrightarrow \mathrm{P}(\mathrm{Ex})$ satisfying the conditions:

1. $\phi(1) = X$
2. for all $i \in K$: if $i + 1 \in K$, then there exists a rule $R \in \mathbf{TR}$ and n-tuple $\langle Y_1, \ldots Y_n \rangle \in R$, that $\phi(i) = Y_1$ and $\phi(i + 1) = Y_k$, for some $1 < k \leq n$.

On–member–branch

From the definition of branch we have a corollary:

Collorary

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Instead of $(\bigcup\{\phi(i) : i \in K\})_1$ we will write $\bigcup \phi_K$, as one–member–branch made by union of a branch $\phi : K \longrightarrow \mathrm{P(Ex)}$.

Closed/open branch

Definition (Closed/open branch)

A branch $\phi : K \longrightarrow \mathrm{P(Ex)}$ is *closed* iff $\phi(i)$ is a t–inconsistent set, for some $i \in K$. A branch is *open* iff is not closed.

Maximal branch (aka complete)

Definition (Maximal branch 1 — for finite cases)

Let $\phi : K \longrightarrow \mathrm{P(Ex)}$ be a branch. We say that ϕ is *maximal* iff

1. $K = \{1, 2, 3, \ldots, n\}$, for some $n \in \mathbb{N}$
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2. there is no branch ψ such that $\phi \subset \psi$.

Definition (Maximal branch 2)

Let $\phi : K \longrightarrow \mathsf{P}(\mathsf{Ex})$ be a branch. We say that ϕ is *maximal* iff there is no branch ψ such that $\bigcup \phi_K \subset \psi$.

Maximal branch (aka complete)

Definition (Maximal branch 1 — for finite cases)

Let $\phi : K \longrightarrow \mathrm{P(Ex)}$ be a branch. We say that ϕ is *maximal* iff

1. $K = \{1, 2, 3, \ldots, n\}$, for some $n \in \mathbb{N}$
2. there is no branch ψ such that $\phi \subset \psi$.

Definition (Maximal branch 2)

Let $\phi : K \longrightarrow \mathrm{P(Ex)}$ be a branch. We say that ϕ is *maximal* iff there is no branch ψ such that $\bigcup \phi_K \subset \psi$.

Fact

Let $\phi : K \longrightarrow \mathrm{P(Ex)}$ be a branch. If ϕ is maximal in respect to definition of maximal branch 1, then it is maximal in respect to definition of maximal branch 2.

Branch consequence relation

Let $Z \subseteq \mathsf{For}$ and $i \in \mathbb{N}$.

By Z^i we mean set $\{\langle A, w_1(i)\rangle : A \in Z\}$.

Definition (Branch consequence relation)

Let $X \subseteq \mathsf{For}$ and $A \in \mathsf{For}$. Formula A is a *branch consequence of* X (in short: $X \rhd_{\mathbf{TR}} A$) iff there exists such a finite set $Y \subseteq X$ and some index $i \in \mathbb{N}$, that any maximal branch starting from the set $Y^i \cup \{\langle \sim A, w_1(i)\rangle\}$ is closed.

Tableau

Definition (Branch maximal in a set of branches)

Let Φ be a set of branches and let $\psi \in \Phi$. Branch ψ is *maximal in set* Φ (in short: Φ–*maximal*) iff there does not exist such a branch $\phi \in \Phi$, that $\psi \subset \phi$.

Tableau

Definition (Tableau)

Let $X \subseteq \mathrm{For}$, $A \in \mathrm{For}$ and Φ be a set of branches. An ordered triple $\langle X, A, \Phi\rangle$ is *a tableau* for $\langle X, A\rangle$ (or just *tableau*) iff there are satisfied conditions:

- Φ is a non-empty subset of the set of branches starting from the set $X^i \cup \{\langle \sim A, w_1(i)\rangle\}$, for some index $i \in \mathbb{N}$ (i.e. if $\psi \in \Phi$, then $\psi(1) = X^i \cup \{\langle \sim A, w_1(i)\rangle\}$)
- each branch in Φ is Φ-maximal

Tableau

Definition (Tableau cont.)

- for any $n, i \in \mathbb{N}$ and any branches $\psi_1, \ldots, \psi_n \in \Phi$, if:
  - i and $i+1$ are in domains of functions $\psi_1, \ldots, \psi_n$
  - for any $1 < k \leq n$ and any $o \leq i$, $\psi_1(o) = \psi_k(o)$

  then there exists such a rule $R \in \mathbf{TR}$ and an ordered m-tuple $\langle Y_1, \ldots, Y_m \rangle \in R$, where $1 < m$, that for all $1 \leq k \leq n$:

  - $\psi_k(i) = Y_1$
  - there exists such $1 < l \leq m$, that $\psi_k(i+1) = Y_l$.

Useless variant of branch

Look at these examples of branches:

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$(\star)$ $\{\langle p \vee q, 1\rangle, \langle \neg\neg p, 1\rangle\} \subset \{\langle p \vee q, 1\rangle, \langle \neg\neg p, 1\rangle, \langle p, 1\rangle\}$

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The one–star–branch was produced by application of rule for double negation $R_{\neg\neg}$:

$R_{\neg\neg}$: $\dfrac{X \cup \{\langle \neg\neg A, i\rangle\}}{X \cup \{\langle \neg\neg A, i\rangle, \langle A, i\rangle\}}$

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$\{\langle p \vee q, 1\rangle, \langle \neg\neg p, 1\rangle, \langle q, 1\rangle, \langle p, 1\rangle\}$

The two– and three–star branches were produced by application of rule for disjunction $R_{\vee}$, and the three–star–branch was additionally produced by rule $R_{\neg\neg}$.

$$R_{\neg\vee}: \frac{X \cup \{\langle \neg(A \vee B), i\rangle\}}{X \cup \{\langle \neg(A \vee B), i\rangle, \langle \neg A, i\rangle, \langle \neg B, i\rangle\}}$$

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If it is open, then the one–star–branch is open, too.

Useless variant of branch

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If it is open, then the one–star–branch is open, too.

Hence all we should know is on the one–star–branch. The presence of the three–star–branch in a given tableau is not necessary.

Useless variant of branch

Definition (Useless variant of branch)

Let ϕ and ψ be such branches that for some numbers i and $i+1$ that belong to their domains and for all $j \leq i$, $\phi(j) = \psi(j)$, but $\phi(i+1) \neq \psi(i+1)$. Branch ψ is *useless variant of branch* ϕ iff:

- there are such a rule $R \in \mathbf{TR}$ and n–tuple $\langle X_1, \ldots, X_n \rangle \in R$, that $\phi(i) = X_1$ and $\phi(i+1) = X_j$, for some $1 < j \leq n$
- there are such a rule $R \in \mathbf{TR}$ and m–tuple $\langle Y_1, \ldots, Y_m \rangle \in R$, where $n < m$, that $\psi(i) = Y_1$ and:
  1. $\psi(i+1) = Y_k$, for some $1 < k \leq m$
  2. for all $1 < l \leq n$ there is such $1 < o \leq m$ that $o \neq k$ and $X_l = Y_o$.

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  1. $\psi(i+1) = Y_k$, for some $1 < k \leq m$
  2. for all $1 < l \leq n$ there is such $1 < o \leq m$ that $o \neq k$ and $X_l = Y_o$.

Let Φ, Ψ be sets of branches and $\Phi \subset \Psi$. Ψ is *useless superset* of Φ iff for any branch $\psi \in \Psi \setminus \Phi$ there is such a branch $\phi \in \Phi$ that ψ is a useless variant of ϕ.

Complete tableau

Definition (Complete tableau)

Let $\langle X, A, \Phi \rangle$ be a tableau. $\langle X, A, \Phi \rangle$ is *complete* iff:

1. all branches in Φ are maximal
2. each such set of branches Ψ that:
   2.1 $\Phi \subset \Psi$
   2.2 $\langle X, A, \Psi \rangle$ is a tableau

   is a useless superset of Φ.

Tableau is *incomplete* iff it is not complete.

Closed/open tableau

Definition (Closed/open tableau)

Let $\langle X, A, \Phi \rangle$ be a tableau. $\langle X, A, \Phi \rangle$ is *closed* iff it satisfies the conditions:

1. $\langle X, A, \Phi \rangle$ is a complete tableau
2. all branches in Φ are closed.

Tableau is *open* iff it is not closed.

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