Tableau metatheory for propositional and syllogistic logics

Part II: General idea of tableau proofs and examples

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Program of lecture

- General idea of tableau proofs
- Examples of successful as well as failed tableau proofs for:
  - propositional logics
  - syllogistic logics.

Tableau proofs: general idea

Tableau proofs have the following properties:

- to prove that in a given logic from $\{A_1, \ldots, A_n\}$ follows formula B we assume that:
  - $A_1, \ldots, A_n$ hold
  - B does not hold
- tableau proofs are then **indirect proofs**
- next we decompose the assumptions by some rules, called **tableau rules**
- and we try to get some kind of **atomic expressions**
- tableau proofs are then **analytic proofs**

Tableau proofs: general idea

- during decomposing various possibilities of the decomposition can appear – they are called **branches**
- if on all branches some kind of set of expressions called **tableau-inconsistent** appears, then the proof is successful
- if on at least one branch all expressions were decomposed and no **tableau-inconsistent** set appeared, then the proof is failed
- if the proof fails, then it is also possible to read off a **counter-model**.

Language of tableau proofs

- The assumption that:
  - $A_1, \dots, A_n$ hold
  - B does not hold
- is made as well as the whole proof is carried out in a language of tableau proofs Ex for a given logic
- set Ex is usually different than the set of formulas of a given logic
- we have two structures:
  - a language on which a given system of logic is defined: For
  - a language in which tableau proofs are conducted: Ex
- because statement:
  - formula A holds (does not hold)
- can express a fact that A has (has not) got some (of many possible) logical value at a world or even more complicated semantic properties that are expressible in Ex, but not in For.

Examples of tableau proofs

- To experience a diversity of tableau proofs we present various examples from the range we would like to cover in our metatheory.
- So, the examples are limited to propositional and syllogistic cases.

Example: language of CPL

Set of symbols of the language of CPL consists of:

1. sentential letters $\mathsf{Var} = \{p_i : i \in \mathbb{N}\}$ (in practice we will write: p, q, r etc.)
2. logical constants: $\mathsf{Con} = \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$
3. brackets:), (.

Example: language of CPL

Set of formulas of CPL is the least set X that satisfies the conditions:

1. $\mathsf{Var} \subseteq X$
2. if $A \in X$, then $\neg A \in X$
3. if A, $B \in X$, then $(A \sharp B) \in X$, where $\sharp \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

The set of formulas will be denoted by For and its members will be called *formulas*.

We accept all conventions about removing external brackets and strength of connectives according to the pattern:

$\neg$
$\wedge \ \vee$
$\rightarrow$
$\leftrightarrow$

Example: semantics of CPL

A *classical valuation* of For is a function $V\colon \mathrm{For} \longmapsto \{0,1\}$ that for all $A, B \in \mathrm{For}$ satisfies the conditions:

1. $V(\neg A) = 1$ iff $V(A) = 0$
2. $V(A \wedge B) = 1$ iff $V(A) = 1$ & $V(B) = 1$
3. $V(A \vee B) = 1$ iff $V(A) = 1$ or $V(B) = 1$
4. $V(A \rightarrow B) = 1$ iff $V(A) = 0$ or $V(B) = 1$
5. $V(A \leftrightarrow B) = 1$ iff $V(A) = V(B)$.

Let $X \subseteq \mathrm{For}$. We assume abbreviation:

- $V(X) = 1$ iff $\forall_{A \in X} V(A) = 1$.

Tableau proofs: semantically determined CPL

CPL is semantically determined as follows:

Let $X \cup \{A\} \subseteq \mathsf{For}$. We say that formula A *is a consequence of* X *in respect of* CPL (shortly: $X \models_{\mathrm{CPL}} A$) iff for all classical valuations V:

$$\text{if } \mathsf{V}(X) = 1, \text{ then } \mathsf{V}(A) = 1.$$

Tableau for CPL

Intuitively, we assume:

- tableau language is: $\mathsf{Ex} = \mathsf{For}$
- *tableau starting inconsistency* is when proving that $\{A_1, \ldots, A_k\} \models_{\mathrm{CPL}} B$, we assume $A_1, \ldots, A_k$ and $\neg B$
- *tableau inconsistency* is when C and $\neg C$ together appear on the same branch, for some formula $C \in \mathsf{For}$
- we also assume some set of tableau rules for CPL.

Tableau rules for CPL

$$(R_\wedge) \quad \frac{A \wedge B}{\begin{array}{c} A \\ B \end{array}} \qquad\qquad (R_\vee) \quad \frac{A \vee B}{A \mid B}$$

$$(R_\rightarrow) \quad \frac{A \rightarrow B}{\neg A \mid B} \qquad\qquad (R_\leftrightarrow) \quad \frac{A \leftrightarrow B}{\begin{array}{c|c} A & \neg A \\ B & \neg B \end{array}}$$

Tableau rules for CPL

$$(R_{\neg\neg}) \quad \frac{\neg\neg A}{A}$$

$$(R_{\neg\wedge}) \quad \frac{\neg(A \wedge B)}{\neg A \mid \neg B} \qquad (R_{\neg\vee}) \quad \frac{\neg(A \vee B)}{\begin{array}{c}\neg A \\ \neg B\end{array}}$$

$$(R_{\neg\to}) \quad \frac{\neg(A \to B)}{\begin{array}{c}A \\ \neg B\end{array}} \qquad (R_{\neg\leftrightarrow}) \quad \frac{\neg(A \leftrightarrow B)}{\begin{array}{c|c}A & \neg A \\ \neg B & B\end{array}}$$

Tableau proofs: example of CPL

(Transitivity)

$$\frac{\begin{array}{c} p \to q \\ q \to r \end{array}}{p \to r}$$

Successful tableau proof in CPL

1.	$p \rightarrow q$		Prem
2.	$q \rightarrow r$		Prem
3.	$\neg(p \rightarrow r)$		$\neg$ Conc
4.	p		
5.	$\neg r$		$(R_{\neg\rightarrow})(3)$
6.	$\neg p$	q	$(R_{\rightarrow})(1)$
7.	$\otimes$		
8.		$\neg q$ r	$(R_{\rightarrow})(2)$
		$\otimes$ $\otimes$	

Successful tableau proof in CPL

So it means, that

$$\{p \to q, q \to r\} \models_{\mathrm{CPL}} p \to r$$

if our tableau system (the set of tableau rules) is sound in respect to $\models_{\mathrm{CPL}}$!

Failed tableau proof in CPL

$$\begin{array}{l} p \to q \\ \neg q \vee r \\ \neg p \\ \hline \neg r \end{array}$$

Failed tableau proof in CPL

1.			$p \to q$		Prem
2.			$\neg q \vee r$		Prem
3.			$\neg p$		Prem
4.			$\neg\neg r$		$\neg$ Conc
5.			r		$(R_{\neg\neg})(4)$
6.	$\neg p$			q	$(R_{\to})(1)$
7.	$\neg q$	r	$\neg q$	r	$(R_{\vee})(2)$
			$\otimes$		

Failed tableau proof in CPL

The failed proof provides three open branches and valuations that falsifies the argument.

If we take a valuation $V(p) = 0$ and $V(r) = 1$, then whatever we take for the remaining letters, we have:

$$V(p \to q) = V(\neg q \vee r) = V(\neg p) = 1,$$

but $V(\neg r) = 0$.

Hence, $\{p \to q, \neg q \vee r, \neg p\} \not\models_{\mathrm{CPL}} \neg r$.

It generally happens, if our tableau system (the set of tableau rules) is complete in respect to $\models_{\mathrm{CPL}}$!

Example: language of Ł3

Logic Ł3 is defined on set of formulas For, too.

Ł3 was motivated by problem of *futura contingentia*.

Jan Łukasiewicz, "O logice trójwartościowej", *Ruch Filozoficzny* 5 (1920), pp. 170–1. Translation: "On three-valued logic", in *Selected works*, ed. L. Borkowski. Amsterdam: North-Holland, 1970, pp. 87–8.

There appears an essential difference in semantics, since the third logical value is additionally introduced: $\frac{1}{2}$.

Example: semantics of Ł3

A Ł3–*valuation* of For is a function V: For $\longmapsto \{0, 1, \frac{1}{2}\}$ that for all $A, B \in$ For satisfies the conditions presented in the matrixes.

A	$\neg A$
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

$\wedge$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

$\vee$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$\rightarrow$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	1	$\frac{1}{2}$
0	1	1	1

$\leftrightarrow$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$
0	0	$\frac{1}{2}$	1

Example: semantics of Ł3

Let $X \subseteq$ For. Again we assume abbreviation:

- $V(X) = 1$ iff $\forall_{A \in X} V(A) = 1$.

Tableau proofs: semantically determined Ł3

Ł3 is semantically determined as follows:

Let $X \cup \{A\} \subseteq \mathrm{For}$. We say that formula A *is a consequence of* X *in respect of* Ł3 (shortly: $X \models_{Ł3} A$) iff for all Ł3-valuations V:

$$\text{if } \mathsf{V}(X) = 1, \text{ then } \mathsf{V}(A) = 1.$$

Tableau for Ł3

Intuitively, we assume:

- tableau language is: $\mathrm{Ex} \neq \mathrm{For}$
- $\mathrm{Ex} = \{\langle A : n\rangle | A \in \mathrm{For}, n \in \{1, 0, \frac{1}{2}, \bullet\}\}$
- when it is clear, we just write: $A : n$ instead of $\langle A : n\rangle$
- *tableau starting inconsistency* is when proving that $\{A_1, \ldots, A_k\} \models_{Ł3} B$ we assume $A_1 : 1, \ldots, A_k : 1$ and $B : \bullet$
- *tableau inconsistency* is when $C : m$ and $C : n$ together appear on the same branch and $n \neq m$, where $n, m \in \{1, 0, \frac{1}{2}\}$, for formula $C \in \mathrm{For}$
- we also assume some set of tableau rules for Ł3.

Tableau rules for Ł3

$$(R_{\bullet}) \quad \frac{A : \bullet}{A : 0 \mid A : \frac{1}{2}}$$

$$(R_{\neg 1}) \quad \frac{\neg A : 1}{A : 0} \qquad (R_{\neg 0}) \quad \frac{\neg A : 0}{A : 1} \qquad (R_{\neg \frac{1}{2}}) \quad \frac{\neg A : \frac{1}{2}}{A : \frac{1}{2}}$$

$$(R_{\wedge 1}) \quad \frac{A \wedge B : 1}{\begin{array}{c} A : 1 \\ B : 1 \end{array}} \qquad (R_{\wedge 0}) \quad \frac{A \wedge B : 0}{A : 0 \mid B : 0}$$

$$(R_{\wedge \frac{1}{2}}) \quad \frac{A \wedge B : \frac{1}{2}}{\begin{array}{c|c|c} A : 1 & A : \frac{1}{2} & A : \frac{1}{2} \\ B : \frac{1}{2} & B : \frac{1}{2} & B : 1 \end{array}}$$

Tableau rules for Ł3

$$(R_{\vee 1}) \quad \frac{A \vee B : 1}{A : 1 \mid B : 1} \qquad\qquad (R_{\vee 0}) \quad \frac{A \vee B : 0}{\begin{array}{c} A : 0 \\ B : 0 \end{array}}$$

$$(R_{\vee \frac{1}{2}}) \quad \frac{A \vee B : \frac{1}{2}}{\begin{array}{c|c|c} A : 0 & A : \frac{1}{2} & A : \frac{1}{2} \\ B : \frac{1}{2} & B : \frac{1}{2} & B : 0 \end{array}}$$

Tableau rules for Ł3

$$(R_{\to 1}) \quad \frac{A \to B : 1}{A : 0 \;\Big|\; B : 1 \;\Big|\; \begin{array}{l} A : \frac{1}{2} \\ B : \frac{1}{2} \end{array}} \qquad (R_{\to 0}) \quad \frac{A \to B : 0}{\begin{array}{c} A : 1 \\ B : 0 \end{array}}$$

$$(R_{\to \frac{1}{2}}) \quad \frac{A \to B : \frac{1}{2}}{\begin{array}{l} A : 1 \\ B : \frac{1}{2} \end{array} \;\Big|\; \begin{array}{l} A : \frac{1}{2} \\ B : 0 \end{array}}$$

Tableau rules for Ł3

$$(R_{\leftrightarrow 1}) \quad \frac{A \leftrightarrow B : 1}{\begin{array}{c|c|c} A : 1 & A : 0 & A : \frac{1}{2} \\ B : 1 & B : 0 & B : \frac{1}{2} \end{array}}$$

$$(R_{\leftrightarrow 0}) \quad \frac{A \leftrightarrow B : 0}{\begin{array}{c|c} A : 1 & A : 0 \\ B : 0 & B : 1 \end{array}}$$

$$(R_{\leftrightarrow \frac{1}{2}}) \quad \frac{A \leftrightarrow B : \frac{1}{2}}{\begin{array}{c|c|c|c} A : 1 & A : \frac{1}{2} & A : 0 & A : \frac{1}{2} \\ B : \frac{1}{2} & B : 1 & B : \frac{1}{2} & B : 0 \end{array}}$$

Successful tableau proof in Ł3

(Contraposition) $\dfrac{\begin{array}{l} p \to q \\ \neg q \end{array}}{\neg p}$

1.	$p \to q : 1$						Prem
2.	$\neg q : 1$						Prem
3.	$\neg p : \bullet$						$\bullet$ Conc
4.	$\neg p : 0$			$\neg p : \frac{1}{2}$			$(R_{\bullet})(3)$
5.	$p : 1$			$p : \frac{1}{2}$			$(R_{\neg 0})(4)$; $(R_{\neg \frac{1}{2}})(4)$
6.	$p : 0$	$q : 1$	$p : \frac{1}{2}$	$p : 0$	$q : 1$	$p : \frac{1}{2}$	
7.	$\otimes$	$q : 0$	$q : \frac{1}{2}$	$\otimes$	$q : 0$	$q : \frac{1}{2}$	$(R_{\to 1})(1)$; $(R_{\neg 1})(2)$
8.		$\otimes$	$\otimes$		$\otimes$	$q : 0$	$(R_{\neg 1})(2)$
						$\otimes$	

Successful tableau proof in Ł3

So it means, that

$\{p \to q, \neg q\} \models_{Ł3} \neg p$

if our tableau system (the set of tableau rules) is sound in respect to $\models_{Ł3}$!

Failed tableau proof in Ł3

$$\begin{array}{l} (p \vee \neg p) \to q \\ \hline q \end{array}$$

1.			$(p \vee \neg p) \to q : 1$				Prem
2.			$q : \bullet$				$\bullet$ Conc
3.		$q : 0$			$q : \frac{1}{2}$		$(R_{\bullet})(2)$
4.	$p \vee \neg p : 0$	$q : 1$	$p \vee \neg p : \frac{1}{2}$				
5.	$p : 0$	$\otimes$	$q : \frac{1}{2}$				$(R_{\to 1})(1)$
6.	$\neg p : 0$		$\otimes$				$(R_{\vee 0})(4)$
7.	$p : 1$						$(R_{\neg 0})(6)$
8.	$\otimes$						
9.				$p \vee \neg p : 0$	$q : 1$	$p \vee \neg p : \frac{1}{2}$	
10.				$p : 0$	$\otimes$	$q : \frac{1}{2}$	$(R_{\to 1})(1)$
11.				$\neg p : 0$		$p : \frac{1}{2}$	$(R_{\vee 0})(9)$
12.				$p : 1$		$\neg p : \frac{1}{2}$	$(R_{\neg 0})(11)$; $(R_{\vee \frac{1}{2}})(9)$
13.				$\otimes$		$p : \frac{1}{2}$	$(R_{\neg \frac{1}{2}})(12)$

Failed tableau proof in Ł3

The failed proof finishes with an open branch and a falsifying valuation that can be read off.

If we take a valuation $V(p) = \frac{1}{2} = V(q)$, then whatever we take for the remaining letters, we have:

$V(p \vee \neg p \to q) = 1$,

but $V(q) = \frac{1}{2}$.

Hence, $\{p \vee \neg p \to q\} \not\models_{\text{Ł}3} q$.

It generally happens, if our tableau system (the set of tableau rules) is complete in respect to $\models_{\text{Ł}3}$!

Example: language of RF

Set of symbols of the language of RF consists of:

1. the same symbols as CPL
2. and additional binary logical constants: $\{\wedge^{\mathsf{w}}, \vee^{\mathsf{w}}, \rightarrow^{\mathsf{w}}, \leftrightarrow^{\mathsf{w}}\}$.

Set of formulas of RF is the least set X that satisfies the conditions:

1. $\mathsf{Var} \subseteq X$
2. if $A \in X$, then $\neg A \in X$
3. if A, $B \in X$, then $(A \sharp B) \in X$, where $\sharp \in \{\wedge, \vee, \rightarrow, \leftrightarrow, \wedge^{\mathsf{w}}, \vee^{\mathsf{w}}, \rightarrow^{\mathsf{w}}, \leftrightarrow^{\mathsf{w}}\}$.

The set of formulas is denoted by $\mathsf{For}^{\mathsf{w}}$, while its members are called *formulas*.

Example: semantics of RF

A *model* for the relating formulas is a pair $\langle v, \mathsf{R}\rangle$, where $v \colon \mathsf{Var} \longmapsto \{0,1\}$ and $\mathsf{R} \subseteq \mathsf{For}^{\mathsf{w}} \times \mathsf{For}^{\mathsf{w}}$.

If two propositions A and B remain in relation R: $\mathsf{R}(A, B)$, then we state they are *related*.

Let $\mathfrak{M} = \langle v, \mathsf{R}\rangle$ be a model for RF. For all $A, B \in \mathsf{For}$ we assume the following truth conditions:

1. $\mathfrak{M} \models A$ iff $v(A) = 1$, if $A \in \mathsf{Var}$
2. classical conditions for $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
3. $\mathfrak{M} \models A \wedge^{\mathsf{w}} B$ iff $\mathfrak{M} \models A$ & $\mathfrak{M} \models B$ & $\mathsf{R}(A, B)$
4. $\mathfrak{M} \models A \vee^{\mathsf{w}} B$ iff $[\mathfrak{M} \models A$ or $\mathfrak{M} \models B]$ & $\mathsf{R}(A, B)$
5. $\mathfrak{M} \models A \rightarrow^{\mathsf{w}} B$ iff $[\mathfrak{M} \not\models A$ or $\mathfrak{M} \models B]$ & $\mathsf{R}(A, B)$
6. $\mathfrak{M} \models A \leftrightarrow^{\mathsf{w}} B$ iff $[\mathfrak{M} \models A$ iff $\mathfrak{M} \models B]$ & $\mathsf{R}(A, B)$.

Example: semantics of RF

Let $X \subseteq \mathsf{For}^{\mathsf{w}}$. We assume abbreviation:

- $\mathfrak{M} \models X$ iff $\forall_{A \in X} \mathfrak{M} \models A$.

Tableau proofs: semantically determined RF

RF is semantically determined as follows:

Let $X \cup \{A\} \subseteq \mathsf{For}^{\mathsf{w}}$. We say that formula A *is a consequence of* X *in respect of* RF (shortly: $X \models_{\mathrm{RF}} A$) iff for all models $\mathfrak{M}$:

$$\text{if } \mathfrak{M} \models X, \text{ then } \mathfrak{M} \models A.$$

Motivations for relating logics: extensionality vs. intentionality

Two formulas can be — for example — related by R: analytically, causally, thematically, temporally etc., or anywise we want.

We can distinguish horizontal, vertical and diagonal conditions that may determine subclasses of models, and consequently define specific relating logics.

Let us note for example that the set of models:

$$\mathbf{M}^U = \{\langle v, \mathsf{R}\rangle : \mathsf{R} = \mathsf{For}^{\mathsf{w}} \times \mathsf{For}^{\mathsf{w}}\},$$

defines CPL as $\models_{\mathbf{M}^U}$, if we reduce the language only to connectives with superscript $^{\mathsf{w}}$.

Historical remarks on relating logic

Jarmużek, T. and Kaczkowski, B. "On some logic with a relation imposed on formulae: tableau system F", *Bulletin of the Section of Logic* 43(1/2)(2014): pp. 53–72.

Epstein, R. L., *The Semantic Foundations of Logic. Vol. 1: Propositional Logics*, Nijhoff International Philosophy Series, 1990.

Walton, D. N., "Philosophical basis of relatedness logic", *Philosophical Studies*, 36/2(1979): pp. 115–136.

Tableau for RF

Intuitively, we assume:

- tableau language is: $\mathsf{Ex} \supset \mathsf{For}^{\mathsf{w}}$
- $\mathsf{Ex} = \mathsf{For}^{\mathsf{w}} \cup \{ARB \colon A, B \in \mathsf{For}^{\mathsf{w}}\} \cup \{A \not R B \colon A, B \in \mathsf{For}^{\mathsf{w}}\}$
- *tableau starting inconsistency* is when proving that $\{A_1, \ldots, A_k\} \models_{\mathrm{RF}} B$ we assume $A_1, \ldots, A_k$ and $\neg B$
- *tableau inconsistency* is when together:
  - C and $\neg C$

  or

  - CRD and $C \not R D$

  appear on the same branch, for $C, D \in \mathsf{For}^{\mathsf{w}}$
- we also assume some set of tableau rules for RF.

Tableau rules for RF

For classical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ we assume CPL tableau rules.

$$(R_{\wedge^{\mathsf{w}}}) \quad \frac{A \wedge^{\mathsf{w}} B}{\begin{array}{c} A \\ B \\ ARB \end{array}} \qquad (R_{\vee^{\mathsf{w}}}) \quad \frac{A \vee^{\mathsf{w}} B}{\begin{array}{c|c} A & B \\ ARB & ARB \end{array}}$$

$$(R_{\rightarrow^{\mathsf{w}}}) \quad \frac{A \rightarrow^{\mathsf{w}} B}{\begin{array}{c|c} \neg A & B \\ ARB & ARB \end{array}} \qquad (R_{\leftrightarrow^{\mathsf{w}}}) \quad \frac{A \leftrightarrow^{\mathsf{w}} B}{\begin{array}{c|c} A & \neg A \\ B & \neg B \\ ARB & ARB \end{array}}$$

Tableau rules for RF

$$(R_{\neg\wedge^{\mathrm{w}}}) \quad \frac{\neg(A \wedge^{\mathrm{w}} B)}{\neg A \;\big|\; \neg B \;\big|\; A \not R B} \qquad\qquad (R_{\neg\vee^{\mathrm{w}}}) \quad \frac{\neg(A \vee^{\mathrm{w}} B)}{\begin{array}{c|c} \neg A & A \not R B \\ \neg B & \end{array}}$$

$$(R_{\neg\to^{\mathrm{w}}}) \quad \frac{\neg(A \to^{\mathrm{w}} B)}{\begin{array}{c|c} A & A \not R B \\ \neg B & \end{array}} \qquad\qquad (R_{\neg\leftrightarrow^{\mathrm{w}}}) \quad \frac{\neg(A \leftrightarrow^{\mathrm{w}} B)}{\begin{array}{c|c|c} A & \neg A & A \not R B \\ \neg B & B & \end{array}}$$

Tableau proofs: example of RF

$$\frac{\begin{array}{l} p \\ p \rightarrow^{\mathsf{w}} q \end{array}}{p \wedge^{\mathsf{w}} q}$$

Successful tableau proof in RF

1.			p				Prem
2.			$p \rightarrow^{\mathrm{w}} q$				Prem
3.			$\neg(p \wedge^{\mathrm{w}} q)$				$\neg$ Conc
4.	$\neg p$		$\neg q$		$p \not{R} q$		$(R_{\wedge^{\mathrm{w}}})(3)$
	$\otimes$						
5.		$\neg p$	q	$\neg p$	q		
6.		pRq	pRq	pRq	pRq		$(R_{\rightarrow^{\mathrm{w}}})(2)$
		$\otimes$	$\otimes$	$\otimes$	$\otimes$		

Successful tableau proof in RF

So it means, that

$\{p, p \to^{\mathsf{w}} q\} \models_{\mathrm{RF}} p \wedge^{\mathsf{w}} q$

if our tableau system (the set of tableau rules) is sound in respect to $\models_{\mathrm{RF}}$!

Tableau proofs: example of RF

$$
\text{(Transitivity)} \qquad \frac{\begin{array}{c} p \rightarrow^{\mathsf{w}} q \\ q \rightarrow^{\mathsf{w}} r \end{array}}{p \rightarrow^{\mathsf{w}} r}
$$

Failed tableau proof in RF

1.	$p \to^{\mathrm{w}} q$	Prem
2.	$q \to^{\mathrm{w}} r$	Prem
3.	$\neg(p \to^{\mathrm{w}} r)$	$\neg$Conc
4.	$p \ \neg r$ | $p \not R r$	$(R_{\neg\to^{\mathrm{w}}})(3)$
5.	$\neg p$ | q | $\neg p$ | q	
6.	pRq $\otimes$ | pRq | pRq | pRq	$(R_{\to^{\mathrm{w}}})(1)$
7.	$\neg q$ | r | $\neg q$ | r | $\neg q$ | r	
8.	qRr $\otimes$ | qRr $\otimes$ | qRr | qRr | qRr $\otimes$ | qRr	$(R_{\to^{\mathrm{w}}})(2)$

Failed tableau proof in RF

The failed proof results in three open branches. For example, we can consider the far-right-open-branch. If we take:

- any such valuation v that $v(q) = v(r) = 1$
- and relation $\mathrm{R} = \{\langle p, q\rangle, \langle q, r\rangle\}$,

then we have model $\mathfrak{M} = \langle v, \mathrm{R}\rangle$ such that:

$\mathfrak{M} \models p \to^{\mathrm{w}} q$ and $\mathfrak{M} \models q \to^{\mathrm{w}} r$,

but $\mathfrak{M} \not\models p \to^{\mathrm{w}} r$.

Hence, $\{p \to^{\mathrm{w}} q, q \to^{\mathrm{w}} r\} \not\models_{\mathrm{RF}} p \to^{\mathrm{w}} r$.

It generally happens, if our tableau system (the set of tableau rules) is complete in respect to $\models_{\mathrm{RF}}$!

Example: Classical Syllogistic (CS)

Logic of categorial propositions:

- All P are Q
- No P are Q
- Some P is/are Q
- Some P is/are not Q

where P and Q are general terms, like: *crocodile*, *spider*, *human being*, *angel* etc.

Traditionally (in Aristotle's syllogistic) terms were supposed to be non-empty.

Here terms can be empty.

So we consider a syllogistic that is almost classical.

Example: language of CS

Symbols of CS are:

- term letters: $\text{Term} = \{P_i : i \in \mathbb{N}\}$ (in fact we will write: P, Q, R etc.)
- logical constants: $\text{Con} = \{\mathbf{a}, \mathbf{i}, \mathbf{o}, \mathbf{e}\}$.

Set of formulas For_{CS} consists of:

1. $\Phi\mathbf{a}\Psi$
2. $\Phi\mathbf{i}\Psi$
3. $\Phi\mathbf{o}\Psi$
4. $\Phi\mathbf{e}\Psi$

where $\Phi, \Psi \in \text{Term}$.

Example: semantics of CS

A *model* for $\mathrm{For}_{\mathrm{CS}}$ is an ordered pair: $\langle D, v\rangle$, where:

- D is non–empty set
- $v : \mathrm{Term} \longmapsto \mathrm{P}(D)$.

Alternatively, we can assume that D is a non–empty set of sets, and then:

- $v : \mathrm{Term} \longmapsto D$.

Example: semantics of CS

We prefer the latter option, but the former one was promoted in the following papers:

Jarmużek T., "Tableau System for Logic of Categorial Propositions and Decidability", *Bulletin of The Section of Logic*, 2008, 37 (3/4), pp. 223–231.

Jarmużek T., Pietruszczak A., "Decidability methods for modal syllogisms", Trends in Logic XIII, (Eds) A. Indrzejczak, J. Kaczmarek, and M. Zawidzki (eds.), Wydawnictwo Uniwersytetu Łódzkiego, Łódź 2014, pp. 95–112.

Pietruszczak A., Jarmużek T., "Pure Modal Logic of Names and Tableau Systems", *Studia Logica*, (2018), https://doi.org/10.1007/s11225-018-9788-6

Example: semantics of CS

Let $\mathfrak{M} = \langle D, v \rangle$ be a model for $\mathrm{For}_{\mathrm{CS}}$. We assume the following truth conditions. For all $\Phi, \Psi \in$ Term:

- $\mathfrak{M} \models \Phi\mathbf{a}\Psi$ iff $v(\Phi) \subseteq v(\Psi)$
- $\mathfrak{M} \models \Phi\mathbf{i}\Psi$ iff $v(\Phi) \cap v(\Psi) \neq \emptyset$
- $\mathfrak{M} \models \Phi\mathbf{o}\Psi$ iff $v(\Phi) \nsubseteq v(\Psi)$
- $\mathfrak{M} \models \Phi\mathbf{e}\Psi$ iff $v(\Phi) \cap v(\Psi) = \emptyset$.

Example: semantics of CS

Let $X \subseteq \mathsf{For}_{\mathrm{CS}}$. We assume abbreviation:

- $\mathfrak{M} \models X$ iff $\forall_{A \in X} \mathfrak{M} \models A$.

Tableau proofs: semantically determined CS

CS is semantically determined as follows:

Let $X \cup \{A\} \subseteq \mathsf{For}_{\mathrm{CS}}$. We say that formula A *is a consequence of* X *in respect of* CS (shortly: $X \models_{\mathrm{CS}} A$) iff for all models $\mathfrak{M}$:

$$\text{if } \mathfrak{M} \models X, \text{ then } \mathfrak{M} \models A.$$

Tableau for CS

Intuitively, we assume:

- tableau language is: $\mathsf{Ex} \supset \mathsf{For}_{\mathrm{CS}}$
- let $\mathbb{N}^{\circ} = \{ni : n \in \{+,-\}, i \in \mathbb{N}\}$
- $\mathsf{Ex} = \mathsf{For}_{\mathrm{CS}} \cup \{\sim A : A \in \mathsf{For}_{\mathrm{CS}}\} \cup \{\Phi^{\circ} : \Phi \in \mathsf{Term}, \circ \in \mathbb{N}^{\circ}\}$
- *tableau starting inconsistency* is when proving that $\{A_1, \ldots, A_k\} \models_{\mathrm{CS}} B$ we assume $A_1, \ldots, A_k$ and $\sim B$
- *tableau inconsistency* is if:
  - C and $\sim C$, for some $C \in \mathsf{For}_{\mathrm{CS}}$

  or
  - Φ^{-i} and Φ^{+i}, for some $\Phi \in \mathsf{Term}$ and $i \in \mathbb{N}$

  appear on the same branch
- we also assume some set of tableau rules for CS.

Tableau rules for CS

$$(R_{\mathbf{a}}) \quad \frac{\begin{array}{c}\Phi\mathbf{a}\Psi \\ \Phi^{+i}\end{array}}{\Psi^{+i}} \qquad\qquad (R_{\mathbf{e}}) \quad \frac{\begin{array}{c}\Phi\mathbf{e}\Psi \\ \Phi^{+i}\end{array}}{\Psi^{-i}}$$

$$(R_{\mathbf{i}}) \quad \frac{\Phi\mathbf{i}\Psi}{\begin{array}{c}\Phi^{+i} \\ \Psi^{+i}\end{array}} \qquad\qquad (R_{\mathbf{o}}) \quad \frac{\Phi\mathbf{o}\Psi}{\begin{array}{c}\Phi^{+i} \\ \Psi^{-i}\end{array}}$$

In case of rules $(R_{\mathbf{i}})$ and $(R_{\mathbf{o}})$ index i must be new, so it has not appeared on the branch yet.

Tableau rules for CS

$$(R_{\sim\mathbf{a}}) \quad \frac{\sim \Phi\mathbf{a}\Psi}{\begin{array}{c}\Phi^{+i}\\ \Psi^{-i}\end{array}} \qquad\qquad (R_{\sim\mathbf{e}}) \quad \frac{\sim \Phi\mathbf{e}\Psi}{\begin{array}{c}\Phi^{+i}\\ \Psi^{+i}\end{array}}$$

$$(R_{\sim\mathbf{i}}) \quad \frac{\sim \Phi\mathbf{i}\Psi}{\Phi\mathbf{e}\Psi} \qquad\qquad (R_{\sim\mathbf{o}}) \quad \frac{\sim \Phi\mathbf{o}\Psi}{\Phi\mathbf{a}\Psi}$$

In case of rules $(R_{\sim\mathbf{a}})$ and $(R_{\sim\mathbf{e}})$ index i must be new, so it has not appeared on the branch yet.

Successful tableau proof in CS

$$\text{(Barbara)} \qquad \frac{\begin{array}{c} P\mathbf{a}Q \\ Q\mathbf{a}R \end{array}}{P\mathbf{a}R}$$

1.	$P\mathbf{a}Q$	Prem
2.	$Q\mathbf{a}R$	Prem
3.	$\sim P\mathbf{a}R$	$\sim$ Conc
4.	P^{+1}	
5.	R^{-1}	$(R_{\sim\mathbf{a}})(3)$
6.	Q^{+1}	$(R_{\mathbf{a}})(1)(4)$
7.	R^{+1}	$(R_{\mathbf{a}})(2)(6)$
	$\otimes$	

Successful tableau proof in CS

So it means, that

$\{P\mathbf{a}Q, Q\mathbf{a}R\} \models_{\mathrm{CS}} P\mathbf{a}R$

if our tableau system (the set of tableau rules) is sound in respect to $\models_{\mathrm{CS}}$!

Failed tableau proof in CS

$$\frac{\begin{array}{c} P\mathbf{o}Q \\ Q\mathbf{a}R \end{array}}{P\mathbf{o}R}$$

1.	$P\mathbf{o}Q$	Prem
2.	$Q\mathbf{a}R$	Prem
3.	$\sim P\mathbf{o}R$	$\sim$ Conc
4.	$P\mathbf{a}R$	$(R_{\sim\mathbf{o}})(3)$
5.	P^{+1}	
6.	Q^{-1}	$(R_{\mathbf{o}})(1)$
7.	R^{+1}	$(R_{\mathbf{a}})(4)(5)$

Failed tableau proof in CS

The failed proof gives one open branch. If we take model $\mathfrak{M} = \langle D, v \rangle$ such that:

- $D = \{\{1\}, \emptyset\}$
- for all $\Phi \in$ Term:

$$v(\Phi) = \begin{cases} \emptyset, & \text{if } \Phi \text{ is } Q \\ \{1\}, & \text{otherwise.} \end{cases}$$

then $\mathfrak{M} \models P\mathbf{o}Q$ and $\mathfrak{M} \models Q\mathbf{a}R$,

but $\mathfrak{M} \not\models P\mathbf{o}R$.

Hence, $\{P\mathbf{o}Q, Q\mathbf{a}R\} \not\models_{\mathrm{CS}} P\mathbf{o}R$.

It generally happens, if our tableau system (the set of tableau rules) is complete in respect to $\models_{\mathrm{CS}}$!

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